An Axiomatization of G'_3

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Abstract. We present a Hilbert-style axiomatization of a paraconsistent logic, called G'_3, recently introduced. G'_3 is based on a 3-valued semantics. We prove a soundness and completeness theorem. The replacement theorem holds in G'_3. As it has already been shown in previous work, G'_3 can express some nonmonotonic semantics.

1 Introduction

A 3-valued logic called G'_3 has been recently introduced to define a new semantics for nonmonotonic reasoning [8]. Hence it is important to study such logic. As far as we know there is no axiomatization of G'_3. Here, we present a Hilbert-style axiomatization of G'_3. We prove a soundness and completeness theorem, and also that the replacement theorem holds in G'_3.

The structure of our paper is as follows. Section 2 describes the general background of the paper including the definition of G'_3 logic. In Section 3 we present our proofs. Finally, in Section 4 we present the conclusions of the paper.

2 Background

We first introduce the syntax of logic formulas considered in this paper. Then we present a few basic definitions of how logics can be built to interpret the meaning of such formulas in order to, finally, give a brief introduction to several of the logics that are relevant for the results of our later sections.

2.1 Syntax of Formulas

We consider a formal (propositional) language built from: an enumerable set $\mathcal{L}$ of elements called *atoms* (denoted a, b, c, ...); the binary connectives $\wedge$ (*conjunction*), $\vee$ (*disjunction*) and $\rightarrow$ (*implication*); and the unary connective $\neg$ (*negation*). Formulas (denoted A, B, C, ...) are constructed as usual by combining these basic connectives together. We also use $A \leftrightarrow B$ to abbreviate $(A \rightarrow B) \wedge (B \rightarrow A)$ and, following the tradition in logic programing, $A \leftarrow B$ as an alternate way of writing $B \rightarrow A$. A *theory* is just a set of formulas and, in this paper, we only consider finite theories. Moreover, if T is a theory, we use the notation $\mathcal{L}_T$ to stand for the set of atoms that occur in the theory T.

2.2 Logic Systems

We consider a *logic* simply as a set of formulas that, moreover, satisfies the following two properties: (i) is closed under modus ponens (i.e. if A and $A \to B$ are in the logic, then so is B) and (ii) is closed under substitution (i.e. if a formula A is in the logic, then any other formula obtained by replacing all occurrences of an atom b in A with another formula B is still in the logic). The elements of a logic are called *theorems* and the notation $\vdash_X A$ is used to state that the formula A is a theorem of X (i.e. $A \in X$). We say that a logic X is *weaker than or equal to* a logic Y if $X \subseteq Y$, similarly we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

Hilbert Style Proof Systems. There are many different approaches that have been used to specify the meaning of logic formulas or, in other words, to define *logics*. In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms (which is usually assumed to be closed by substitution). This set of axioms specifies, so to speak, the 'kernel' of the logic. The actual logic is obtained when this 'kernel' is closed with respect to the inference rule of modus ponens. Examples of Hilbert style definitions will be given in Section 2.3.

Given a theory T, the notation $\vdash_X F$ for provability of a logic formula F in the logic X, denotes the fact that the formula F can be derived from the axioms of the logic and the formulas contained in T by a sequence of applications of modus ponens. The well known result of the *deduction theorem*, which is valid in the logics considered in this paper as explained in 2.3, gives an alternate interpretation to this notation: A formula F is a logical consequence of T, i.e. $T \vdash_X F$, if and only if $\vdash_X (F_1 \wedge \cdots \wedge F_n) \to F$ for some formulas $F_i \in T$.

We furthermore extend this notation, for any pair of theories T and U, using $T \vdash_X U$ to state the fact that $T \vdash_X F$ for every formula $F \in U$. If M is a set of atoms we also write $T \Vdash_X M$ when: $T \vdash_X M$ and M is a classical 2-valued model of T (i.e. atoms in M are set to true, and atoms not in M to false; the set of atoms is a classical model of T if the induced interpretation evaluates T to true).

Recall that, in all these definitions, the logic connectives are parameterized by some underlying logic, e.g. the expression $\vdash_X (F_1 \wedge \cdots \wedge F_n) \to F$ actually stands for $\vdash_X (F_1 \wedge_X \cdots \wedge_X F_n) \to_X F$.

Multivalued Logics. An alternative way to define the semantics for a logic is by the use of truth values and interpretations. Multivalued logics generalize the idea of using truth tables to determine the validity of formulas in classical logic. The core of a multivalued logic is its *domain* of values $\mathcal{D}$, where some of such values are special and identified as *designated*. Logic connectives (e.g. $\wedge$, $\vee$, $\to$, $\neg$) are then introduced as operators over $\mathcal{D}$ according to the particular definition of the logic.

An *interpretation* is a function $I \colon \mathcal{L} \to \mathcal{D}$ that maps atoms to elements in the domain. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in $\mathcal{D}$, and then evaluating the resulting expression in terms of the connectives of the logic (which are defined over $\mathcal{D}$). A formula is said to be a *tautology* if, for every possible interpretation, the formula evaluates to a designated value. The most simple example of a multivalued logic is classical logic where: $\mathcal{D} = \{0, 1\}$, 1 is the unique designated value, and connectives are defined through the usual basic truth tables.

Note that in a multivalued logic, so that it can truly be a *logic*, the implication connective has to satisfy the following property: for every value $x \in \mathcal{D}$, if there is a designated value $y \in \mathcal{D}$ such that $y \to x$ is designated, then x must also be a designated value. This restriction enforces the validity of modus ponens in the logic. The inference rule of substitution holds without further conditions because of the functional nature of interpretations and how they are evaluated.

2.3 Basic Logics

In this subsection we introduce positive and C_ω logics.

Positive Logic Pos, is defined by the following set of axioms:

Pos1 $a \to (b \to a)$
Pos2 $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$
Pos3 $a \wedge b \to a$
Pos4 $a \wedge b \to b$
Pos5 $a \to (b \to (a \wedge b))$
Pos6 $a \to (a \vee b)$
Pos7 $b \to (a \vee b)$
Pos8 $(a \to c) \to ((b \to c) \to (a \vee b \to c))$

Note that this axioms somewhat constraint the meaning of the $\to$, $\wedge$ and $\vee$ connectives to match our usual intuition. Positive logic however, as its name suggests, does not contain formulas with negation.

It is a well known result that in any logic satisfying axioms **Pos1** and **Pos2**, and with *modus ponens* as its unique inference rule, the *deduction theorem* holds [5]. This theorem holds, in particular, for all the logics considered in this paper.

The C_ω logic, the weakest paraconsistent logic due to daCosta [3], is defined as positive logic plus the following two axioms:

Cw1 $a \vee \neg a$
Cw2 $\neg\neg a \to a$

Note that $a \vee \neg a$ is a theorem of C_ω (it is an axiom of the logic), while the formula $(\neg a \wedge a) \to b$ is not. This non-theorem shows one of the motivations of paraconsistent logics: they do allow, so to speak, 'local inconsistencies' (global inconsistencies are disallowed as usual). All the paraconsistent logics that we will consider in this paper share the same property. It follows that results such as the contrapositive of implication, i.e. $(a \to b) \to (\neg b \to \neg a)$, are no longer valid in paraconsistent logics.

2.4 Defining G_3 and G'_3

Both logics, G_3 and G'_3 are 3-valued logics where their truth values are 0,1 and 2 where 2 is the unique designated value.

We first define the truth tables for the $\to$ and $\neg$ connectives of the G_3 and G'_3 logics in Table 1. For more information about G_3 read [9].

Table 1. Truth tables of connectives in G_3 and G'_3.

x	$\neg_{G_3} x$	$\neg_{G'_3} x$
0	2	2
1	0	2
2	0	0

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

Conjunction and disjunction are defined as the min and max functions, respectively.

In [2], G'_3 is introduced only to prove that $a \vee (a \to b)$ is not a theorem of C_ω.

It is quite obvious but still interesting to observe that in G'_3 one can still express the G_3 logic, since $\neg_{G_3} a = a \to_{G'_3} (\neg_{G'_3} a \wedge \neg_{G'_3} \neg_{G'_3} a)$.

3 Axiomatization of G'_3

We present a Hilbert-style axiomatization of G'_3. Our logic has 3 primitive logical conectives, namely $\to$, $\wedge$, and $\neg$. We also have several defined conectives.

1. $A \vee B := ((A \to B) \to B) \wedge ((B \to A) \to A)$.
2. $\triangle A := (A \to \neg\neg A)$.
3. $\nabla A := \neg \triangle A$.

From now on the symbol $\vdash$ will stand for $\vdash_{G'_3}$, unless otherwise stated. Logic G'_3 has all the axioms of C_ω logic plus the following:

E1 $(\triangle\beta \wedge \beta) \to (\triangle\neg\beta \wedge \neg\neg\beta)$
E2 $\nabla\beta \to (\triangle\neg\beta \wedge \neg\beta)$
E3 $\neg\neg\neg\beta \leftrightarrow \neg\beta$
E4 $(\triangle\beta \wedge \neg\beta) \to (\triangle(\beta \to \theta) \wedge (\beta \to \theta))$
E5 $(\triangle\theta \wedge \theta) \to (\triangle(\beta \to \theta) \wedge (\beta \to \theta))$
E6 $(\nabla\beta \wedge \triangle\theta \wedge \neg\theta) \to (\triangle(\beta \to \theta) \wedge \neg(\beta \to \theta))$
E7 $(\nabla\beta \wedge \nabla\theta) \to (\triangle(\beta \to \theta) \wedge (\beta \to \theta))$
E8 $(\triangle\beta \wedge \beta \wedge \nabla\theta) \to \nabla(\beta \to \theta)$
E9 $(\triangle\beta \wedge \beta \wedge \triangle\theta \wedge \neg\theta) \to (\triangle(\beta \to \theta) \wedge \neg(\beta \to \theta))$
E10 $(\triangle\beta \wedge \neg\beta) \to (\triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta))$
E11 $(\triangle\theta \wedge \neg\theta) \to (\triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta))$
E12 $(\nabla\beta \wedge \nabla\theta) \to \nabla(\beta \wedge \theta)$
E13 $(\nabla\beta \wedge (\triangle\theta \wedge \theta)) \to \nabla(\beta \wedge \theta)$

E14 $(\nabla\theta \wedge (\triangle\beta \wedge \beta)) \to \nabla(\beta \wedge \theta)$
E15 $((\triangle\beta \wedge \beta) \wedge (\triangle\theta \wedge \theta)) \to (\triangle(\beta \wedge \theta) \wedge (\beta \wedge \theta))$

A simple but useful result is the following.

Theorem 1. *Let Γ and Δ be two set of formulas. Let θ, θ_1, θ_2, α, and ψ be arbitrary formulas. Then the following basic properties hold.*

1. $\Gamma \vdash \alpha$ *implies* $\Gamma \cup \Delta \vdash \alpha$
2. $\Gamma, \theta \vdash \alpha$ *iff* $\Gamma \vdash \theta \to \alpha$
3. $\Gamma \vdash \theta_1 \wedge \theta_2$ *iff* $\Gamma \vdash \theta_1$ *and* $\Gamma \vdash \theta_2$
4. $\Gamma, \theta \vdash \alpha$ *and* $\Gamma, \neg\theta \vdash \alpha$ *iff* $\Gamma \vdash \alpha$
5. $\Gamma \vdash \alpha$ *and* $\Delta, \alpha \vdash \psi$ *then* $\Gamma \cup \Delta \vdash \psi$

Proof.

1. It follows directly from the definition of proof.
2. Since we have axioms **Pos 1** and **Pos 2** and modus ponens, then The Deduction Theorem holds. So the implication "$\Gamma, \theta \vdash \alpha$ then $\Gamma \vdash \theta \to \alpha$" follows from The Deduction Theorem and the converse follows from monotonicity (part 1 in this theorem).
3. The first implication follows from axioms **Pos 3**, **Pos 4** and modus ponens; The converse follows from axiom **Pos 5** and modus ponens.
4. Suppose that $\Gamma, \theta \vdash \alpha$ and $\Gamma, \neg\theta \vdash \alpha$, by part 2 on this theorem, we have $\Gamma \vdash \theta \to \alpha$ and $\Gamma \vdash \neg\theta \to \alpha$. Then, by **Pos 8** we obtain $\Gamma \vdash (\theta \vee \neg\theta) \to \alpha$. Since $a \vee \neg a$ is an axiom of C_ω, we obtain the result. The converse follows from monotonicity (part 1 on this theorem).
5. Since $\Delta, \alpha \vdash \phi$, then there exist formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_n = \phi$ and each β_i is an axiom or $\beta_i \in \Delta$ or $\beta_i = \alpha$ or β_i is a direct consequence of preceding formulas. Also, since $\Gamma \vdash \alpha$, then there exist formulas $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that that $\gamma_m = \alpha$ and each γ_i is an axiom or $\gamma_i \in \Gamma$ or γ_i is a direct consequence of preceding formulas. Then we have that $\beta_1, \beta_2, \cdots, \gamma_1, \gamma_2, \cdots, \gamma_m, \cdots, \beta_n$ is a proof in $\Gamma \cup \Delta$ of ϕ. Therefore, $\Gamma \cup \Delta \vdash \phi$.

Theorem 2. *(Soundness) Every theorem in* G'_3 *is a tautology in* G'_3.

Proof. Every axiom is logically valid and modus ponens preserves validity.

3.1 Completeness

Definition 1. *Given a 3-valuation v of G'_3, we define for each formula A an associated formula A_v as follows:*

1. $A_v := \triangle A \wedge A$, *if* $v(A) = 2$.
2. $A_v := \nabla A$, *if* $v(A) = 1$.
3. $A_v := \triangle A \wedge \neg A$, *if* $v(A) = 0$.

For a set Γ of formulas, we write Γ_v to denote the set of formulas $\{\alpha_v : \alpha \in \Gamma\}$.

Lemma 1. *Given a formula α, whose set of atomic formulas is Δ, the following holds: $\Delta_v \vdash \alpha_v$.*

Proof. The proof is by induction on the size of α, and is modeled after Kalmar's Lemma [5].

Base Case: α is an atomic formula, say p. Hence we need to show that $p_v \vdash p_v$, but this immediately true.
Inductive step: Suppose that α is a non atomic formula. Then, we have 3 cases:

(Case $\neg$) Suppose that α is of the form $\neg\beta$. By inductive hypothesis we know that $\Delta_v \vdash \beta_v$. We need to consider 3 subcases:
$v(\beta) = 2$. Hence $\Delta_v \vdash \triangle\beta \wedge \beta$. Since $v(\alpha) = 0$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \neg\alpha$, that is $\Delta_v \vdash \triangle\neg\beta \wedge \neg\neg\beta$. It suffices to show that $\triangle\beta \wedge \beta \vdash \triangle\neg\beta \wedge \neg\neg\beta$. It follows directly from the axioms.
$v(\beta) = 1$. Hence $\Delta_v \vdash \nabla\beta$. Since $v(\alpha) = 2$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \alpha$, that is $\Delta_v \vdash \triangle\neg\beta \wedge \neg\beta$. It suffices to show that $\nabla\beta \vdash \triangle\neg\beta \wedge \neg\beta$. It follows directly from the axioms.
$v(\beta) = 0$. Hence $\Delta_v \vdash \triangle\beta \wedge \neg\beta$. Since $v(\alpha) = 2$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \alpha$, that is $\Delta_v \vdash \triangle\neg\beta \wedge \neg\beta$. It suffices to show that $\triangle\beta \wedge \neg\beta \vdash \triangle\neg\beta \wedge \neg\beta$. It follows directly from the axioms.

(Case $\rightarrow$) Suppose that α is of the form $\beta \rightarrow \theta$. By inductive hypothesis we know that $\Delta_v \vdash \beta_v$, and $\Delta_v \vdash \theta_v$. We need to consider 6 subcases:
$v(\beta) = 0$. Hence $\Delta_v \vdash \triangle\beta \wedge \neg\beta$. Since $v(\alpha) = 2$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \alpha$, that is $\Delta_v \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$ It suffices to show that $\triangle\beta \wedge \neg\beta \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$. It follows directly from the axioms.
$v(\theta) = 2$. Hence $\Delta_v \vdash \triangle\theta \wedge \theta$. Since $v(\alpha) = 2$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \alpha$, that is $\Delta_v \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$ It suffices to show that $\triangle\theta \wedge \theta \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$. It follows directly from the axioms.
$v(\beta) = 1, v(\theta) = 0$. Hence $\Delta_v \vdash \nabla\beta$ and $\Delta_v \vdash \triangle\theta \wedge \neg\theta$. Since $v(\alpha) = 0$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \neg\alpha$, that is $\Delta_v \vdash \triangle(\beta \rightarrow \theta) \wedge \neg(\beta \rightarrow \theta)$. It suffices to show that $(\nabla\beta \wedge \triangle\theta \wedge \neg\theta) \vdash \triangle(\beta \rightarrow \theta) \wedge \neg(\beta \rightarrow \theta)$. It follows directly from the axioms.
$v(\beta) = 1, v(\theta) = 1$. Hence $\Delta_v \vdash \nabla\beta$ and $\Delta_v \vdash \nabla\theta$. Since $v(\alpha) = 2$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \alpha$, that is $\Delta_v \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$. It suffices to show that $(\nabla\beta \wedge \nabla\theta) \vdash \triangle(\beta \rightarrow \theta) \wedge (\beta \rightarrow \theta)$. It follows directly from the axioms.
$v(\beta) = 2, v(\theta) = 1$. Hence $\Delta_v \vdash \triangle\beta \wedge \beta$ and $\Delta_v \vdash \nabla\theta$. Since $v(\alpha) = 1$, we need to show that $\Delta_v \vdash \nabla\alpha$, that is $\Delta_v \vdash \nabla(\beta \rightarrow \theta)$. It suffices to show that $\triangle\beta \wedge \beta \wedge \nabla\theta \vdash \nabla(\beta \rightarrow \theta)$. It follows directly from the axioms.
$v(\beta) = 2, v(\theta) = 0$. Hence $\Delta_v \vdash \triangle\beta \wedge \beta$ and $\Delta_v \vdash \triangle\theta \wedge \neg\theta$. Since $v(\alpha) = 0$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \neg\alpha$, that is $\Delta_v \vdash \triangle(\beta \rightarrow \theta) \wedge \neg(\beta \rightarrow \theta)$. It suffices to show that $\triangle\beta \wedge \beta \wedge \triangle\theta \wedge \neg\theta \vdash \triangle(\beta \rightarrow \theta) \wedge \neg(\beta \rightarrow \theta)$. It follows directly from the axioms.

(Case $\wedge$) Suppose that α is of the form $\beta \wedge \theta$. By inductive hypothesis we know

that $\Delta_v \vdash \beta_v$, and $\Delta_v \vdash \theta_v$. We need to consider 6 subcases:
$v(\beta) = 0$. Hence $\Delta_v \vdash \triangle\beta \wedge \neg\beta$. Since $v(\alpha) = 0$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \neg\alpha$, that is $\Delta_v \vdash \triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta)$. It suffices to show that $\triangle\beta \wedge \neg\beta \vdash \triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta)$. It follows directly from the axioms.
$v(\theta) = 0$. Hence $\Delta_v \vdash \triangle\theta \wedge \neg\theta$. Since $v(\alpha) = 0$, we need to show that $\Delta_v \vdash \triangle\alpha \wedge \neg\alpha$, that is $\Delta_v \vdash \triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta)$. It suffices to show that $\triangle\theta \wedge \neg\theta \vdash \triangle(\beta \wedge \theta) \wedge \neg(\beta \wedge \theta)$. It follows directly from the axioms.
$v(\beta) = 1, v(\theta) = 1$. Hence $\Delta_v \vdash \nabla\beta$ and $\Delta_v \vdash \nabla\theta$. Since $v(\alpha) = 1$, we need to show that $\Delta_v \vdash \nabla\alpha$, that is $\Delta_v \vdash \nabla(\beta \wedge \theta)$. It suffices to show that $(\nabla\beta \wedge \nabla\theta) \vdash \nabla(\beta \wedge \theta)$. It follows directly from the axioms.
$v(\beta) = 1, v(\theta) = 2$. Hence $\Delta_v \vdash \nabla\beta$ and $\Delta_v \vdash \triangle\theta \wedge \theta$. Since $v(\alpha) = 1$, we need to show that $\Delta_v \vdash \nabla\alpha$, that is $\Delta_v \vdash \nabla(\beta \wedge \theta)$. It suffices to show that $\nabla\beta \wedge (\triangle\theta \wedge \theta) \vdash \nabla(\beta \wedge \theta)$. It follows directly from the axioms.
$v(\beta) = 2, v(\theta) = 1$. Hence $\Delta_v \vdash \nabla\theta$ and $\Delta_v \vdash \triangle\beta \wedge \beta$. Since $v(\alpha) = 1$, we need to show that $\Delta_v \vdash \nabla\alpha$, that is $\Delta_v \vdash \nabla(\beta \wedge \theta)$. It suffices to show that $\nabla\theta \wedge (\triangle\beta \wedge \beta) \vdash \nabla(\beta \wedge \theta)$. It follows directly from the axioms.
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Theorem 3. *(Completeness) Every tautology in* G'_3 *is a theorem in* G'_3.

Proof. Let A ba a tautology en G'_3 and let $B_1, ..., B_n$ its atoms. Let $\{B_1, , , , B_{n-1}\} = \Delta$. For any values assigned to $B_1, ...B_n$, we have: $B_{1v}, ..., B_{nv} \vdash A \wedge \triangle A$
Let B_n take the values 0,1 and 2 respectively, according to Lemma 1 we obtain:
$\Delta_v, \neg B_n \wedge (B_n \to \neg\neg B_n) \vdash A$
$\Delta_v, \neg(B_n \to \neg\neg B_n) \vdash A$
$\Delta_v, B_n \wedge (B_n \to \neg\neg B_n) \vdash A$
Making use of $A \wedge \triangle A \vdash A$ and property 4 in Theorem 1, twice, we obtain: $\Delta_v \vdash A$. The result follows after repeating the same argument several times.

It is also important to notice that our logic is different from other paraconsitent logics, for example the one by da Costa's C_1. This can be seen from the fact that the axiom scheme $\neg(\alpha \wedge \neg\alpha)$ is not valid in C_1 , but it is a theorem in G'_3. Furthermore, Pierce's Law is valid in C_1 but not in G'_3. Also, we have already seen in [8] that G'_3 is also different from Pac [1]. Finally, G'_3 is also different from a paraconsisten four-valued logic introduced by [7], as it was shown in [8].

3.2 A Relevant property of G'_3

We define $\leftrightarrow$ as usual, that means $\alpha \leftrightarrow \beta$ is $(\alpha \to \beta) \wedge (\beta \to \alpha)$. For a pair of formulas θ, α and an atom p, we write $\theta[\alpha/p]$ for the formula obtained from θ after replacing every occurrence of atom p in θ by the formula α.

Lemma 2. *Let* α_1 *and* α_2 *be two formulas such that* $\vdash \alpha_1 \leftrightarrow \alpha_2$. *Let* θ *be a formula and* p *an atom. Then* $\vdash \theta[\alpha_1/p] \leftrightarrow \theta[\alpha_2/p]$.

Proof. By soundness and completeness it is enough to check that $\models \alpha_1 \leftrightarrow \alpha_2$ then $\models \theta[\alpha_1/p] \leftrightarrow \theta[\alpha_2/p]$. This proof is done by induction on the size of θ and taking into account the following property: If β_1 and β_2 are formulas, then $\models \beta_1 \leftrightarrow \beta_2$ iff for every 3-valuation v of G_3' it holds that $v(\beta_1) = v(\beta_2)$. The rest is a simple exercise.

4 Conclusions

The proof presented here for the soundness and completeness theorem uses a result analogous to Kalmar′s Lemma, as in the proof of the soundness and completeness theorem for Classical Logic in the introductory book by Mendelson [5]. We do not know of other axiomatization for G_3' and so far, our work has not gone further to find out about the possibility of reducing the number of axioms presented here.

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