On associated q-orthogonal polynomials with a class of discrete q-distributions

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Abstract

The aim of this work is twofold, on the one hand the associated q-orthogonal polynomials with a class of discrete q-distributions, by their weight functions are derived and on the other hand the combinatorial interpretation of these q-orthogonal polynomials is presented. Specifically, we derive the associated q-orthogonal polynomials with some deformed types of the q-negative Binomial of the second kind, q-binomial of the second kind and Euler distributions. The derived q-orthogonal polynomials are based on the little q-Jacobi, affine q-Krawtchouk and little q-Laguerre/Wall orthogonal polynomials, respectively. Also, we provide a combinatorial interpretation of these q-orthogonal polynomials, as applications of a generalization of matching extensions in paths, already presented by the authors.

1 Introduction

Kemp [Kem92a, Kem92b], introduced Heine and Euler, q-Poisson distributions, with probability functions given respectively by

$$f_X^H(x) = e_q(-\lambda)\frac{q^{\binom{x}{2}}\lambda^x}{[x]_q!}, x = 0, 1, 2, \ldots,\ \ 0 < q < 1,\ \ 0 < \lambda < \infty \tag{1}$$

and

$$f_X^E(x) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!}, x = 0, 1, 2, \ldots,\ \ 0 < q < 1,\ \ 0 < \lambda(1-q) < 1, \tag{2}$$

where

$$e_q(z) := \sum_{n=0}^{\infty}\frac{(1-q)^n z^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty},\ \ |z| < 1 \tag{3}$$

and

$$E_q(z) := \sum_{n=0}^{\infty}\frac{(1-q)^n q^{\binom{z}{2}} z^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{\binom{z}{2}} z^n}{[n]_q!} = ((1-q)z;q)_\infty,\ \ |z| < 1. \tag{4}$$

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Charalambides [Cha10, Cha16], derived Heine as direct approximation, as $n \to \infty$, of the q-Binomial I and the q-negative Binomial I, with probability functions given respectively by

$$f_X^B(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1}, \ x = 0, 1, \ldots, n, \tag{5}$$

and

$$f_X^{NB}(x) = \binom{n+x-1}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n+x} (1 + \theta q^{j-1})^{-1}, \ x = 0, 1, \ldots, \tag{6}$$

where $\theta > 0, \ 0 < q < 1$.

Moreover, Charalambides [Cha10, Cha16], derived Euler distribution as direct approximation, as $n \to \infty$, of the q-Binomial II and the negative q-Binomial II, with probability functions given respectively by

$$f_X^{BS}(x) = \binom{n}{x}_q \theta^x \prod_{j=1}^{n-x} (1 - \theta q^{j-1}), \ x = 0, 1, \ldots, n, \tag{7}$$

and

$$f_X^{NBS}(x) = \binom{n+x-1}{x}_q \theta^x \prod_{j=1}^{n} (1 - \theta q^{j-1}), \ x = 0, 1, \ldots, \tag{8}$$

where $0 < \theta < 1$ and $0 < q < 1$ or $1 < q < \infty$ with $\theta q^{n-1} < 1$.
Kyriakoussis and Vamvakari [KV10] introduced deformed types of the q-negative Binomial of the first kind, q-binomial of the first kind and of the Heine distributions and derived the associated q-orthogonal polynomials, based on discrete q-Meixner, q-Krawtchouk and q-Charlier orthogonal polynomials respectively.
Moreover, Kyriakoussis and Vamvakari [KV12] established families of terminating and non-terminating q-Gauss hypergeometric series discrete distributions and associated them with defined classes of generalized q-Hahn and big q-Jacobi orthogonal polynomials, respectively.
Also, Kyriakoussis and Vamvakari[KV05] presented generalization of matching extensions in graphs and provided combinatorial interpretation of wide classes of orthogonal and q-orthogonal polynomials as generating functions of matching sets in paths.
In this paper, we derive the associated q-orthogonal polynomials with some deformed types of the q-negative Binomial of the second kind, q-binomial of the second kind and Euler distributions. The derived q-orthogonal polynomials are based on the little q-Jacobi, affine q-Krawtchouk and little q-Laguerre/Wall orthogonal polynomials respectively. Also, we provide a combinatorial interpretation of these q-orthogonal polynomials, as applications of a generalization of matching extensions in paths, already prresented by the authors.
For the needs of this paper the class of discrete q-distributions, q-negative Binomial I, q-Binomial I and Heine will be called *class of discrete q-distributions of type I*, while the class of discrete q-distributions, q-negative Binomial II, q-Binomial II and Euler will be called *class of discrete q-distributions of type II.*

2 Preliminaries

Let v be a probability measure in R with finite moments of all orders

$$s_m = \int_R x^m dv(x).$$

Then there exist a sequence of normalized orthogonal polynomials $\{p_m(x)\}$ with respect to the measure v satisfying the recurrence relation

$$xp_m(x) \quad = \quad p_{m+1}(x) + a_m p_m(x) + b_m p_{m-1}(x), \ m \geq 1, \tag{9}$$

with initial conditions

$$p_0(x) = 1, \;\; p_1(x) = x - a_0.$$

Moreover, they satisfy the orthogonality relation

$$\int_S p_m(x)p_\nu(x)dv(x) = \delta_{m\nu} b_1 b_2 \cdots b_m, \quad m, \nu \geq 0 \tag{10}$$

where $\delta_{m\nu}$ the Kronecker delta.
The polynomials $\{p_m(x)\}$ depend on the moment sequence $\{s_m\}_{m\geq 0}$ and they can be obtain from the formula

$$p_m(x) = \sqrt{\frac{b_1 b_2 \cdots b_m}{D_{m-1}D_m}} \begin{vmatrix} s_0 & s_1 & \dots & s_m \\ s_1 & s_2 & \dots & s_{m+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ s_{m-1} & s_m & \dots & s_{2m-1} \\ 1 & x & \dots & x^m \end{vmatrix}, \tag{11}$$

where $D_m = \det(\{s_{i+j}\}_{0\leq i,j\leq m})$ denotes the Hankel determinant.
Conversely, Favard's (1935) theorem ensures the existence of a probability measure v on R for which the sequence of polynomials determined by the recurrence relation (9) are orthogonal. It can also be shown that the probability measure v is supported only in finitely many points if and only if $b_m = 0$ for some m on, thus the sequence of polynomials is essentially finite. The mean value and the variance of the random variable X in R with probability density function $v(x)$ are given respectively by

$$\mu = a_0 \;\text{ and }\; \sigma^2 = b_1.$$

If $a_m = 0$ then all moments of odd order are zero

$$s_{2m+1} = \int_{x\in R} x^{2m+1} dv(x) = 0.$$

Also, from the recurrence relation (9) the following representation of orthogonal polynomials is derived
$p_m(x) =$

$$p_0(x) \begin{vmatrix} x+a_1 & b_2^{1/2} & 0 & \dots & \dots & 0 \\ b_2^{1/2} & x+a_2 & b_3^{1/2} & 0 & & \vdots \\ 0 & b_3^{1/2} & x+a_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & x+b_{m-1} & c_m^{1/2} \\ 0 & \dots & \dots & 0 & b_m^{1/2} & x+b_m \end{vmatrix}. \tag{12}$$

(see Szegö([Sze59], p.374).

Note that the probability measure v is uniquely determined if the coefficinets a_m and b_m in the reccurence relation (9) are bounded when $m \to \infty$ (see Christiansen [Chr04]).

The q-orthogonal polynomials Little q-Jacobi, affine q-Krawtchouk and little q-Laguerre/Wall satisfy the recurrence relation (9) with a_m and b_m given in the next table respectively.

Table 1 : q-Classical Orthogonal Polynomials: Little q-Jacobi, affine q-Krawtchouk and little q-Laguerre/Wall

Little q-Jacobi	
$p_m^{LitJ}(x;a,b;q)$	
a_m	$\frac{q^m\left(1+a^2bq^{m+1}+a\left(1-(1+b)q^m-(1+b)q^m-(1+b)q^{m+2}+bq^{2m+1}\right)\right)}{(1-abq^{2m})(1-abq^{2m+2})}$
b_m	$\frac{-aq^{m+1}(1-q^m)(1-aq^m)(1-bq^m)(1-abq^m)(c-abq^m)(1-cq^m)}{(1-abq^{2m})^2(1-abq^{2m-1})(1-abq^{2m+1})}$
Affine q-Krawtchouk	
$p_m^{Aff}(x;p,n,q)$	
a_m	$1-\left[(1-q^{m-n})(1-pq^{m+1})-pq^{m-n}(1-q^m)\right]$
b_m	$pq^{m-n}(1-q^m)(1-pq^m)(1-q^{m-n-1})$
Little q-Laguerre/Wall	
$p_m^{LLW}(x,a;q)$	
a_m	$q^m(1-aq^{m+1})+aq^m(1-q^m)$
b_m	$aq^{2m-1}(1-q^m)(1-aq^m)$

3 Main Results

3.1 Associated q-Orthogonal Polynomials with the class of Discrete q-Distributions of type II

In this section we derive the associated q-orthogonal polynomials with the class of discrete q- distributions of type II, (8),(7) and (2), in respect to their weight functions. We begin by transfering from the random variable X of the q-negative Binomial of the second kind distribution (8) to the equal-distributed deformed random variable $Y=[X]_q$, and we obtain a deformed q-negative Binomial II distribution defined in the spectrum $S=\{[x]_q, x=0,1,\ldots\}$ with p.f.

$$\begin{aligned} f_Y^{NBS}(y) &= \binom{n+g(y)-1}{g(y)}_q \theta^{g(y)}\prod_{j=1}^{n}(1-\theta q^{j-1}),\ 0<\theta<1,\ 0<q<1, \\ y &= [0]_q,[1]_q,[2]_q,\ldots, \end{aligned} \tag{13}$$

where

$$g(y)=\frac{\ln(1-(1-q)y)}{\ln q}.$$

Using the orthogonality relation of the normalized little q-Jacobi orthogonal polynomials [Ism05, KS98] and the linear transformation of orthogonal polynomials [Sze59], we easily derive the following result.

Proposition 3.1. *The probability distribution with p.f. $f_Y^{NBS}(y)$ is induced by the normalized linear transformation of the little q-Jacobi orthogonal polynomials, say $p_m^J(y;a,b,q)$, $0<q<1$, given by*

$$p_m^J(y;a,b,q) = \frac{(-1)^m(abq^{m+1};q)_m}{q^{\binom{m}{2}}(1-q)^m(aq;q)_m}p_m^{LitJ}(y;a,b,q), \tag{14}$$

where $y=[x]_q,\ x=0,1,\ldots,$ and $p_m^{LitJ}(y;a,b,q)$ the little q-Jacobi orthogonal polynomials with parameter $a=\theta/q$ and $b=q^{n-1}$.

Next, we transfer from the random variable X of the q-Binomial of the second kind distribution (7) to the equal-distributed deformed random variable $Y=[X]_q$, and we obtain a deformed q-Binomial II distribution defined in the spectrum $S=\{[x]_q,\ \ x=0,1,\ldots\}$ with p.f.

$$\begin{aligned} f_Y^{BS}(y) &= \binom{n}{g(y)}_q \theta^{g(y)}\prod_{j=1}^{n-g(y)}(1-\theta q^{j-1}),\ 0<\theta<1,\ 0<q<1, \\ y &= [0]_q,[1]_q,[2]_q,\ldots, \end{aligned} \tag{15}$$

where

$$g(y) = \frac{\ln(1-(1-q)y)}{\ln q}.$$

Using the orthogonality relation of the normalized affine q-Krawtchouk orthogonal polynomials [Ism05, KS98] and the linear transformation of orthogonal polynomials [Sze59], we easily derive the following result.

Proposition 3.2. *The probability distribution with p.f. $f_Y^{BS}(y)$ is induced by the normalized linear transformation of deformed affine q-Krawtchouk orthogonal polynomials, say $p_m^{AK}(y;p,q,n)$, $0<q<1$, given by*

$$p_m^{AK}(y;p,n,q) \quad = \quad (1-q)^{-m}(pq,q^{-n};q)_m p_m^{Aff}(q^{-n}y;p,n,q), \tag{16}$$

where $y=[x]_q,\ x=0,1,\ldots,$ and $p_m^{Aff}(q^{-n}y;p,n,q)$ the deformed affine q-Krawtchouk orthogonal polynomials with parameter $p=\theta/q$.

Finally, we transfer from the random variable X of the Euler distribution (2) to the equal-distributed deformed random variable $Y=[X]_q$, and we obtain a deformed Euler distribution defined in the spectrum $S=\{[x]_q,\ x=0,1,\ldots\}$ with p.f.

$$\begin{aligned} f_Y^E(y) \quad &= \quad E_q(-\lambda)\frac{\lambda^{g(y)}}{[g(y)]_q!},\ 0<q<1,\ 0<\lambda(1-q)<1, \\ y \quad &= \quad [0]_q,[1]_q,[2]_q,\ldots, \end{aligned} \tag{17}$$

where

$$g(y) = \frac{\ln(1-(1-q)y)}{\ln q}.$$

Using the orthogonality relation of the normalized little q-Laguerre/Wall orthogonal polynomials [Ism05, KS98] and the linear transformation of orthogonal polynomials [Sze59], we easily derive the following result.

Proposition 3.3. *The probability distribution with p.f. $f_Y^E(y)$ is induced by the normalized linear transformation of the little q-Laguerre/Wall orthogonal polynomials, say $p_m^L(y;a,q)$, $0<q<1$, given by*

$$p_m^L(y;a,q) \quad = \quad (-1)^m(aq;q)_m(1-q)^{-m}q^{\binom{m}{2}}p_m^{LLW}(y;a,q), \tag{18}$$

where $y=[x]_q,\ x=0,1,\ldots,$ and $p_m^{LLW}(y;a,q)$ the little q-Laguerre/Wall orthogonal polynomials with parameter $a=\lambda(1-q)$.

Remark 3.4. The approximation, as $n\to\infty$, of the q-Binomial I and the q-negative Binomial I to the Heine distribution, can alternatively be concluded by the limit of the associated q-orthogonal polynomials, q-Krawtchouk and q-Meixner to the q-Charlier ones. Also, the approximation, as $n\to\infty$, of the q-Binomial II and the q-negative Binomial II to the Euler distribution, can also be concluded by the limit of the associated little q-Jacobi and affine q-Krawtchouk to the little q-Laguerre/Wall ones. The above mentioned conclusions can be justified since the coefficients in the recurrence relation of the associated q-orthogonal polynomials are bounded in m.

3.2 Combinatorial Interpetation of the Associated q-Orthogonal Polynomials

Combinatorial interpretation of orthogonal polynomials using matchings in graphs has received much attention by several authors over the last decades. Among them we refer to Feinsilver et al [FSS96], Godsil [God81], Godsil and Gutman [GG81], Viennot [Vie83], and Heilmann and Lieb [HL72], Kyriakoussis and Vamvakari [KV05].
Let G be a simple graph on m vertices with vertex labels 1 to m, having edge weight $W(i,j)$ a non-negative real number for each unordered pair of vertices $<i,j>$, $i=1,2,\ldots,m,\ j=1,2,\ldots,m,\ \ i<j$ and vertex weight $w_i,\ i=1,2,\ldots,m$. Also, let M be a matching set of G consisting of disjoint edges pairwise having no vertex in common. Then the weight of M, say $W_G(M)$, is defined by

$$W_G(M) = \prod_{\langle i,j\rangle \epsilon M} W(i,j)\prod_{i\notin M} w_i$$

and the corresponding generating function in m variables including the vertex and edge weights is defined by

$$P(G; w_1, w_2, \ldots, w_m) = \sum_M (-1)^{|M|} \prod_{\langle i,j\rangle \epsilon M} W(i,j) \prod_{i \notin M} w_i$$

with $|M|$ the number of edges in M, summing over all matchings M of G .
Let L_m be a path on m vertices with edge weight $W(i,j) > 0$ when $|i-j| = 1$, $W(i,j) = 0$ otherwise and with vertex weight w_i, $i = 1, 2, \ldots, m$. Note that w_i and $W(i, i+1)$ are bounded sequences in i, $i = 1, 2, \ldots$.
Kyriakoussis and Vamvakari [KV05], setting

$$\begin{aligned} \mathcal{L}_n &= P(L_m; w_1, w_2, \ldots, w_m) \\ &= \sum_M (-1)^{|M|} \prod_{\langle i,j\rangle \epsilon M} W(i,j) \prod_{i \notin M} w_i \end{aligned} \quad (19)$$

where $|M|$ the number of edges in M, have proved the following proposition.

Proposition 3.5. *The generating function of matching sets in paths, $\mathcal{L}_m$, satisfies the recurrence relation*

$$\mathcal{L}_{m+1} = w_{n+1}\mathcal{L}_m - W(m, m+1)\mathcal{L}_{m-1}, \quad m = 0, 1, 2, \ldots \quad (20)$$

with initial conditions $\mathcal{L}_{-1} = 0$ and $\mathcal{L}_0 = 1$.

Remark 3.6. Setting in (20) vertex weight $w_m = x - a_m$ and edge weight $W(m, m+1) = b_m$, where a_m and b_m are bounded sequences in m, and comparing (20) with (9), we have a wide class of generating functions of matching sets in paths identified with q-orthogonal polynomials. Between them the little q-Jacobi, affine q-Krawtchouk and little q-Laguerre/Wall polynomials where the bounded sequences a_m and b_m are given respectively in the Table 1.

Remark 3.7. 3. Setting in (20) vertex weight $w_m = [x]_q - d_m$ and edge weight $W(m, m+1) = g_m$, where d_m and g_m are bounded sequences in m and comparing (20) with (9) we have a wide class of generating functions of matching sets in paths identified with transformed q-orthogonal polynomials. Between them the associated transformed little q-Jacobi, affine q-Krawtchouk, and little q-Laguerre/Wall polynomials with the deformed q-negative binomial of type II, q-binomial of type II and Euler distributions respectively.

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