On the largest part size and its multiplicity of a random integer partition

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Abstract

Let λ be a partition of the positive integer n chosen uniformly at random among all such partitions. Let $L_n = L_n(\lambda)$ and $M_n = M_n(\lambda)$ be the largest part size and its multiplicity, respectively. For large n, we focus on a comparison between the partition statistics L_n and $L_n M_n$. In terms of convergence in distribution, we show that they behave in the same way. However, it turns out that the expectation of $L_n M_n - L_n$ grows as fast as $\frac{1}{2}\log n$. We obtain a precise asymptotic expansion for this expectation and conclude with an open problem arising from this study.

1 Introduction and Statement of the Results

Partitioning integers into summands (parts) is a subject of intensive research in combinatorics, number theory and statistical physics. If n is a positive integer, then, by a partition λ of n, we mean the representation

$$\lambda : \quad n = \lambda_1 + \lambda_2 + . + \lambda_k, \quad k \geq 1, \tag{1}$$

where the positive integers λ_j satisfy $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$. Let $\Lambda(n)$ be the set of all partitions of n and let $p(n) = |\Lambda(n)|$ (by definition $p(0) = 1$ regarding that the one partition of 0 is the empty partition). The number $p(n)$ is determined asymptotically by the famous partition formula of Hardy and Ramanujan [HR18]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \to \infty.$$

A precise asymptotic expansion for $p(n)$ was found by Rademacher [Rad37] (see also [And76, Chapter 5]). Further on, we assume that, for fixed integer $n \geq 1$, a partition $\lambda \in \Lambda(n)$ is selected uniformly at random. In other words, we assign the probability $1/p(n)$ to each $\lambda \in \Lambda(n)$. We denote this probability measure on $\Lambda(n)$ by $\mathbf{P}$. Let $\mathbf{E}$ be the expectation with respect to $\mathbf{P}$. In this way, each numerical characteristic of $\lambda \in \Lambda(n)$ can be regarded as a random variable, or, a statistic produced by the random generation of partitions of n.

In this paper, we focus on two statistics of random integer partitions $\lambda \in \Lambda(n)$: $L_n = L_n(\lambda) = \lambda_1$, which is the largest part size in representation (1) and $M_n = M_n(\lambda)$, equal to the multiplicity of the largest part λ_1 (i.e., $M_n(\lambda) = m, 1 \leq m \leq k-1$, if $\lambda_1 = ... = \lambda_m > \lambda_{m+1} \geq ... \geq \lambda_k$ in (1), and $M_n(\lambda) = k$, if $\lambda_1 = ... = \lambda_k$).

Each partition $\lambda \in \Lambda(n)$ has a unique graphical representation called Ferrers diagram [And76, Chapter 1]. It illustrates (1) by the two-dimensional array of dots, composed by λ_1 dots in the first (most left) row, λ_2 dots

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in the second row,..., and so on, λ_k dots in the kth row. Therefore, a Ferrers diagram may be considered as a union of disjoint blocks (rectangles) of dots whose side lengths represent the part sizes and their multiplicities of the partition λ, respectively. For instance, Figure 1 illustrates the partition $7+5+5+5+4+2+1+1+1$ of $n = 31$ in which $L_{31} = 7$ and $M_{31} = 1$.

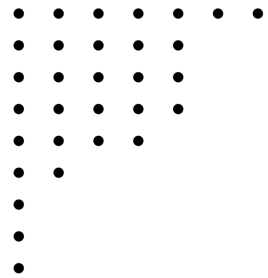

Figure 1

The earliest asymptotic results on random integer partition statistics have been obtained long ago by Husimi [Hum38] and Erdös and Lehner [EL41]. Husimi has derived an asymptotic expansion for $\mathbf{E}(L_n)$ in the context of a statistical physics model of a Bose gas. Erdös and Lehner were apparently the first who have studied random partition statistics in terms of probabilistic limit theorems. In fact, they showed that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{L_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u\right) = H(u), \tag{2}$$

where

$$H(u) = \exp\left(-\frac{1}{c}e^{-cu}\right), \quad -\infty < u < \infty \tag{3}$$

and

$$c = \frac{\pi}{\sqrt{6}}. \tag{4}$$

Later on, Szekeres has studied in detail the asymptotic behavior of the number of integer partitions of n whose largest part is $\le k$ and $= k$ in the whole range of values of $k = k(n)$. In particular, he has obtained in [Sze53] Erdös and Lehner's limiting distribution (2) using an entirely different method of proof. Husimi's asymptotic result was subsequently reconfirmed by Kessler and Livingston [KL76]. Higher moments of L_n were studied in [Ric74]. A general method providing asymptotic expansions of expectations of integer partition statistics was recently proposed by Grabner et al. [GKW14]. Among the numerous examples, they derived a rather complete asymptotic expansion for $\mathbf{E}(L_n)$, namely,

$$\begin{aligned}\mathbf{E}(L_n) &= \frac{\sqrt{n}}{2c}(\log n + 2\gamma - 2\log c) + \frac{\log n}{2c^2} + \frac{1}{4} \\ &+ \frac{1+2\gamma-2\log c}{4c^2} + O\left(\frac{\log n}{n}\right), \quad n \to \infty,\end{aligned} \tag{5}$$

where c is given by (4) and $\gamma = 0.5772...$ denotes the Euler's constant (see [GKW14, Proposition 4.2]). Notice that by conjunction of the Ferrers diagram the largest part and the total number of parts in a random partition of n are identically distributed for any n. The sequence $\{p(n)\mathbf{E}(L_n)\}_{n\ge1}$ is given in [Slo18] as A006128.

There are serious reasons to believe that the multiplicity M_n of the largest part of a random partition of n behaves asymptotically in a much simpler way than many other partition statistics. Grabner and Knopfmacher [GK06] used the Erdös-Lehner limit theorem (2) to establish that

$$\lim_{n\to\infty} \mathbf{E}(M_n) = 1. \tag{6}$$

In addition, among many other important asymptotic results, Fristedt, in his remarkable paper [Fri93], showed that, with probability tending to 1 as $n \to \infty$, the first m_n largest parts in a random partition of n are distinct if $m_n = o(n^{1/4})$. Hence it may not be that L_n constitutes the main contribution to n by a single part size and some smaller part sizes may occur with sufficient multiplicity so that the products of these part sizes with their multiplicities could be much larger than L_n. In terms of the Ferrers diagram, this means that its first block (the block on the highest position in the Ferrers diagram) has typically smaller area than the areas of several next blocks with larger heights (multiplicities of parts).

Our aim in this paper is to study the asymptotic behavior of the area $L_n M_n$ of the first block in the Ferrers diagram of a random partition of n. We show some similarities and differences between the single part size L_n and its corresponding block area $L_n M_n$. As a first step, we obtain a distributional result for M_n that confirms the limit in (6).

Theorem 1.1. *For any $n \geq 1$, we have*

$$\mathbf{P}(M_n = 1) = \frac{p(n-1)}{p(n)}. \tag{7}$$

In addition, if $n \to \infty$, then

$$\mathbf{P}(M_n = 1) = 1 - \frac{c}{\sqrt{n}} + \frac{1 + c^2/2}{n} + O(n^{-3/2}), \tag{8}$$

where the constant c is given by (4).

Combining the Erdös-Lehner limit theorem (2) with (8), one can easily observe that the limiting distributions of L_n and $L_n M_n$ coincide under the same normalization.

Corollary 1.2. *For any real u, we have*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c}\log n \leq u\right) = H(u),$$

where $H(u)$ and c are given by (3) and (4), respectively.

Although L_n and $L_n M_n$ follow the same limiting distribution, the difference in means $\mathbf{E}(L_n M_n) - \mathbf{E}(L_n)$ grows as fast as $\frac{1}{2}\log n$. A complete estimate is given by the following

Theorem 1.3. *If $n \to \infty$, then, as $n \to \infty$,*

$$\mathbf{E}(L_n M_n) = \mathbf{E}(L_n) + \frac{1}{2}\log n - C + O\left(\frac{1}{\log n}\right),$$

where $C = \log c + 1 - \gamma = 0.67165...$ and $\mathbf{E}(L_n)$ and c are given by (5) and (4), respectively.

Remark 1.4. The sequence $\{p(n)\mathbf{E}(L_n M_n)\}_{n\geq 1}$ is given as A092321 in [Slo18].

The proofs of Theorems 1.1 and 1.3 and are based on generating function identities established in [ABBKM16] that involve products of the form $P(x)G(x)$, where $P(x)$ is the Euler partition generating function

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty}(1 - x^j)^{-1} \tag{9}$$

and $G(x)$ is a function which is analytic in the open unit disk and does not grow too fast as $x \to 1$. Our asymptotic expansions in (8) and Theorem 1.3 are obtained using a general asymptotic result of Grabner et al. [GKW14] for the nth coefficient $x^n[P(x)G(x)]$. In the proof of Theorem 1.3 we also apply a classical approach for estimating the growth of a power series around its main singularity.

We organize the paper as follows. Section 2 contains some auxiliary facts related to generating functions and the asymptotic analysis of their coefficients. In Section 3 we sketch the proofs of Theorems 1.1 and 1.3. Finally, in Section 4 we conclude with an open problem on the position of $L_n M_n$ in the sequence of ordered block areas of a random Ferrers diagram.

2 Preliminaries: Generating Functions and an Asymptotic Scheme

We start with two generating function identities. In the next two lemmas $P(x)$ is the generating function given by (9) and by definition $\prod_1^0 := 1$.

Lemma 2.1. *(i) For any positive integer m, we have*

$$\sum_{n=1}^{\infty} p(n)\mathbf{P}(M_n = m)x^n = x^m \prod_{j\geq m} (1-x^j)^{-1} = P(x)x^m \prod_{j=1}^{m-1} (1-x^j).$$

(ii) We also have

$$\sum_{n=1}^{\infty} p(n)\mathbf{E}(L_n M_n)x^n = \sum_{k=1}^{\infty} \frac{kx^k}{1-x^k} \prod_{j=1}^{k} (1-x^j)^{-1} = P(x)F(x),$$

where

$$F(x) = \sum_{k=1}^{\infty} \frac{kx^k}{1-x^k} \prod_{j=k+1}^{\infty} (1-x^j). \tag{10}$$

Sketch of the proof. Part (i) is the last conclusion of Theorem 2.3 from [ABBKM16]. Part (ii) is given in A092321 of [Slo18]. It also follows from Proposition 4.1 in [ABBKM16].

We shall essentially use the main result from [GKW14, Theorem 2.3]. We present here only slight modifications of those parts of this theorem that we will need in our further asymptotic analysis. Furthermore, by $\log x$ we denote the main branch of the logarithmic function that satisfies the inequality $\log x < 0$ if $0 < x < 1$.

Lemma 2.2. *Suppose that, for some constants $K > 0$ and $\eta < 1$, the function $G(x)$ satisfies*

$$G(x) = O(e^{K/(1-|x|)^\eta}), \quad |x| \to 1. \tag{11}$$

(i) Let $G(e^{-t}) = at^b + O(|f(t)|)$ as $t \to 0$, $\Re t > 0$, where $b \geq 0$ is an integer and a is real number. Then, we have

$$\frac{1}{p(n)} x^n[P(x)G(x)] = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \frac{s}{s-1} \sum_{j=0}^{b+1} \frac{(b+j+1)!}{j!(b+j-1)!} \left(-\frac{1}{2s}\right)^j$$
$$+O(e^{-2s}) + O\left(e^{-n^{1/2-\epsilon}} + f\left(c/\sqrt{n} + O(n^{-1/2-\epsilon})\right)\right)$$

for any $\epsilon \in (0, (1-\eta)/2)$, where

$$s = \sqrt{\frac{2\pi^2}{3}\left(n - \frac{1}{24}\right)} = 2c\sqrt{n - \frac{1}{24}} \tag{12}$$

and c is given by (4).

(ii) Suppose that $G(x)$ satisfies condition (11) and, for $t = u + iv$, let $G(e^{-t}) = a \log \frac{1}{t} + O(f(|\ t\ |))$ as $t \to 0$, where $u > 0$, $v = O(u^{1+\epsilon})$ as $u \to 0^+$ and ϵ and a are as in part (i). Then, we have

$$\frac{1}{p(n)} x^n[P(x)G(x)] = a \log\left(\frac{\sqrt{24n-1}}{2\pi}\right) + O\left(n^{-1/2} + f\left(c/\sqrt{n} + O(n^{-1/2-\epsilon})\right)\right)$$

with c given by (4).

As in [GKW14], we remark that parts (i) and (ii) can be combined so that Lemma 2.2 generalizes to mixed asymptotic expansions involving sums of powers of t and logarithms of $1/t$. The proof of Lemma 2.2, based on the saddle point method, is presented in [GKW14, Section 3].

3 On the Method of Proof and Some Technical Details

Sketch of the Proof of Theorem 1.1. First, we set $m = 1$ in Lemma 2.1 (i). We have

$$\sum_{n=1}^{\infty} \mathbf{P}(M_n = 1)x^n = xP(x). \tag{13}$$

This implies (7) at once. The asymptotic behavior of the quotient in (7) may be found using Rademacher's "exact-asymptotic" formula [Rad37] (see also [And76, Chapter 5]). It seems that a quicker way is to apply the result of Lemma 2.2 (i). Here we have $G(x) = x$, which obviously satisfies (11). Setting $x = e^{-t}$ in (13) and expanding e^{-t} as a Taylor series, we can take into account as many powers of t as we wish. This will be transferred into powers of $n^{-1/2}$ in the asymptotic expansion of $\mathbf{P}(M_n = 1)$. We decide to bound the error of estimation up to a term of order $O(n^{-3/2})$ and write

$$e^{-t} = 1 - t + \frac{1}{2}t^2 + f(t), \tag{14}$$

with

$$f(t) = \sum_{j=3}^{\infty} \frac{t^j}{j!}. \tag{15}$$

The representation (14) requires to apply Lemma 2.2(i) twice: for the term $-t$ with $a = -1$ and $b = 1$ and for the term $\frac{1}{2}t^2$ with $a = 1/2$ and $b = 2$. Furthermore, (15) implies that $f(c/\sqrt{n} + O(n^{-1/2-\epsilon})) = O(n^{-3/2})$. Thus, from (13) it follows that

$$\mathbf{P}(M_n = 1) = \frac{x^n[xP(x)]}{p(n)} = 1 - A_1(n) + A_2(n) + O(n^{-3/2}). \tag{16}$$

The computation of $A_1(n)$ and $A_2(n)$ is based on Lemma 2.2(i) with s given by (12). We skip all technical details and present here the final evaluations:

$$A_1(n) = -\frac{c}{\sqrt{n}} - \frac{1}{n} + O(n^{-3/2}), \tag{17}$$

$$A_2(n) = \frac{c^2}{2n} + O(n^{-3/2}). \tag{18}$$

The proof is completed by substituting (17) and (18) into (16).

Proof of the Corollary. The total probability formula and the asymptotic estimate given by Theorem 1.1 imply that

$$\begin{aligned}
&\mathbf{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u\right) = \mathbf{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u \mid M_n = 1\right)\mathbf{P}(M_n = 1)\\
&+\mathbf{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c}\log n \le u \mid M_n \ne 1\right)\mathbf{P}(M_n \ne 1)\\
&= \mathbf{P}\left(\frac{L_n}{\sqrt{n}} - \frac{1}{2}\log n \le u\right)(1 + O(1/\sqrt{n})) + O(1/\sqrt{n}).
\end{aligned}$$

Hence the Corollary follows easily from Erdös and Lehner's result (2).

Sketch of the Proof of Theorem 1.3. First, we represent the function $F(x)$ given by (10) as

$$F(x) = F_1(x) + F_2(x), \tag{19}$$

where

$$F_1(x) = \sum_{k=1}^{\infty} kx^k \prod_{j=k+1}^{\infty} (1 - x^j), \tag{20}$$

$$F_2(x) = \sum_{k=1}^{\infty} k(\sum_{l=2}^{\infty} x^{kl}) \prod_{j=k+1}^{\infty} (1-x^j)$$
$$= \sum_{k=1}^{\infty} \frac{kx^{2k}}{1-x^k} \prod_{j=k+1}^{\infty} (1-x^j). \tag{21}$$

Grabner and Knopfmacher [GK06, formula 6.2] found a simpler alternative representation for $F_1(x)$. They showed that the right-hand side of (20) yields

$$F_1(x) = \sum_{k=1}^{\infty} \frac{x^k}{1-x^k}.$$

It is also known that

$$\sum_{n=1}^{\infty} p(n)\mathbf{E}(L_n)x^n = P(x)F_1(x)$$

(see, e.g., [GKW14, p. 1059]). In addition, Grabner et al. [GKW14, p. 1084] used Mellin trasform technique to show that

$$F_1(e^{-t}) = \frac{\log(1/t)+\gamma}{t} + \frac{1}{4} - \frac{t}{144} + O(|\, t\,|^3), \quad t \to 0.$$

From this expansion and their main result (see also both parts of Lemma 2.2) they derived asymptotic formula (5) for $\mathbf{E}(L_n)$. From (19) it follows that $x^n[F(x)] = x^n[F_1(x)] + x^n[F_2(x)]$, which in turn implies that

$$\mathbf{E}(L_n M_n) = \mathbf{E}(L_n) + \frac{x^n[F_2(x)]}{p(n)}. \tag{22}$$

The asymptotic analysis of the second summand in the right-hand side of (22) is based on Lemma 2.2 (ii). It requires a suitable expansion for $F_2(e^{-t})$ given by the next lemma.

Lemma 3.1. *If $t = u + iv$ and u and v satisfy the conditions of Lemma 2(ii), then*

$$F_2(e^{-t}) = \log\frac{1}{t} + \gamma - 1 + O\left(1/\log\frac{1}{t}\right), \quad t \to 0.$$

The proof of Lemma 3.1 contains lengthy technical details. We will skip them including here only a brief description.

First, we focus on an asymptotic estimate for $F_2(e^{-u})$ as $u \to 0^+$. We set $x = e^{-u}$ in (21) and partition the range of summation in its right-hand side into four intervals. It turns out that the main contribution is given by the sum over all integers $k \in \left(\frac{1}{u}\left(\log\frac{1}{u} - \log\log\frac{1}{u} - \log 3\right), \frac{1}{u}\left(\log\frac{1}{u} + \log\log\frac{1}{u} + \log 2\right)\right]$, while the other three sums are negligible (of maximum order $O\left(1/\log\frac{1}{u}\right)$ as $u \to 0^+$). Finally, we transfer the variable u into $t = u+iv$ using Taylor's formula. For the reminder term we apply the same approach and the relationship between u and v from Lemma 2.2 (ii).

Hence, applying Lemma 2.2 (ii) with $G(x) := F_2(x)$ and $f(t) := 1/\log\frac{1}{t}$, we obtain

$$\frac{1}{p(n)} x^n[F_2(x)] = \log\frac{c}{\sqrt{n}} + \gamma - 1 + O\left(\frac{1}{\log n}\right), \quad n \to \infty. \tag{23}$$

The proof of Theorem 1.3 is completed combining (22) with (23).

4 Concluding Remarks

The main goal of this study is the comparison between the typical growths of the first block area $L_n M_n$ and its base length L_n in the Ferrers diagram of a random integer partition n. It turns out that the leading terms in the asymptotic expansions of the expectations of these two statistics are the same for large n; both are equal to $\frac{\sqrt{n}}{2c}\log n$. Erdös and Lehner's limit theorem (2) and Theorem 1 show that this leading terms control the weak convergence of L_n and L_nM_n. Both statistcs, after one and the same normalization, tend to a Gumbel distributed random variable. The expectations of L_n and L_nM_n are, however, different for large n. In fact, by Theorem 1.3

$$\lim_{n\to\infty}\left(\mathbf{E}(L_nM_n)-\mathbf{E}(L_n)-\frac{1}{2}\log n\right)=-C=-0.67165....$$

This phenomenon suggests a question related to the typical shape of a random Ferrers diagram of n. To state the problem in a more precise way, we denote by $X_n^{(k)}$ the multiplicity of part k ($k=1,...,n$) in a random integer partition of n. Let $Z_n^{(r)}$ be the rth largest member of the sequence $\{kX_n^{(k)}\}_{k=1}^n$. Erdös and Szalay [ES84] showed that

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{c}{\sqrt{n}}Z_n^{(1)}-\frac{1}{2}\log\frac{n}{c^2}-\log\log\log n\le u\right)=e^{-e^{-u}},\quad -\infty<u<\infty.$$

Fristedt [Fri93, Theorem 2.7] has also studied this kind of rearrangements of the Ferrers diagrams and generalized Erdös and Szalay's result to $Z_n^{(r)}$, where $r\ge 1$ is fixed. He showed that

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{P}\left(\frac{c}{\sqrt{n}}Z_n^{(r)}-\frac{1}{2}\log\frac{n}{c^2}-\log\log\log n\le u\right)\\&=\int_{-\infty}^{u}\frac{\exp\left(-e^{-w}-rw\right)}{(r-1)!}dw,\quad -\infty<u<\infty.\end{aligned}\tag{24}$$

So, it might be interesting to determine the typical position of L_nM_n among all ordered areas $Z_n^{(r)}$. If R_n denotes the smallest value of r such that $Z_n^{(r)}=L_nM_n$, then we conjecture that

$$\mathbf{E}(R_n)\asymp\log\log n,\quad n\to\infty.\tag{25}$$

This claim is supported by the following non-rigorous argument. From the result of Theorem 1.3 it follows that

$$\mathbf{E}(L_nM_n)=\frac{\sqrt{n}}{2c}\log n+O(\sqrt{n}),\quad n\to\infty.\tag{26}$$

A calculation of the expectation of the distribution in the right-hand side of (24) demonstrated in [Fri93, p. 708] shows that if $r=r(n)\to\infty$ as $n\to\infty$, then

$$\mathbf{E}(Z_n^{(r)})=\frac{\sqrt{n}}{2c}\left(\log n+2\log\log\log n-2\log r\right)+O(\sqrt{n}).\tag{27}$$

Combining (26) with (27), one may conclude that $\log r(n)$ is of order $\log\log\log n$, which supports the claim in (25). We hope to return to this question in a future study.

The full text of this work may be found in [Mut17].

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