Using Grassmann calculus in combinatorics: Lindström-Gessel-Viennot lemma and Schur functions

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Abstract

Grassmann (or anti-commuting) variables are extensively used in theoretical physics. In this paper we use Grassmann variable calculus to give new proofs of celebrated combinatorial identities such as the Lindström-Gessel-Viennot formula for graphs with cycles and the Jacobi-Trudi identity. Moreover, we define a one parameter extension of Schur polynomials that obey a natural convolution identity.

1 Introduction - Grassmann variables and calculus

Grassmann (or anti-commuting) variables $\chi_1, ..., \chi_m$ ($m \in \mathbb{N}$) are defined through their anti-commutation relations:

$$\chi_i \chi_j = -\chi_j \chi_i, \quad \forall i, j = 1, \dots, m. \tag{1}$$

As an immediate consequence, one has the following crucial identity:

$$\chi_i^2 = 0, \quad \forall i = 1, \dots, m. \tag{2}$$

More precisely, the Grassmann algebra Λ_m over the m anti-commuting variables $\{\chi_1, \dots, \chi_m\}$ is defined as the linear span of the 2^m independent products of the χ_i's. Its elements are functions of the form

$$f(\chi) = \sum_{n=0}^{m} \frac{1}{m!} \sum_{1 \le i_1, \dots i_n \le n} a_{i_1 \dots i_n} \chi_{i_1} \cdots \chi_{i_n}, \tag{3}$$

where $a_{i_1 \dots i_n}$ are complex coefficients, antisymmetric with respect to their indices, $a_{i_{\sigma(1)}, \dots i_{\sigma(n)}} = \epsilon(\sigma) a_{i_1, \dots, i_n}$, and the vector space structure is simply defined by addition and scalar multiplication of the coefficients. A function f which is a sum of only even (resp. odd) monomials is called even (resp. odd). The multiplication rule for monomials is

$$(\chi_{i_1} \dots \chi_{i_n})(\chi_{j_1} \dots \chi_{j_p}) = \begin{cases} 0 & \text{if} \quad \{i_1, \dots, i_n\} \cap \{j_1, \dots, j_p\} \neq \emptyset \\ \mathrm{sgn}(k)\, \chi_{k_1} \dots \chi_{k_{n+p}} & \text{otherwise} \end{cases}, \tag{4}$$

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with $k = (k_1, \ldots, k_{n+p})$ the permutation of $(i_1, \ldots, i_n, j_1, \ldots, j_p)$ such that $k_1 < \ldots < k_{n+p}$. It is then extended to the whole Grassmann algebra by distributivity. For notational convenience, we will furthermore commute complex and Grassmann variables, and to permute the Grassmann variables themselves following the defining rule (1).

One then defines the exponential of a Grassmann function f by

$$e^{f(\chi)} := \sum_{p=0}^{+\infty} \frac{1}{p!} f(\chi)^p, \tag{5}$$

which, following (1), is a simple polynomial expression. In particular one immediately finds that $e^{\chi_{i_1} \cdots \chi_{i_n}} = 1 + \chi_{i_1} \ldots \chi_{i_n}$. Another interesting property is that $e^A e^B = e^{A+B}$ for any even Grassmann functions A and B (since A and B therefore commute).

Due to these multiple properties, Grassmann variables are extensively used in quantum field theory to describe the physics of fermions[1], which are particles obeying the so-called Fermi-Dirac statistics, statistics which is based on anti-commutation laws (unlike bosons[2], which are particles obeying the Bose-Einstein statistics (statistics based on commutation laws), and which are described by physicists using usual commuting variables) – the interested reader is reported to quantum field theory textbooks such as [FKT02] for more details.

The **Grassmann integral** $\int d\chi \equiv \int d\chi_m \ldots d\chi_1$ is the unique linear map from Λ_m to $\mathbb{C}$ s. t.

$$\int d\chi \, \chi_1 \ldots \chi_m = 1.$$

Moreover, $\int d\chi \, \chi_{i_1} \ldots \chi_{i_n} = 0 \quad \text{whenever} \quad n < m.$

Example 1.1. Let χ and $\bar{\chi}$ be two independent Grassmann variables (the bar has nothing to do with any complex conjugation) and let $a \in \mathbb{C}$. One computes:

$$\int d\bar{\chi} d\chi \, e^{-\bar{\chi} a \chi} = \int d\bar{\chi} d\chi \, (1 - \bar{\chi} a \chi) = \int d\bar{\chi} d\chi \, (-\bar{\chi} a \chi) = a \int d\bar{\chi} d\chi \, \chi \bar{\chi} = a. \tag{6}$$

Similarly, one computes: $\int d\bar{\chi} d\chi \, \chi \bar{\chi} \, e^{-\bar{\chi} a \chi} = 1$.

Example 1.2. Consider N independent Grassmann variables $\{\chi_i \,|\, 1 \leq i \leq N\}$. Then, for any permutations σ of $\{1, \ldots, N\}$, one has:

$$\int d\chi_N \ldots d\chi_1 \, \chi_{\sigma(1)} \ldots \chi_{\sigma(N)} = \text{sgn}(\sigma). \tag{7}$$

Let M be an $N-$dimensional square matrix whose entries are commuting variables (such as complex numbers). Its **determinant** can be expressed as a Grassmann Gaussian integral over $2N$ Grassmann variables $\bar{\chi}_i, \chi_i$, $i = 1, \ldots, N$. As above, the conjugate notation conveniently accounts for the doubling of variables. Using the morphism property of the exponential on even functions, one proves:

$$\det M = \int d\bar{\chi}_N d\chi_N \ldots d\bar{\chi}_1 d\chi_1 \, \exp\left(- \sum_{i,j=1}^{N} \bar{\chi}_i M_{ij} \chi_j \right). \tag{8}$$

Similarly, one can express any **minor** of M using Grassmann calculus. Let $0 \leq p \leq N$ and let $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_p\}$ be two subsets of indices of $\{1, \ldots, N\}$, where $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_p$. We denote by $M_{I^c J^c}$ the matrix obtained by deleting from M the rows with indices in I and the column with indices in J. One has

$$(-1)^{\Sigma I + \Sigma J} \, \det(M_{I^c J^c}) = \int d\bar{\chi} d\chi \, \chi_J \bar{\chi}_I \, \exp\left(- \sum_{i,j=1}^{N} \bar{\chi}_i M_{ij} \chi_j \right), \tag{9}$$

where we have used the notation

$$d\bar{\chi} d\chi := d\bar{\chi}_N d\chi_N \ldots d\bar{\chi}_1 d\chi_1.$$

[1]Example of fermions are: the electrons, the neutrinos, the quarks.

[2]Examples of bosons are: photons, gluons, Higgs bosons.

Moreover, $\Sigma I = \sum_{k=1}^{p} i_k$, $\Sigma J = \sum_{k=1}^{p} j_k$ and

$$\chi_J \bar{\chi}_I := \chi_{j_1} \bar{\chi}_{i_1} \cdots \chi_{j_p} \bar{\chi}_{i_p} . \tag{10}$$

One can prove more general **Grassmann Gaussian integral** formulas[3], such as:

$$\int d\eta d\bar{\eta} \exp \left(\sum_{k,\ell=1}^{N} \bar{\eta}_k M^{-1}_{k\ell} \eta_\ell + \sum_{k=1}^{N} (\bar{\psi}_k \eta_k + \bar{\eta}_k \psi_k) \right) = \det(M^{-1}) \exp \left(- \sum_{k,\ell=1}^{N} \bar{\psi}_k M_{k\ell} \psi_\ell \right) . \tag{11}$$

The integrand here leaves in the Grassmann algebra $\{\bar{\eta}_i, \eta_i, \bar{\chi}_i, \chi_i \,|\, i = 1, \ldots, N\}$.

In this paper, we use Grassmann calculus to prove the Lindström-Gessel-Viennot (LGV) lemma and the Jacobi-Trudi identity. Note that the LGV lemma proof we give does not require the use of any involution arguments. The vanishing contribution of intersecting paths will appear as a simple consequence of the Grassmann nilpotency identity (2)!

2 Grassmann proof of the LGV lemma

Let G be a finite directed graph. Note that we allow loops and multiple edges. Let $V = \{v_1, \ldots, v_N\}$ be the set of vertices of G. One assigns to each edge e a **weight** w_e. One further assumes that the variables w_e commute with each others.

A **path** P from v to v' is a collection of edges $(e_1, e_2, \ldots, e_k)$ such that one can reach v' from v by successively traversing $e_1, \ldots, e_k$ in the specified order. Following [Tal12], let us recall the following definitions. The **weight** of a given path $P = (e_1, \ldots, e_k)$ is:

$$\mathrm{wt}(P) := \prod_{k=1}^{m} w_{e_k} . \tag{12}$$

The **weight path matrix** of the graph G is the matrix $M = (m_{ij})_{1 \le i,j \le N}$, whose entries are:

$$m_{ij} := \sum \mathrm{wt}(P). \tag{13}$$

The sum above is taken on paths P from v_i to v_j.

These quantities are considered as formal power-series in the weights. A crucial remark is that

$$M = (\mathrm{Id} - A)^{-1}, \tag{14}$$

where $A = (A_{ij})$ is the **weighted adjacency matrix** of the graph ($A_{ij} = w_{ij}$ if there is an edge from v_i to v_j, and 0 otherwise).

A **cycle** is a path from a vertex v to itself (or more precisely an equivalent class thereof up to change of source vertex). We denote by $\mathcal{C}$ the set of all possible collections of self-avoiding and pairwise vertex-disjoint cycles, including the empty collection. Given $\mathbf{C} = (C_1, \ldots, C_k) \in \mathcal{C}$, we define its weight and sign as

$$\mathrm{wt}(\mathbf{C}) := \prod_{i=1}^{k} \mathrm{wt}(C_i) , \qquad \text{and} \qquad \mathrm{sgn}(\mathbf{C}) := (-1)^k, \tag{15}$$

while, by convention, $\mathrm{wt}(\mathbf{C}) = \mathrm{sgn}(\mathbf{C}) = 1$ for the empty collection.

Lemma 2.1. *The determinant of M^{-1} is:*

$$\det(M^{-1}) = \sum_{\mathbf{C} \in \mathcal{C}} \mathrm{sgn}(\mathbf{C}).\mathrm{wt}(\mathbf{C}). \tag{16}$$

Proof. Equations (14) and (8) yield:

$$\det(M^{-1}) = \int d\bar{\chi} d\chi \exp \left(- \sum_{i,j=1}^{N} \bar{\chi}_i (\delta_{ij} - A_{ij}) \chi_j \right) = \int d\bar{\chi} d\chi \prod_{i=1}^{N} (1 + \chi_i \bar{\chi}_i) \prod_{k,l=1}^{N} (1 + A_{kl} \bar{\chi}_k \chi_l). \tag{17}$$

[3]See, for example, [FKT02], for more details on Grassmann integration and Grassmann changes of variables.

The integrand decomposes as sums of terms of the form

$$A_{k_1l_1}\dots A_{k_sl_s}\times\bar\chi_{k_1}\chi_{l_1}\dots\bar\chi_{k_s}\chi_{l_s}\ \chi_{i_1}\bar\chi_{i_1}\dots\chi_{i_r}\bar\chi_{i_r}, \tag{18}$$

giving a non-zero contribution to the integral if and only if: all Grassmann variables appear exactly once and $A_{k_jl_j}\neq 0$ for all $1\le j\le s$. Let us assume that this is the case. The inequality $A_{k_jl_j}\neq 0$ implies that there is a directed edge e_j from v_{k_j} to v_{l_j}; this means that $A_{k_jl_j}=w_{k_jl_j}$. Let us call H_s the subgraph made out of the edges $e_1,\dots,e_s$. Each ingoing (resp. outgoing) edge at a vertex $v_\ell\in H_s$ is associated to a variable χ_ℓ (resp. $\bar\chi_\ell$). Hence there cannot be more than one ingoing (resp. outgoing) edge of H_s at each v_ℓ. On the other hand, if there were only say one ingoing but no outgoing edge at v_ℓ, this would require that $\bar\chi_{i_k}=\bar\chi_\ell$ for some $1\le k\le r$. This would however necessarily bring a second factor χ_{i_k} and therefore cancel the integrand. We conclude that there must be exactly one ingoing at one outgoing edge at each vertex of H_s. This means that H_s must decompose into a collection $\mathbf{C}$ of self-avoiding and pairwise vertex-distinct cycles. Furthermore there is in this case a unique choice of indices $1\le i_1<\dots<i_r\le N$ yielding a non-vanishing monomial of degree $2N$. Each collection of cycles $\mathbf{C}$ is weighted by $\mathrm{wt}(\mathbf{C})$, up to a sign. Moreover, the integral is of the form of (7), with σ a product of $|\mathbf{C}|$ disjoint cycles of even length, the other cycles being trivially of length 1. The signature of σ is therefore $\mathrm{sgn}(\mathbf{C})$, and we conclude that $\mathbf{C}$ contributes with a term $\mathrm{sgn}(\mathbf{C})\mathrm{wt}(\mathbf{C})$. □

One considers now the minor $M_{\mathcal{AB}}$, where $\mathcal{A}=\{a_1,\dots,a_p\}$ and $\mathcal{B}=\{a_1,\dots,a_p\}$ are p-dimensional sets of indices in $\{1,\dots,n\}$. A **p-path** from $\mathcal{A}$ to $\mathcal{B}$ is a collection of paths $\mathbf{P}=(P_1,\dots,P_k)$ s. t. P_i connects a_i to $b_{\sigma_{\mathbf{P}}(i)}$, for some permutation $\sigma_{\mathbf{P}}$. The weight and sign of $\mathbf{P}$ are furthermore given by:

$$\mathrm{wt}(\mathbf{P}):=\prod_{i=1}^{k}\mathrm{wt}(P_i)\,,\qquad\text{and}\qquad \mathrm{sgn}(\mathbf{P}):=\mathrm{sgn}(\sigma_{\mathbf{P}}). \tag{19}$$

The p-path $\mathbf{P}$ is **self-avoiding** if: 1) each P_i is self-avoiding; 2) P_i and P_j are vertex-disjoint whenever $i\neq j$. We denote by $\mathcal{P}_{\mathcal{A},\mathcal{B}}$ the set of self-avoiding p-paths from $\mathcal{A}$ to $\mathcal{B}$.

Finally, a **self-avoiding flow** from $\mathcal{A}$ to $\mathcal{B}$ is a pair $(\mathbf{P},\mathbf{C})$ such that: 1) $\mathbf{P}\in\mathcal{P}_{\mathcal{A},\mathcal{B}}$; 2) $\mathbf{C}\in\mathcal{C}$; and 3) $\mathbf{P}$ and $\mathbf{C}$ are vertex disjoint. We denote the set of self-avoiding flow from $\mathcal{A}$ to $\mathcal{B}$ by $\mathcal{F}_{\mathcal{A},\mathcal{B}}$.

The LGV formula for graph with cycles is:

Theorem 2.2. *One has [Tal12]*

$$\det(M_{\mathcal{AB}})=\frac{\sum\limits_{(\mathbf{P},\mathbf{C})\in\mathcal{F}_{\mathcal{A},\mathcal{B}}}\mathrm{sgn}(\mathbf{P})\mathrm{wt}(\mathbf{P})\,\mathrm{sgn}(\mathbf{C})\mathrm{wt}(\mathbf{C})}{\sum\limits_{\mathbf{C}\in\mathcal{C}}\mathrm{sgn}(\mathbf{C})\mathrm{wt}(\mathbf{C})}. \tag{20}$$

In particular, if G is acyclic then [GV85, Lin73]:

$$\det(M_{\mathcal{AB}})=\sum_{\mathbf{P}\in\mathcal{P}_{\mathcal{A},\mathcal{B}}}\mathrm{sgn}(\mathbf{P})\mathrm{wt}(\mathbf{P}). \tag{21}$$

Let us now give the Grassmann calculus proof of this identity.

Proof. The left-hand side of (20) is a minor of the matrix M. We need to re-express it as a minor of M^{-1}. To this purpose, we could directly use $\det((M^{-1})_{\mathcal{AB}})=(-1)^{\Sigma\mathcal{A}+\Sigma\mathcal{B}}\frac{\det(M_{\mathcal{B}^c\mathcal{A}^c})}{\det(M)}$. Nevertheless, in this paper we instead rely exclusively on Grassmann calculus. One can thus use formula (9) to express $\det(M_{\mathcal{AB}})$ as

$$(-1)^{\Sigma\mathcal{A}^c+\Sigma\mathcal{B}^c}\int d\bar\psi d\psi\,\psi_{\mathcal{B}^c}\bar\psi_{\mathcal{A}^c}\,\exp\left(-\sum_{i,j=1}^{N}\bar\psi_iM_{ij}\psi_j\right). \tag{22}$$

We now re-express the exponential above using the Grassmann Gaussian integral formula (11). This leads to

$$\frac{(-1)^{\Sigma\mathcal{A}^c+\Sigma\mathcal{B}^c}}{\det(M^{-1})}\int d\bar\psi d\psi\,\psi_{\mathcal{B}^c}\bar\psi_{\mathcal{A}^c}\int d\eta d\bar\eta\exp\left(\sum_{k\ell=1}^{N}\bar\eta_kM^{-1}_{k\ell}\eta_\ell+\sum_{k=1}^{N}(\bar\psi_k\eta_k+\bar\eta_k\psi_k)\right). \tag{23}$$

The denominator is given by Lemma 2.1. The integral in the numerator writes:

$$(-1)^{\Sigma\mathcal{A}^c+\Sigma\mathcal{B}^c}\int d\eta d\bar{\eta}\,\exp\left(\sum_{k,\ell=1}^{N}\bar{\eta}_k M^{-1}_{k\ell}\eta_\ell\right)\int d\bar{\psi}d\psi\,\psi_{\mathcal{B}^c}\bar{\psi}_{\mathcal{A}^c}\,\exp\left(\sum_k(\bar{\psi}_k\eta_k+\bar{\eta}_k\psi_k)\right). \tag{24}$$

In order to perform the Grassmann integral on the sets of variables $\bar{\psi}$ and ψ in (24), we use the following result:

Lemma 2.3. *The following identity holds*

$$\int d\bar{\psi}d\psi\,\psi_{\mathcal{B}^c}\bar{\psi}_{\mathcal{A}^c}\,\exp\left(\sum_{k=1}^{N}(\bar{\psi}_k\eta_k+\bar{\eta}_k\psi_k)\right)=(-1)^{\Sigma\mathcal{A}+\Sigma\mathcal{B}}\,\bar{\eta}_{\mathcal{B}}\eta_{\mathcal{A}}. \tag{25}$$

Proof. When developing the exponential in (25) above, the only term which leads to a non-vanishing contribution is the one containing:

$$\left(\prod_{i=1}^{p}\bar{\psi}_{a_i}\eta_{a_i}\right)\left(\prod_{i=1}^{p}\bar{\eta}_{b_i}\psi_{b_i}\right)=\prod_{i=1}^{p}\bar{\psi}_{a_i}\eta_{a_i}\bar{\eta}_{b_i}\psi_{b_i}=(\psi_{\mathcal{B}}\bar{\psi}_{\mathcal{A}})(\bar{\eta}_{\mathcal{B}}\eta_{\mathcal{A}}). \tag{26}$$

The Grassmann integration on the lef-hand side of (25) leads to $(\eta_{\mathcal{A}}\bar{\eta}_{\mathcal{B}})$ multiplied by the sign:

$$\int d\bar{\psi}d\psi\,(\psi_{\mathcal{B}^c}\bar{\psi}_{\mathcal{A}^c})\,(\psi_{\mathcal{B}}\bar{\psi}_{\mathcal{A}})=(-1)^{\Sigma\mathcal{A}+\Sigma\mathcal{B}}. \tag{27}$$

This concludes the proof. □

Expression (24) above thus becomes:

$$\int d\eta d\bar{\eta}\,\bar{\eta}_{\mathcal{B}}\eta_{\mathcal{A}}\,\exp\left(\sum_{k,\ell=1}^{N}\bar{\eta}_k M^{-1}_{k\ell}\eta_\ell\right)=\int d\eta d\bar{\eta}\,\bar{\eta}_{\mathcal{B}}\eta_{\mathcal{A}}\prod_{i=1}^{N}e^{\bar{\eta}_i\eta_i}\prod_{k,\ell=1}^{N}e^{-\bar{\eta}_k A_{k\ell}\eta_\ell}. \tag{28}$$

Using now (14), this rewrites as

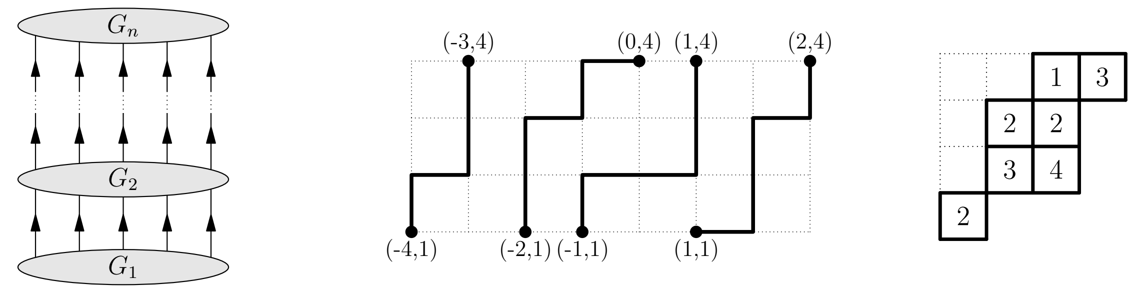
$$(-1)^p\int d\bar{\eta}d\eta\;\bar{\eta}_{\mathcal{B}}\eta_{\mathcal{A}}\prod_{i=1}^{N}(1+\eta_i\bar{\eta}_i)\prod_{k,l=1}^{N}(1+A_{kl}\bar{\eta}_k\eta_l). \tag{29}$$

A similar analysis as the one of Lemma 2.1 then shows that the non-zero contributions to the integral are labelled by self-avoiding flows $(\mathbf{P},\mathbf{C})\in\mathcal{F}_{\mathcal{A},\mathcal{B}}$. Indeed, open paths are now allowed, but their source (resp. sink) vertices must be associated to a Grassmann variable $\bar{\eta}_{a_i}$ (resp. η_{b_i}) and therefore be in $\mathcal{A}$ (resp. in $\mathcal{B}$). The key argument is that, because of the Grassmann nilpotency condition (2), the paths and cycles must be self-avoiding and pairwise vertex-disjoint!

The term indexed by the flow $(\mathbf{P},\mathbf{C})$ is equal to $\mathrm{wt}(\mathbf{P})\,\mathrm{wt}(\mathbf{C})$, up to a sign. By the same argument as in Lemma 2.1, the term associated to $(\mathbf{P},\mathbf{C})$ differs from the one associated to $(\mathbf{0},\emptyset)$ by a factor $\mathrm{sgn}(\mathbf{C})$. In the latter situation, one can relabel the variables η_{b_i} and assume without loss of generality that P_i connects a_i to b_i (for all i), and that a factor $\mathrm{sgn}(\mathbf{P})$ is included. The only difference with respect to the case studied in Lemma 2.1 is that we have now a permutation with $|\mathbf{P}|=p$ even cycles, yielding an extra factor $(-1)^p$ which cancels the one of formula (29). Finally, the sign associated to a general $(\mathbf{P},\mathbf{C})\in\mathcal{P}_{\mathcal{AB}}$ is equal to $\mathrm{sgn}(\mathbf{C})\mathrm{sgn}(\mathbf{P})$, which concludes the proof. □

3 Transfer matrix approach

In quantum field theory, the path integral represents a space time approach to the time evolution of a system, represented as a sum over paths. Accordingly, the LGV lemma is interpreted as the evolution of a system of fermions on a lattice that represents a discrete analogue of space-time. In some instances, it turns out that this evolution can also be described in another formalism based on singling out a time direction in space-time. In our case, this formalism applies to a particular class of graphs which are described below. The sum over paths



1.a. Graph $G_1 \to G_2 \to \cdots \to G_n$ 1.b. Non intersecting lattice paths 1.c. Skew Young table

Figure 1: Some illustrations

is then interpreted as a matrix element of an operator between an initial and a final state which are elements of a Hilbert space constructed as follows. We refer the reader to [DI89] for some background on statistical field theory.

Let us consider N Grassmann variables $\chi_1, \ldots, \chi_N$. The scalar product is defined in analogy with the standard scalar product on holomorphic functions, using an integration over Grassmann variables

$$\langle f, g\rangle = \int d\overline{\chi} d\chi \exp\big(-\overline{\chi}\chi\big)\, \overline{f}(\overline{\chi}) g(\chi) = \sum_{k=0}^{N} \frac{1}{k!} \sum_{1 < i_1, \cdots, i_k \leq N} \overline{a}_{i_1 \ldots i_k} b_{i_1 \ldots i_k}. \tag{30}$$

Moreover, given an $N \times N$ matrix $\tilde{T}$, one has:

$$\tilde{T}\cdot f(\chi) = \sum_{k=0}^{N} \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq N} a_{i_1 \ldots i_k} \Big(\sum_{1 \leq j_1 \leq N} \tilde{T}_{i_1 j_1} \chi_{j_1} \Big) \ldots \Big(\sum_{1 \leq j_k \leq N} \tilde{T}_{i_k j_k} \chi_{j_k} \Big). \tag{31}$$

This action can also be written in terms of Grassmann integration as

$$\tilde{T}\cdot f(\chi) = \int d\overline{\eta} d\eta\, \exp\big(-\overline{\eta}\eta\big)\, \exp\big(\overline{\eta}\, \tilde{T}\, \chi\big)\, f(\eta), \tag{32}$$

where we have used the notation:

$$\overline{\eta}\, \tilde{T}\, \chi = \sum_{i,j=1}^{N} \overline{\eta}_i \tilde{T}_{ij} \chi_j.$$

Moreover, if $\tilde{S}$ is another $N \times N$ matrix,

$$(\tilde{S}\tilde{T})\cdot f(\chi) = \int d\overline{\psi} d\psi d\overline{\eta} d\eta\, \exp\big(-\overline{\eta}\eta\big) \exp\big(\overline{\eta}\, \tilde{S}\psi\big) \exp\big(-\overline{\psi}\psi\big)\, \exp\big(\overline{\psi}\, \tilde{T}\chi\big) f(\eta). \tag{33}$$

Consider now a sequence of n weighted directed graphs $G_1, \ldots, G_n$ each having N vertices labeled by an integer $i \in \{1, \ldots, N\}$. Loops, multiple edges and isolated vertices are allowed. We denote by $w_{m,ij}$ the weight of an edge oriented from vertex i to vertex j in G_m, with the convention that the weight vanishes if there is no such an edge. We label the nN vertices of the disjoint union $G_1 \cup \cdots \cup G_n$ by pairs (i, m) where the second index refers to the graph G_m and the first one to the vertex i in G_m.

We define the graph $G_1 \to G_2 \to \cdots \to G_n$ by adding $N(n-1)$ edges to the disjoint union $G_1 \cup \cdots \cup G_n$ see Fig. 1.a. These $N(n-1)$ edges connect the vertex (i, m) to the vertex $(i, m+1)$, for all $m \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, N\}$ with a weight 1. The weighted adjacency matrix of $G_1 \to G_2 \to \cdots \to G_n$ is given by

$$A_{(i,m),(j,p)} := \begin{cases} w_{i,j} & \text{if } p = m \\ 1 & \text{if } p = m+1 \text{ and } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

The previous construction is motivated by the following theorem, relating k non intersecting paths in $G_1 \to G_2 \to \cdots \to G_n$, starting at vertices $\mathcal{A} \in G_1$ and ending at vertices $\mathcal{B} \in G_n$, to a $k \times k$ minor in a $N \times N$ matrix constructed using the weighted adjacency matrices A_i of G_i.

Theorem 3.1. *One has*

$$\sum_{\substack{\text{non intersecting paths } \mathcal{P}_1,\dots,\mathcal{P}_k\ \mathcal{A}\to\mathcal{B} \\ \text{and cycles } \mathcal{C}_1,\dots,\mathcal{C}_r \text{ in } G_1\to G_2\to\cdots\to G_n}} (-1)^{\epsilon(\mathcal{A},\mathcal{B})}(-1)^r w(\mathcal{P}_1)\cdots w(\mathcal{P}_k)w(\mathcal{C}_1)\cdots w(\mathcal{C}_r)$$
$$= \det(1-A_n)\cdots\det(1-A_1)\det\left[(1-A_n)^{-1}\cdots(1-A_1)^{-1}\right]_{\mathcal{AB}}, \quad (35)$$

with $\epsilon(\mathcal{A},\mathcal{B})$ *the signature of the permutation of the labels of the vertices in* $\mathcal{B}$ *with respect to those in* $\mathcal{A}$ *and* $\det M_{\mathcal{A},\mathcal{B}}$ *is the determinant restricted to the lines corresponding to* $\mathcal{A}$ *and columns to* $\mathcal{B}$*.*

Proof. The result is proved by induction on n. For $n = 1$, the statement corresponds to Theorem 2.2. Then, one passes from n to $n+1$ by integrating pairs of variables between vertices (i, m) and $(i, m+1)$ and the use of (33). □

In statistical physics, a homogeneous term of degree k in $\mathcal{H}$ represents a state of k fermions occupying the vertices of G_m at time m. The anti-commutation relations express Pauli exclusion principle that states that two fermions cannot occupy the same vertex. The operator $(1-A_m)^{-1}$ (multiplied by a power of its determinant) transforms this state into another k fermion state at time $m+1$, on the vertices of G_{m+1}. Thus, T_m represents a discrete time evolution; this matrix is known in physics as the *transfer matrix*.

The interest of this result comes from the evaluation of the sum over paths by a minor in an $N \times N$ matrix instead of an $nN \times nN$ matrix as would result from an application of the LGV lemma. In the next two sections we show how this result can be used in the theory of Schur functions. Other related applications of fermionic techniques can be found in [LLN09] and [Zin09].

4 An application to Schur functions

Given an integer k, a partition of k is a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_r$ of r integers such that $\lambda_1+\cdots+\lambda_r = k$. A partition is conveniently represented by a Young diagram denoted λ and made of r left justified rows, the k^{th} row containing λ_k boxes, with the longer rows on the top of the shorter ones. We set $|\lambda| := \lambda_1 + \cdots + \lambda_r$.

Given a second Young diagram μ with r' rows, we write $\mu \leq \lambda$ if $r' \leq r$ and if for all $i\,\{1,\dots,r\}$, $\mu_i \leq \lambda_i$. When $\mu \leq \lambda$, the skew Young diagram λ/μ is constructed by removing the μ_i first left boxes in the line i of λ for all i. We also consider the empty Young diagram and $\lambda/\emptyset = \lambda$ while $\lambda/\lambda = \emptyset$. We further set $\mu_i = 0$ for $i \geq r'$.

A semi standard (skew) Young tableau (SSYT) of shape λ/μ is a filling of the Young diagram λ/μ by some integers in $\{1,\dots,n\}$ in such a way that they are increasing along the columns and non decreasing along the rows. To each of these integers we associate an indeterminate x_m and the Schur function is defined as

$$s_{\lambda/\mu}(x) := \sum_{\substack{\text{skew Young Tableau} \\ \text{of shape } \lambda/\mu}} \prod_{1\leq m\leq n} x_m^{k_m}, \quad (36)$$

where k_m is number of times the integer m appears in the SSYT, see Fig. 1.c.

It is known (see [GV85]) that $s_{\lambda/\mu}(x)$ can be constructed using r non intersecting lattice paths as follows. Define a graph G with vertices labelled (i, m) with i and m positive integers and oriented edges from (i, m) to $(i+1, m)$ and from (i, m) to $(i, m+1)$. The graph G is conveniently visualized as a two dimensional square lattice with arrows pointing upwards and rightwards. Although infinite, at any stage of the computation only a finite number of vertices are involved. We leave the precise range of i unspecified for notational convenience and assume $1 \leq m \leq n$ unless otherwise stated. Then, the skew Schur functions can be written as a sum over r non intersecting paths on G,

$$s_\lambda(x) = \sum_{\substack{\text{non intersecting lattice paths } \mathcal{P}_1,\dots,\mathcal{P}_r \\ \mathcal{P}_i:(\mu_i-i+l,1)\to(\lambda_i-i+l,n)}} W(\mathcal{P}_1)\cdots W(\mathcal{P}_i). \quad (37)$$

where l is a global translation parameter that does not affect the result, because of translation invariance. The weight of a path is again given by the product of the weight of its edges. The weight of an horizontal edge from (i, m) to $(i+1, m)$ is x_m and the weight of all vertical edges is 1.

The graph G can be written as $G = G_1 \to G_2 \to \cdots \to G_n$ with all G_m isomorphic to a one dimensional lattice with edges oriented to the right, i.e. from i to $i+1$. The weighted adjacency matrix is made of right translations T (defined by $T_{ij}=1$ if $j=i+1$ and 0 otherwise) multiplied by x_m, such that $1-A_m=1-x_mT$. Its inverse reads $(1-A_m)^{-1}=\sum_{p\geq 0}(x_m)^p\,T^p$. One then has

$$(1-A_n)^{-1}\cdots(1-A_1)^{-1}=\sum_k h_k(x)T^k, \tag{38}$$

with $h_k(x)$ the complete symmetric functions of $x_1,\dots,x_n$ of degree k,

$$h_k(x)=\sum_{k_1+\cdots k_n=k} x_1^{k_1}\cdots x_n^{k_n}. \tag{39}$$

We can apply Theorem 3.1 (with all G_m acyclic so that there is no contribution of cycles) and equation (35) yields the celebrated Jacobi-Trudi identity

$$s_{\lambda/\mu}(x)=\det\left(h_{\lambda_j-\mu_i+i-j}(x)\right)_{1\leq,i,j\leq r}, \tag{40}$$

for a skew partition with r rows, with the convention that $h_{j-i}(x)=0$ if $i>j$.

From a physical point of view, we may associate to a partition an element of $\mathcal{H}$ defined by $|\lambda\rangle=\chi_{_{\lambda_1-1+l}}\cdots\chi_{_{\lambda_r-r+l}}$. Introducing $U(x)=(1-x_nT)^{-1}\cdots(1-x_1T)^{-1}$, Schur functions are transition amplitudes between two such states, $s_{\lambda/\mu}(x)=\langle\lambda|U(x)|\mu\rangle$, which is non zero only if $\mu\leq\lambda$.

If we separate the variables x into two disjoints sets denoted x' and x'', one has: $U(x)=U(x')U(x'')$. This comes from the fact that all these operators commute. The relation

$$\langle\lambda|U(x)|\mu\rangle=\sum_{\mu\leq\nu\leq\lambda}\langle\lambda|U(x')|\nu\rangle\langle\nu|U(x'')|\mu\rangle \tag{41}$$

then leads to the convolution identity:

$$s_{\lambda/\mu}(x)=\sum_{\mu\leq\nu\leq\lambda}s_{\lambda/\nu}(x')s_{\nu/\mu}(x''). \tag{42}$$

This identity follows from the LGV lemma. From a lattice point of view, this is a vertical composition. In the next section, we will derive an horizontal composition from the multiplication law

$$U^{a+b}(x)=U^a(x)U^b(x).$$

5 A one parameter extension of Schur polynomials

Let us introduce the following symmetric polynomials

$$S_k(a,x)=\sum_{k_1+\cdots k_n=k}x_1^{k_1}\dots x_n^{k_n}\prod_{1\leq m\leq n}\frac{a(a+1)\dots(a+k_m-1)}{k_m!}. \tag{43}$$

For $a=1$ we recover the complete homogeneous polynomials $S_k(1,x)=h_k(x)$. For example,

$$S_1(a,x)=a\sum_{1\leq m\leq n}x_m,\qquad S_2(a,x)=\frac{a(a+1)}{2}\sum_{1\leq m\leq n}x_m^2+a^2\sum_{1\leq p<m\leq n}x_mx_p, \tag{44}$$

$$S_3(a,x)=\frac{a(a+1)(a+2)}{6}\sum_{1\leq m\leq n}x_m^3+\frac{a^2(a+1)}{2}\sum_{1\leq p<m\leq n}(x_m^2x_p+x_mx_p^2)\;+\;a^3\sum_{1\leq q<p<m\leq n}x_mx_px_q. \tag{45}$$

These polynomials appear in the expansion of $U^a(x)=(1-x_nT)^{-a}\cdots(1-x_1T)^{-a}$, generalizing (38),

$$U^a(x)=\sum_{k\geq 0}S_k(a,x)T^k, \tag{46}$$

which follows from writing $(1-x_mT)^{-a}=\frac{1}{\Gamma(a)}\int_0^\infty dt_m t_m^{a-1}\exp -t_m(1-x_mT)$. Using $U^a(x)=(1-x_nT)^{-a}\cdots(1-x_1T)^{-a}$ in equation (35) instead of $U(x)=(1-x_nT)^{-1}\cdots(1-x_1T)^{-1}$ leads to a one parameter generalization of the Schur function. The latter are defined by replacing the $h_k(x)$ by $S_k(a,x)$ in the Jacobi-Trudi identity (40).

Definition 5.1 (One parameter extension of Schur polynomials)**.** Let

$$s_{\lambda/\mu}(a,x) := \det \left(S_{\lambda_j-\mu_i+i-j}(a,x)\right)_{1\le i,j\le r}. \tag{47}$$

We use here the convention $S_0(a,x)=1$ and $S_k(a,x)=0$ for $k<0$.

Schur functions are recovered for $a=1$, $s_{\lambda/\mu}(1,x)=s_{\lambda/\mu}(x)$. Theorem 3.1 then implies that $S_{\lambda/\mu}$ can also be written as a sum over r non intersecting lattice paths for a skew diagram with r rows. However, since we use $(1-x_mT)^a$ instead of $(1-x_mT)$, for $j>i$ there is an edge from (i,m) to (j,m) weighted by

$$w_{(i,m)\to(j,m)} = (-1)^{j-i+1}\frac{a(a-1)\dots(a-j+i+1)}{(j-i)!}\,x_m^{j-i}. \tag{48}$$

In that case, the paths $(i,m)\to(i,p)\to(j,p)\to(j,q)$ and $(k,m)\to(k,q)$ for $i<k<j$ do not intersect but contribute with an extra -1 because the order of their endpoints have been reversed.

Example 5.2 ($s_{(2,1)}(a,x)$ as a sum over paths)**.** The paths contributing to $s_{(2,1)}(a,x)$ join vertices $(1,1)$ and $(2,1)$ on on side and $(2,n)$ and $(4,n)$ on the other side.

$(1,1)\to(1,m)\to(2,m)\to(2,n)$ $(2,1)\to(2,p)\to(3,p)\to(3,q)\to(4,q)\to(4,n)$	$a^3\sum\limits_{\substack{1\le p<m\le n,\\ 1\le p\le q\le n}} x_mx_px_q$
$(1,1)\to(1,m)\to(2,m)\to(2,n)$ $(2,1)\to(2,p)\to(4,p)\to(4,n)$	$-\dfrac{a^2(a-1)}{2}\sum\limits_{1\le p<m\le n} x_mx_p^2$
$(1,1)\to(1,m)\to(4,m)\to(4,n)$ $(2,1)\to(2,n)$	$-\dfrac{a(a-1)(a-2)}{6}\sum\limits_{1\le m\le n} x_m^3$
$(1,1)\to(1,m)\to(3,m)\to(3,p)\to(4,p)\to(4,n)$ $(2,1)\to(2,n)$	$+\dfrac{a^2(a-1)}{2}\sum\limits_{1\le m\le p\le n} x_m^2x_p$

For Schur functions, the last three contributions are absent ($a=1$), since they involve horizontal segments of length 2 and 3. In the last two rows there is a extra sign because of the interchange of endpoints.

$$s_{(2,1)}(a,x) = \det\begin{pmatrix} S_2(a,x) & 1\\ S_3(a,x) & S_2(a,x)\end{pmatrix} = \frac{a(a^2-1)}{3}\sum_{1\le m\le n} x_m^3 + a^2 \sum_{\substack{1\le p<m\le n\\ 1\le p\le q\le n}} x_mx_px_q. \tag{49}$$

The main interest of this extension of Schur polynomials is the following convolution identity:

Theorem 5.3 (Convolution identity)**.** *One has*

$$s_{\lambda/\mu}(a+b,x) = \sum_{\substack{\nu \text{ partition}\\ \mu\le\nu\le\lambda}} s_{\lambda/\nu}(a,x)s_{\nu/\mu}(b,x).. \tag{50}$$

Note that, for the empty partition, one has: $s_\emptyset(a,x)=1$.

Proof. The proof relies on the multiplication law $U^a(x)U^b(x)=U^{a+b}(x)$. This translates to

$$S_k(a+b,x) = \sum_{p+q=k} S_p(a,x)S_q(b,x). \tag{51}$$

The result then follows from the expansion of the determinant in (47), expansion which uses the Cauchy-Binet formula.

□

Example 5.4. The convolution identity for $(2,1)$ reads

$$s_{(2,1)}(a+b,x) = s_{(2,1)}(a,x) + s_{(1)}(a)s_{(1,1)}(b)(x) + s_{(2,1)/(1)}(a,x)s_{(1)}(b,x) + s_{(1)}(a,x)s_{(2)}(b,x) + s_{(2,1)}(b,x). \tag{52}$$

Other identities satisfied by $s_\lambda(a,x)$ can easily be proven. For example, for the conjugate diagrams (obtained by symmetry with respect to the main diagonal), one has:

$$s_{\lambda^*/\mu^*}(a,x) = (-1)^{|\lambda|-|\mu|}s_{\lambda/\mu}(-a,x).$$

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