Introducing the Mass-Based Rough Mereology*

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Dedicated to the memory of Jan Łukasiewicz on the Centenary of the creation by Him of many-valued logics

Abstract. We investigate a model for rough mereology based reasoning in which things in the universe of mereology are endowed with positive masses. We define mass based rough inclusions and establish its properties. This model does encompass inter alia set theoretical universes of finite sets with masses as cardinalities, probability universes with masses as probabilities of possible events, sets of satisfiable formulas with values of satisfiability. We give an abstract version of the Bayes theorem which does extend the classical Bayes theorem as well as the Łukasiewicz logical version of the Bayes formula. We define the mass-based rough mereological logic (m-RM logic) and we show that m-RM logic contains as a particular case the Łukasiewicz many-valued logic and that m-RM logic becomes the Łukasiewicz logic of probability in the case when masses are satisfiability values of indefinite formulas. We also establish an abstract form of the betweenness relation which has proved itself important in problems of data analysis and behavioral robotics. We point finally to applications in clustering and in the evidence theory.

Keywords: mass, rough inclusion, the Bayes theorem, the Łukasiewicz many-valued logic, the Łukasiewicz logic of probability, betweenness, evidence theory.

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1. Introduction

In our investigations into problems of data analysis and behavioral robotics as a model for intelligent agents, we have come at the theory of rough mereology as a useful environment for these investigations. Rough mereology (or, as Achille Varzi has called it, the fuzzified mereology) rests on the notion of a rough inclusion $\mu(x, y, r)$, a relation of being a part to a degree. In our studies of rough mereology, we have applied some forms of rough inclusions, e.g., derived from Archimedean t-norms via their Hilbertian representations or as residua of continuous t-norms, see, e.g. [11], [12], [17]. Basic forms of applied by us rough inclusions are linked to well-known idea of Pascal-Galileo of the relation of the number of favourable outcomes to the number of all outcomes of a random trial, the idea which in modern times was exploited, e.g., by Łukasiewicz [6] in assigning fractional truth values to indefinite formulas. We apply this idea of a fractional value towards a definition of an abstract class of rough inclusions based on a notion of a mass assigned to things in a considered universe. This idea allows us to relate to each pair x, y of things degrees of partial containment of x in y and of y in x, a fortiori leading to a mass-based form of the Bayes theorem. The Bayes theorem known well from Probability Calculus, was investigated in the framework of rough set theory as well, notably by Pawlak [9] due to its ability of relating two-sided dependencies between pairs of things.

We define the mass based rough mereological logics and show that it translates into the Łukasiewicz logic of probability calculus in [6]. We study relations of mass based rough mereology to many-valued logics and show that a mass based rough mereological logic offers a generalization of the Łukasiewicz many-valued logic based on the Łukasiewicz implication and negation.

We regard this work as a preliminary study into a new area of rough mereology which offers an enriching of the repertoire of rough inclusions with applications in Data Analysis.

2. An outline of mereology

We accept here the standard version of mereology as proposed in Leśniewski [5] in his pioneering work. The interested reader may as well consult, e.g., Casati and Varzi [2] or Polkowski [11], [12]. Given some collection U of things regarded as individuals in ontological sense, a *relation of a part* is a binary relation π on U which is required to be

M1 *Irreflexive*: *For each thing* x, *it is not true that* $\pi(x, x)$.

M2 *Transitive*: *For each triple* x, y, z *of things, if* $\pi(x, y)$ *and* $\pi(y, z)$*, then* $\pi(x, z)$.

The relation of a part does induce the relation of an *improper part* $\Pi(x, y)$, defined as

$$\Pi(x, y) \Leftrightarrow \pi(x, y) \vee x = y, \tag{1}$$

which is clearly a partial order on things in U. The basic relation involving the notion of an improper part is the relation of *overlapping*, $Ov(x, y)$ in symbols, defined as follows.

$$Ov(x, y) \Leftrightarrow \exists z.\Pi(z, x) \wedge \Pi(z, y). \tag{2}$$

The notion of overlapping in turn is instrumental in definition of the class operator in the sense of Leśniewski [5]. This operator assigns to each non-empty collection of things F in the universe (U, π) its class, $ClsF$ which is a thing satisfying the two conditions:

(C1) If $x \in F$ then $\Pi(x, ClsF)$.

(C2) If $\Pi(x, ClsF)$ then for each y with $\Pi(y, x)$ there exists $z \in F$ such that $Ov(y, z)$.

One more important fact in the thery of mereology is the Leśniewski inference rule:

(IR) for each pair x, y of things, if for each thing z such that $\Pi(z, x)$ there exists a thing w such that $\Pi(w, y)$ and $Ov(z, w)$, then $\Pi(x, y)$.

From (C1-2) and (IR) it follows directly

Proposition 2.1. For each thing x, it is true that $x = Cls\{z : \Pi(z, x)\}$.

We are now in a position to recall here two *fusion operators* due to Tarski [18]. These operators are the *sum* $x + y$ and the *product* $x \cdot y$ defined by means of

$$x + y = Cls\{z : \Pi(z, x) \vee \Pi(z, y)\} \tag{3}$$

and

$$x \cdot y = Cls\{z : \Pi(z, x) \wedge \Pi(z, y)\} \tag{4}$$

The things x, y are *disjoint*, $dis(x, y)$ in symbols, whenever there is no thing z such that $\Pi(z, x)$ and $\Pi(z, y)$ (a fortiori, the product of x and y is not defined). Hence, $dis(x, y)$ holds true if and only if $Ov(x, y)$ is false.

The *difference* $x - y$ is defined as follows

$$x - y = Cls\{z \in U : \Pi(z, x) \wedge \neg\Pi(z, y)\}. \tag{5}$$

We denote with the symbol V the thing defined as

$$V = ClsU. \tag{6}$$

which we call the universal thing.

Helped by V, for each thing $x \in U$, we define the complement to x denoted $-x$ as follows

$$-x = V - x. \tag{7}$$

We recall that those operations induce in the mereological universe the structure of a complete Boolean algebra without the null element $(U, +, \cdot, -, V)$.

Finally, we introduce the operation denoted $x \hookrightarrow y$ which we call the *mereological implication* and define as

$$x \hookrightarrow y = -x + y. \tag{8}$$

We define the truth condition for the mereological implication by declaring that

(T) $x \hookrightarrow y$ holds true if and only if $-x + y = V$.

3. An outline of rough mereology

Rough (fuzzified, see Varzi [20]) mereology is a theory of *rough inclusions*. A rough inclusion on a mereological universe U endowed with a part relation π is the relation $\mu(x, y, r)$ on the product $U \times U \times [0, 1]$ cf. Polkowski and Skowron [17] and Polkowski [11],[12], Polkowski and Artiemjew [14].

Rough inclusions satisfy the following postulates, relative to a given part relation π and the induced by π relation Π of an improper part on U:

RINC1 $\mu(x, y, 1)$ if and only if $\Pi(x, y)$.
This postulate asserts that parts to degree of 1 are improper or proper parts.

RINC2 if $\mu(x, y, 1)$ then for each thing z and $r \in [0, 1]$ [if $\mu(z, x, r)$ then $\mu(z, y, r)$].

This postulate does express a feature of partial containment that a 'bigger' thing contains a given thing 'more' than a 'smaller' thing. It can be called the *monotonicity condition* for rough inclusions.

RINC3 $\mu(x, y, r)$ and $s < r$ then $\mu(x, y, s)$.
This postulate specifies the meaning of the phrase 'a part to a degree at least of r'.

We mention here the fact that rough inclusions have been extensively studied from the point of view of granular computing and granular algorithms for classification in data in Polkowski and Artiemjew [14].

4. Mass assignment on a mereological universe. The mass based rough mereological logic (m-RM logic)

We introduce a new scheme for inducement of rough inclusions on the mereological unverse (U, Π) by means of the mass assignment $m : (0, 1] \to U$ which assigns the value of mass $m(x)$ to each thing x in U. The notion of a mass is known from the evidence theory of Dempster and Schaffer, see, e.g., Dempster [3], where it is also called the basic probability assignment (b.p.a.). For a general discusuion of mass expressions see Nicolas [7].

We assume that masses on things in U are positive. We introduce the symbol 0 to denote non-existing things and we extend the mass assignment by letting

$$m(0) = 0. \tag{9}$$

m-RM-logic We define the logic which settles the properties of mass assignments. This logic is a generalization, inter alia, of the logic of probability calculus in Łukasiewicz [6]. The implication symbol $\Rightarrow$ is understood as the symbol of implication in propositional calculus.

Axioms

(A1) $(x = V) \Leftrightarrow (m(x) = 1)$.

(A2) $(x = 0) \Leftrightarrow (m(x) = 0)$.

(A3) $(x \hookrightarrow y) \Rightarrow [m(y) = m(x) + m(-x \cdot y)]$.

Theses

(T1) $m(V) = 1$. By substitution x/V into A1.

(T2) $m(0) = 0$. By substitution $x/0$ into A2.

(T3) $(x = y) \Rightarrow [m(x) = m(y)]$. Proof: $x = y$ is equivalent to $\Pi(x, y) \wedge \Pi(y, x)$, hence, to $x(\hookrightarrow y) \wedge (y \hookrightarrow x)$ which implies by A3 and positiveness of m that $m(y) \geq m(x)$ and $m(x) \geq m(y)$ and finally $m(y) = m(x)$.

(T4) $m(x) + m(-x) = 1$. Proof: substitute x/V in A3. Get $m(V) = m(x) + m(-x \cdot V)$, i.e., $1 = m(x) + m(-x)$ by T1.

(T5) $m(x + y) = m(x) + m(-x \cdot y)$. Proof: Substitute $y/x + y$ into A3 and get $m(x + y) = m(x) + m(-x \cdot x + -x \cdot y) = m(x) + m(-x \cdot y)$.

(T6) $m(y) = m(x \cdot y) + m(-x \cdot y)$. Proof: Substitute $x/x \cdot y$ into A3 and obtain $m(y) = m(x \cdot y) + m(-(x \cdot y) \cdot y)$=$m(x \cdot y) + m((-x + -y) \cdot y)$= $m(x \cdot y) + m(-x \cdot y)$.

(T7) $m(x + y) = m(x) + m(y) - m(x \cdot y)$. Proof: hint in Łukasiewicz [6]: subtract thesis in T5 from thesis in T6.

(T8) $dis(x, y) \Rightarrow m(x + y) = m(x) + m(y)$. Proof: $dis(x, y)$ is equivalent to $x \cdot y = 0$, hence, $m(x \cdot y) = 0$ by T2 and the thesis follows by T6.

(T9) $m(x + y) = m(x) + m(y) \Rightarrow x \cdot y = 0$. Proof: by T7, $m(x \cdot y) = 0$ so by T2 $x \cdot y = 0$.

(T10) $m(x) + m(-x \cdot y) = m(y) \Rightarrow (x \hookrightarrow y)$. Proof: by T8, $m(y) = m(x) + m(-x \cdot y) = m(x + -x \cdot y) = m(x + y)$ so $-x + y = V$, i.e. $x \hookrightarrow y$ holds true.

We define the mass assignment of x relative to y, denoted $m_y(x)$ and defined as follows

$$Def.1.\ m_y(x) = \frac{m(x \cdot y)}{m(x)}. \tag{10}$$

(T11) $m_x(V) = m(x)$. Proof: $m_x(V) = \frac{m(V \cdot x)}{m(V)} = \frac{m(x)}{1} = m(x)$.

(T12) $m(x \cdot y) = m(x) \cdot m_y(x) = m(y) \cdot m_x(y)$. Proof: by definition (10)

(T13) [The Bayes theorem in mass mereology] $m_y(x) = \frac{m(y) \cdot m_x(y)}{m(x)}$. Proof: from T12.

Following Łukasiewicz [6](Df(2), p. 29 in Selected Works), we define independence of things as the relation $I(x, y)$:

$$Def2.\ I(x, y) \Leftrightarrow m_y(x) = m_y(-x). \tag{11}$$

(T14) $I(x, y) \Leftrightarrow \frac{m(x \cdot y)}{m(x)} = \frac{m(-x \cdot y)}{m(-x)}$. Proof: by Def 2 and Def. 1.

(T15) $I(x, y) \Leftrightarrow m(x \cdot y) = m(x) \cdot m(y)$. Proof: Łukasiewicz (loc.cit.) shows the most elegant proof so we apply his idea here. From $\frac{m(x \cdot y)}{m(x)} = \frac{m(-x \cdot y)}{m(-x)}$, on assumption that $I(x, y)$, it follows that

$$\frac{m(x \cdot y)}{m(x)} = \frac{m(-x \cdot y)}{m(-x)} = \frac{m(x \cdot y) + m(-x \cdot y)}{m(x) + m(-x)} = \frac{m(y)}{1} = m(y).$$

It follows from above that mass m is additive: $m(x + y) = m(x) + m(y)$ when x and y are disjoint.

5. The Bayes formula

The simple Bayes formula has been proved as T13 in the preceding section. Now, we prove its general form

$$(T16)\ \ +_{i\neq j} y_i \cdot y_j = 0 \wedge +_i y_i = V \Rightarrow m_z(x) = \frac{m(z)m_x(z)}{\sum_{j=1}^{k} m(y_j)m_{y_j}(x)}.$$

Proposition 5.1. There exists a set $Y = y_1, y_2, \cdots, y_k$ of things in U (the reference set) such that for each thing x it is true that

$$m(x) = \sum_{j=1}^{k} m(y_j)m_{y_j}(x).$$

Proof:
Let Y be a maximal set of pairwise disjoint things. Then, $+Y = V$. Given an arbitrarily chosen thing x, there exists the set $Y(x) \subseteq Y$ such that if $Ov(x, y_j)$ then $y_j \in Y$ for $j = 1, 2, \cdots, k$. We need to establish the truth of some claims.

Claim 1. $x = Cls\{x \cdot y_j : y_j \in Y(x)\}$.

Proof:
We recall that $x = Cls\{z : \Pi(z, x)\}$ and we use the inference rule (IR) to establish that things on both sides of Claim 1 are improper elements of each other, hence, they are equal.

Let $\Pi(z, x)$; there exists y_j in $Y(x)$ such that z overlaps with y_j and $\Pi(z \cdot y_j, x \cdot y_j)$ holds true, hence, by (IR), $\Pi(x, Cls\{x \cdot y_j : y_j \in Y(x)\})$ is true.

For the converse, let $\Pi(z, Cls\{x\cdot y_j : y_j \in Y(x)\})$; there exist things u, v such that $\Pi(u, z)$, $\Pi(u, v)$, $w = x \cdot y_p$ for some $y_p \in Y(x)$, hence $\Pi(w, x)$ and, by (IR), $\Pi(Cls\{x \cdot y_j : y_j \in Y(x)\}, x)$.

It follows finally that $x = Cls\{x \cdot y_j : y_j \in Y(x)\}$. □

Claim 2. $Cls\{x \cdot y_j : y_j \in Y(x)\} = +_{y_j \in Y(x)}(x \cdot y_j)$.

Proof:
It goes on the same lines as the proof of Claim 1. □

Claim 3. $x = +_{j=1}^{k}(x \cdot y_j)$.

Proof:
By Claims 1 and 2. □

By additivity of the mass m and by Claim 3 it follows that
Claim 4.

$$m(x) = \sum_{j=1}^{k} m(x \cdot y_j) = \sum_{j=1}^{k} m(y_j) \cdot m_{y_j}(x).$$

Hence, by T13 and Claim 4,

$$m_z(x) = \frac{m(z) \cdot m_x(z)}{\sum_{j=1}^{k} m(y_j) \cdot m_{y_j}(x)}.$$

The Bayes formula is proved. □

6. Rough inclusions induced by mass assignments

For things in a universe U and a mass assignment m, we define the relation $\mu_m(x, y, r)$, where $x, y \in U$ and $r \in [0, 1]$, as follows

$$\mu_m(x, y, r) \Leftrightarrow \frac{m(x \cdot y)}{m(x)} \geq r. \tag{12}$$

Hence, $\mu_m(x, y, r)$ coincides on the left-hand side with already discussed relative mass $m_y(x)$.We denote with the symbol $\mu_m(x, y)$ the greatest value of r possible, i.e.,

$$\mu_m(x, y) = \frac{m(x \cdot y)}{m(x)}. \tag{13}$$

For things $x, y \in U$, we define the notion of an improper part $\Pi(x, y)$ as follows

Definition 6.1. $\Pi(x, y) \Leftrightarrow (x \cdot y = x)$.

We establish the following properties of the relation Π and of function μ.

(T17) $\Pi(x, y) \Rightarrow \mu_m(x, y) = 1$. Proof: by definition6.1 and equation (12).

(T18) $\Pi(x, y) \wedge \Pi(y, x) \Rightarrow (x = y)$. Proof: by definition 6.1, $x \cdot y = x$ and $x \cdot y = y$, hence, $x = y$.

(T19) $\Pi(x, x)$. Proof: $x \cdot x = x$.

(T20) $\Pi(x, y) \wedge \Pi(y, z) \Rightarrow \Pi(x, z)$. Proof: from $x \cdot y = x$ and $y \cdot z = y$, we obtain $x \cdot z = (x \cdot y) \cdot z$=$x \cdot (y \cdot z) = x \cdot y = x$.

(T18), T(19) and (T20) show that Π is a genuine notion of an improper part and $\pi(x, y) \Leftrightarrow \Pi(x, y) \wedge x \neq y$ is the corresponding notion of a part.

(T21) $(\mu_m(x, y) = 1) \Rightarrow \Pi(x, y)$. Proof: as $x \cdot y \hookrightarrow x$, we have by A3: $m(x) = m(x \cdot y) + m(x - x \cdot y)$, hence, $m(x - x \cdot y) = 0$ so $x - x \cdot y = 0$, hence, $x = x \cdot y$, i.e., $\Pi(x, y)$.

(T22) $\Pi(x, y) \Rightarrow x \hookrightarrow y$. Proof: from $x \cdot y = x$, we obtain $-x + y = -(x \cdot y) + y = -x + (-y) + y = -x + V = V$.

(T23) $(x \hookrightarrow y) \Rightarrow \Pi(x, y)$. Proof: from $-x + y = V$ it follows that $-x \cdot x + y \cdot x = x$, i.e., $y \cdot x = x$.

(T24) $\Pi(x, y) \Leftrightarrow x \hookrightarrow y \Leftrightarrow \mu_m(x, y) = 1$. Proof: by T17, T21, T22, T23.

(T25) $\Pi(x, y) \Rightarrow m(x) \leq m(y)$. Proof: by T22 and A3, we have $m(y) = m(x) + m(y \cdot (-x))$, hence, $m(y) \geq m(x)$.

(T26) $[(\mu_m(x, y) = 1) \wedge (\mu_m(z, x) = r)] \Rightarrow (\mu_m(z, y) \geq r)$. Proof: as $\Pi(x, y)$, hence, $\Pi(z \cdot x, z \cdot y)$, T25 implies that $m(z \cdot x) \leq m(z \cdot y)$, hence, $\mu_m(z, y) \geq \mu_m(z, x)$.

T24, T26 imply that μ defined by us is a genuine rough inclusion satisfying (RINC1), (RINC2) and in an obvious way (RINC3).

Let us recall the simple form of the Bayes theorem T13 rendered with the help of the rough inclusion μ:

(T13)* $\mu_m(x, y) = \frac{m(y) \cdot \mu_m(y, x)}{m(x)}$.

We infer from this theorem some new facts on the rough inclusion μ.

(T27) $\Pi(x, y) \Rightarrow \mu_m(y, x) = \frac{m(x)}{m(y)}$. Proof: let $\mu_m(x, y) = 1$ in T13*.

(T28) $\frac{\mu_m(x, y)}{\mu_m(y, x)} = \frac{m(y)}{m(x)}$. Proof. Direct from T13*.

(T29) [Transitivity formula] $\frac{\mu_m(x, y)}{\mu_m(, x)} \cdot \frac{\mu_m(y, z)}{\mu_m(z, y)} = \frac{\mu_m(x, z)}{\mu_m(z, x)}$. Proof: from T28.

(T30) $\mu_m(x,-y) = 1 - \mu_m(x,y)$. Proof: as $x \cdot y \hookrightarrow x$, we have by A3 that $m(x) = m(x \cdot y) + m(-(x \cdot y) \cdot x) = m(x \cdot y) + m((-y) \cdot x)$ and thus $m((-y) \cdot x) = m(x) - m(x \cdot y)$. Hence, $\mu_m(x,-y) = \frac{m((-y)\cdot x)}{m(x)}$=$\frac{m(x)-m(x\cdot y)}{m(x)}$=$\frac{m(x)}{m(x)} - \frac{m(x\cdot y)}{m(x)}$=$1 - \mu_m(x,y)$.

7. The Łukasiewicz logic of probability as a particular case of m-RM logic. Relations of m-RM logic to many-valued logics and a generalization of the Łukasiewicz 'fuzzy' logic

In Łukasiewicz [6], the rendering of rules of probability in the logical form was proposed. This work brought for the idea of fractional values of logical formulas, the idea applied in Computer Science, notably in Machine Learning and Data Analysis,e.g., in the form of rough inclusions in decision systems [11], in rough mereology [17] or in definition of the rough memebership functions in Pawlak and Skowron [10]. Łukasiewicz considered a collection of indefinite formulas of the form $q(x)$ over a finite universe U and for a formula $q(x)$ he defined its value $w(q)$ as the fraction

$$w(q) = \frac{|\{u \in U : q(u)\ is\ true\}|}{|U|}, \tag{14}$$

where $|X|$ denotes the cardinality of the finite set X.

We define the function tr from symbols of arguments and operations of $m-RM$ logic onto symbols and operations of the logic of probability. For a thesis λ of $m-RM$ logic, the formula δ such that $\lambda\ tr\ \delta)$ is a thesis of the logic of probability.

For symbols: $x\ tr\ a, y\ tr\ b, m\ tr\ w, 1\ tr\ 1, 0\ tr\ 0, V\ tr\ 1$.

For operations: $-x\ tr\ a, -y\ tr\ b, x+y\ tr\ a+b, xy\ tr\ ab, \mu(x,y)\ tr\ w_a(b)$. Hence, $x \hookrightarrow y\ tr\ a > b$.

7.1. Many-valued 'fuzzy' logics

The name for those many-valued logics comes from their importance for Artificial Intelligence and Machine Learning, see Hájek [4] for the theory of t-norms and 'fuzzy' logics. We recall here the 'classical' ones studied fom theoretical point of view and in applications.

They are induced by t-norms, respectively, the minimum t-norm $M(x,y)$, the product t-norm $P(x,y)$ and the Łukasiewicz t-norm $L(x,y)$ defined as follows on the unit square $[0,1]^2$

$$M(x,y) = min\{x,y\}, \tag{15}$$

$$P(x,y) = x \cdot y, \tag{16}$$

and,

$$L(x,y) = max\{0, x+y-1\}. \tag{17}$$

Each of those t-norms does induce the implication in the form of its residuum $x \rightarrow_T y$, where T stands for the corresponding t-norm. When $x \leq y$, each residuum takes on the value 1, so only the case $x > y$ does require definitions:

$$x \to_M y = y, \quad (18)$$

$$x \to_P y = \frac{y}{x}, \quad (19)$$

and,

$$x \to_L y = min\{1, 1 - x + y\}. \quad (20)$$

Along with negations defined as $-x = x \to 0$, the residua $\to_M, \to_P, \to_L$ define logics of, respectively, Goedel, Goguen, Łukasiewicz.

Each of these residua defines the corresponding rough inclusion μ_{Goedel}, μ_{Goguen}, $\mu_{Lukasiewicz}$ by the formula $\mu(x, y, r)$ if and only if $x \to y \geq r$, see [11].

7.2. m-RM logic against many-valued logics

We observe the following relationships showing that m-RM logic coincides to a substantial extent with logics of Goedel, Goguen and Łukasiewicz and it offers a generalization of the last one.

(T31) If $\Pi(x, y)$ then Goedel, Goguen, Łukasiewicz rough inclusions, and, the mass based rough inclusion all give the value 1.

(T32) If x, y are independent, i.e., $m(x \cdot y) = m(x) \cdot m(y)$, then $\mu_m(x, y) = m(y)$, i.e., $\mu_m(x, y) = \mu_{Goedel}(x, y)$: on collections of pairwise independent things, the degree of inclusion is given by the Goedel logic; e.g. on collections of pairwise independent events in a finite probability space (or, more generally, on collections of pairwise independent sets).

(T33) If $\Pi(x, y)$ then $m(y) \geq m(x)$, hence, $\mu_m(y, x) = \frac{m(x)}{m(y)}$, i.e., $\mu_m(y, x) = \mu_{Goguen}(y, x)$, so when x is an improper part in y with respect to the rough inclusion μ_m, then the reciprocal rough inclusion $\mu_m(y, x)$ of y into x is defined also by the Goguen logic.

Finally, we consider the rough mereological implication $x \hookrightarrow y$. We have

(T34) (i) If $\Pi(x, y)$ then $m(x \hookrightarrow y) = 1$;

(ii) If $\Pi(y, x)$ them $m(x \hookrightarrow y) = 1 - m(x) + m(y)$;

(iii) If neither $\Pi(x, y)$ nor $\Pi(y, x)$ then $m(x \hookrightarrow y) = 1 - m(x - y)$.

Proofs. For (i): T24. For (ii): $m(x \hookrightarrow y) = m(-x + y)$=$m(V - (x - y)) = 1 - m(x - y)$ = $1-[m(x)-m(y)]$=$1-m(x)+m(y)$. For (iii): $m(x \hookrightarrow y) = m(-x+y)$=$m(V-(x-y)) = 1-m(x-y)$.

Letting for $x \in [0, 1]$: $m(x) = x$, we obtain the logic of Łukasiewicz: $-x = 1 - x, x \to_L y = min\{1, 1 - x + y\}$. The case (iii) was not considered by Łukasiewicz as in his case the set of values was the linearly ordered unit interval [0,1].

8. Betweenness and granularity in mass based rough mereology

The notion of betweenness relation due to Tarski [19], modified by van Benthem [1] and adapted by us to the needs of data analysis and behavioral robotics will acquire here an abstract formulation in the framework of the mass mereology.

We introduce first the notion of distance $\delta(x, y)$ between two things x, y in the universe U. We let

$$\delta(x, y) = m(x - y) + m(y - x). \quad (21)$$

We say that the thing z *is between* things x, y, in symbols $Btw(z, x, y)$, when the following condition is satisfied

(B) For each thing w not identical to z,and for each thing u such that $\Pi(u, w)$ there exists a thing t such that $\Pi(t, z)$ and either $\delta(t, x) < \delta(u, x)$ or $\delta(t, y) < \delta(u, y)$.

We have the following proposition

Proposition 8.1. For each pair x, y of things, the sum $x + y$ is the only thing between x and y.

Proof:
Proof: assume that $w \neq x + y$ so w does not satisfy the class definition for $x + y$, hence, there exists a thing u such that $\Pi(u, w)$ and u does overlap neither with x nor with y. Letting v as x or y, we have: $\delta(u, x) = m(u) + m(x) > 0$ while $\delta(x, x) = 0$, similarly in case of y. □

The condition (B) as well as the notion of betweenness can be extended for finite sets of things to the notion $GBtw(z, T)$, where T is a finite set of things, of the *generalized betweenness relation* which holds when the codition (GB) is satisfied

(GB) For each thing $w \neq z$ and each thing u with $\Pi(u, w)$ there exist a thing v such that $\Pi(v, z)$ and a thing $t \in T$ such that $\delta(u, t) > \delta(v, t)$.

Remark 8.1. In particular cases, important for applications, the mereological sum acquires specific renditions in the context of betweenness.

In behavioral robotics, when mobile robots, or more generally intelligent agents, are modeled as planar rectangles,the mereological sum of two things a, b is the extent $ext(a, b)$, i.e., the smallest rectangle containing a and b cf. Polkowski and Ośmiałowski [15], [16]. Hence, the notion of the meeological sum should be modified: for a given context C, the sum $x + y$ of two things satisfying the context C is the smallest thing satisfying the context C and such that it contains each thing being an improper part of either x or y (smallest, containment are understood in terms of the relation Π).

In the case of information/decision systems, where things are represented by means of their information sets, i.e., for a system with attributes in the set A and with the values of attributes in the set V, the information set for a thing u is the set $Inf(u) = \{a(u) : a \in A\}$, the mereological sum of things u and v relative to a partition $P = \{A_1, A_2\}$ is a thing w such that $Inf(w) = \{a(u) : a \in A_1\} \cup \{a(v) : a \in A_2\}$ cf. Polkowski [13].

8.1. Granular computing in mass-based rough mereology

For a thing x and a real number $r \in [0, 1]$ called the *granularity radius*, we define the *granule about x of radius r* denoted $g_r(x)$ as the class

$$g_r(x) = Cls\{y : \mu_m(y, x) \geq r\}. \tag{22}$$

From the class definition condition (C2), we obtain the following characterization of the improper part for granules.

Proposition 8.2. $\Pi(u, g_r(x))$ iff there exist things w and v with the properties that (i) $\mu_m(u, w) = \frac{m(w)}{m(u)}$ (ii) $\mu_m(v, w) = \frac{m(w)}{m(v)}$ (iii) $\mu_m(v, x) \geq r$.

Proof:
It does follow directly from the class definition (C1), (C2). □

The algorithm for checking whether $\Pi(u, g_r(x)$
1. Find the ordered set $W(u) = \{w : \mu_m(u, w) = \frac{m(w)}{m(u)} = \{w : m(u \cdot w) = m(w)\}$.
2. For each $w \in W(u)$ according to the order find the set $V_W(w) = \{v : m(v \cdot w) = m(w) \wedge \mu_m(v, x) \geq r\}$ going in the order of the set $\{v : \mu(v, x) \geq r\}$.
3. If the first $V_W(w)$ non–empty then return YES else NO (u is not any improper part of $g_r(x)$).

9. In search of an application: Clustering

Let us consider a possible mechanism for clustering based on mass rough inclusions. To this end, we introduce another distance function $\Delta(x, y)$ given by the formula

$$\Delta(x, y) = |\mu(x, y) - \mu(y, x)| = m(x \cdot y) \cdot |\frac{1}{m(x)} - \frac{1}{m(y)}|. \tag{23}$$

Given $\varepsilon > 0$, we consider the tolerance relation

$$\tau(x, y) \Leftrightarrow \Delta(x, y) \leq \varepsilon. \tag{24}$$

We define clusters as tolerance classes, i.e. maximal collections of things with the property that each pair of things in the collection are in the relation τ.

Let us provide a simple example.

Example 9.1. Consider things in a collection U being landscapes or photographs of a countryland on which we have trees, figures of people, houses. For a particular thing x we assign the mass $m(x)$ as the sum $m_1(x) + m_2(x) + m_3(x)$, where $m_1(x) =$ if and only if there are at least 3 trees on x, $m_2(x) = 1$ if and only if there are at least 2 people on x, and $m_3(x) = 1$ if and only if there is at least one house on x. We assume that $m(x)$ is at least 1 for each x in U. Figure 1 shows possible outcomes for pairs of things and values of ε for which these things may fall into one cluster. We have three possible types of things: Type I with m=3, Type II with m=2, and, Type III with m=1. We include into $x \cdot y$ a unit if and only if both x, y satisfy conditions for this unit, for instance if both x, y have $m_1 = 1$ then $m_1(x \cdot y) = 1$. We do not consider in this example the sum operation.

It follows that for $\varepsilon < \frac{1}{3}$, clustering makes into clusters things of the same type: cluster 1 with things of Type I, cluster 2 with things of Type II, and, cluster 3 with things of Type III.

10. In search of an application: Making evidence theory decisive

In evidence theory (cf. Dempster, loc.cit.), mass assignments are also called basic probability assignments (b.p.a.'s) and they are assigned to all subsets of a set of possible outcomes called the frame of discernment. We illustrate our approach with an example.

Table 1. Types of mass assignment towards clustering

$Type\ x$	$Type\ y$	$m(x \cdot y)$	$\Delta(x, y)$	$\varepsilon\ clustering$
I	I	3	$\Delta = 0$	$any\ positive$
I	II	2	$\Delta = \frac{1}{3}$	$\varepsilon \geq \frac{1}{3}$
I	III	1	$\Delta = \frac{2}{3}$	$\varepsilon \geq \frac{2}{3}$
II	II	$2\ or\ 1$	$\Delta = 0$	$any\ positive$
II	III	$0\ excluded\ or\ 1$	$\Delta = \frac{1}{2}$	
III	III	$40\ excluded\ or\ 1$	$\Delta = 0$	$any\ positive$

Example 10.1. Imagine a car accident - a collision at the road crossing endowed with traffic lights. It is crucial to establish what light was on for the driver on the main road. Witnesses gave combined evidences resulting in the following b.p.a. m:

$m(red) = 0.25$,
$m(yellow) = 0.35$,
$m(green) = 0.20$,
$m(red\ or\ yellow) = 0.08$,
$m(red\ or\ green) = 0.02$,
$m(yellow\ or\ green) = 0.08$,
$m(red\ or\ yellow\ orgreen) = 0.02$.

From this assignment, values of the belief function, $Bel(A) = \sum_{B \subseteq A} m(B)$, are computed:

$Bel(red) = 0.25$,
$Bel(yellow) = 0.35$,
$Bel(green) = 0.20$,
$Bel(red\ or\ yellow) = 0.68$,
$Bel(red\ or\ green) = 0.47$,
$Bel(yellow\ or\ green) = 0.63$,
$Bel(red\ or\ yellow\ or\ green) = 1.0$.

We now compute values of rough inclusion taking as new masses for rough inclusions the computed values of Belief function. Hence, $\mu(x, y) = \frac{Bel(x \cap y)}{Bel(x)}$. These computed values of rough inclusions are collected in Figure 2. We omit the full set $\{r, y, g\}$ as the least decisive.

We introduce the measure of *independent evidence* $M(y)$ as the sum

$$\sum_{all\ non-singleton\ sets\ x \neq y} \mu(y, x). \tag{25}$$

These values are therefore : $M(red,\ yellow) = 0.54$, $M(red,\ green) = 1.0$, $M(yellow,\ green) = 0.85$. It follows that the maximally independent, having the smallest intersection/dependence on other sets is $red,\ yellow$. One decides that the light on the main road at the moment of crossing the crossroads was either red or yellow. Now in this set, the proportion of evidence for red to evidence for yellow is like 0.27:0.5 so the decision is on yellow light.

Table 2. Values of rough inclusions between sets of traffic lights

set	red	$yellow$	$green$	$red, yellow$	$red, green$	$yellow, green$
red	1.0	0.0	0.0	1.0	1.0	0.0
$yellow$	0.0	1.0	0.0	1.0	0.0	1.0
$green$	0.0	0.0	1.0	0.0	1.0	1.0
$red, yellow$	0.27	0.5	0.0	1.0	0.27	0.27
$red, green$	0.5	0.0	0.4	0.5	1.0	0.5
$yellow, green$	0.0	0.7	0.4	0.55	0.3	1.0

11. Conclusion

We have introduced the notion of a mass into rough mereology which has allowed us to express the reciprocal relations of partial containment in the form characteristic to the Bayes formula in probability theory. We have expressed the betweenness relation in an abstract mass-based framework. We proposed an application to clustering that allows for inducing various sets of clusters dependent on the threshold distance ε.We demonstrated the mass based rough mereological logic which does encompass the Łukasiewicz logic of probability and we extended the 'fuzzy', i.e., many-valued logic of Łukasiewicz to the logic based on properties of mass assignments. We hope that this abstract formulation will prove a convenient vehicle for some forms of approximate reasoning to be developed in future. At the end, we proposed a decision procedure involving mass based rough inclusions derived from belief values in evidence theory.

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