The Linear Algebra of Abstract Argumentation

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Abstract. In Abstract Argumentation, the task of modeling and analyzing semantics is a hot problem. An alternative representation of computational models of argument, based on the matrix theory, is proposed, in order to obtain a deeper understanding of extension-based semantics and of ranking semantics too. In this paper, we start from the concept of matrix representation of an argumentation graph and develop a general strategy to model both extension-based semantics and ranking-based ones, and both classical and new ones, in terms of matrix operations. This line of research is strongly based on linear algebra to evaluate arguments. Such procedures confirm the ability to evaluate arguments and open to new perspectives in the field of ranking semantics.

Keywords: Abstract Argumentation · Semantics · Matrix Theory.

1 Introduction and Related Work

Abstract Argumentation Frameworks (AFs) are usually represented as directed graphs not only because of the feature of visualization, but also because they play a significant role in modeling and analyzing the semantics of AFs. Mathematically speaking, studies in Abstract Argumentation semantics are concerned with graph-theoretic measures on directed graphs. Matrix theory is an important field of Linear Algebra used in particular for representing and handling graphs. Adjacency matrices, representing adjacent nodes, are capable of representing undirected and directed graphs. So, matrix representations provide a bridge to linear algebra-based algorithms for graph computation. Therefore, the potential application to AFs is worth to expect, since efficient and versatile methods for Abstract Argumentation are important for further advances in the field.

To the best of our knowledge, there are only few works in the literature of matrix representation and computation of argument graphs. An initial work in defining a matrix representation of argumentation graphs is made in [12], in which the matrices and some operations on them are introduced into the study of Dung's theory of argumentation, showing that every AF can be represented by a matrix, and the basic extensions of an AF can be determined by sub-blocks of its matrix. Similarly to this approach, we exploit the matrix representation of an argumentation graph, but we rely on a more intuitive strategy which is directly compatible with the theory of Abstract Argumentation. Another similar approach is defined in [8], in which it is introduced a matrix-based mathematical approach for determining whether a set of arguments is an extension. This differs in the fact that it does not use matrix sub-blocks but it creates sets of arguments in a matrix representation to define tests for the different argumentation semantics as they do not try to find all arguments passing a given criteria. In [11], it is proposed a Boolean matrix approach to encode Dung's acceptability semantics. Each semantics is encoded into one or more Boolean constraint models, which can be solved by Boolean constraint solvers. Recently, in [2], authors represent a Weighted Argumentation Framework (WAF) by a non-binary matrix, and characterize the basic extensions by analysing sub-blocks of this matrix.

While, in [5], it is investigated the relationship between semantics for formal argumentation and measures from social networking theory. This is done by considering matrix exponentials in which measures used for link prediction in social networks are exploited in the same way to measure acceptability of arguments for AFs. The intuition behind authors' matrix representation is similar to the idea presented in the present work, but it is only used to devise a new ranking-based semantics. Moreover, it does not take into account problems like self-attacking or unattacked arguments, and there is no discussion about how the exponential evaluation behaves with cyclic graphs.

The contributions of this paper are to use algebraic methods to obtain a deeper understanding of extension-based semantics, and to apply these ideas to bipolar and weighted AFs and to ranking semantics too. The matrix-based approach is related to the development of efficient techniques for computing extensions. Moreover, well-established techniques for the incremental computation of matrix-product could be profitably used to address the incremental computation of extensions in dynamic AFs. Starting with tools coming from Linear Algebra, we show that it is possible to give a uniform (matrix-based) representation method of an argumentation graph, and a continuity in modeling both extension-based semantics and ranking-based ones, and both classical and new ones, by developing of a single general matrix-based strategy.

The paper is organized as follows. After quickly discussing background and the basic concept of Argumentation Matrix in the next section, Section 3 defines the problem of developing a final matrix which is able to represent all the defeats and defenses between arguments, Section 4 describes the ability to find classical extension-based semantics in terms of matrices, and sections 5, 6 and 7 show its applicability also to generalizations of the standard AF. With the matrix operations in place, we then provide a new ranking-based semantics, coming from the field of Linear Algebra, that gives insights on some interesting properties. Finally, Section 8 concludes the paper.

2 Background

Let us start by providing the basics of Abstract Argumentation.

Definition 1 *An* ***AF*** *is a pair $F = \langle \mathcal{A}, \mathcal{R} \rangle$, where $\mathcal{A}$ is a finite set of arguments and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. Given $\alpha, \beta \in \mathcal{A}$, the relation $\alpha \mathcal{R} \beta$ means that α attacks β.*

An argumentation semantics is the formal definition of a method ruling the argument evaluation process. The most basic concepts shared by all argumentation semantics in the literature are *conflict-freeness* and *defense*. Then, standard acceptability semantics, introduced by Dung [6], characterize conflict-free and defended sets of arguments.

Definition 2 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF, and $S \subseteq \mathcal{A}$:*

- *S is* conflict-free *(*cf *for short) if $\nexists \alpha, \beta \in S$ s.t. $\alpha \mathcal{R} \beta$;*
- *$\alpha \in \mathcal{A}$ is* defended by *S if $\forall \beta \in \mathcal{A}\colon \beta \mathcal{R} \alpha \Rightarrow \exists \gamma \in S$ s.t. $\gamma \mathcal{R} \beta$;*
- *$f_F\colon 2^{\mathcal{A}} \mapsto 2^{\mathcal{A}}$ s.t. $f_F(S) = \{\alpha \mid \alpha$ is defended by $S\}$ is called the* characteristic function *of F;*
- *S is* admissible *if S is* cf *and S is defended by itself, i.e. $\forall \alpha \in S, \forall \beta \in \mathcal{A}\colon \beta \mathcal{R} \alpha \Rightarrow \exists \gamma \in S$ s.t. $\gamma \mathcal{R} \beta$.*
- *S is a* grounded extension *if S is the admissible least fixed point of f_F;*
- *S is a* stable extension *if S is* cf *and $\forall \alpha \in \mathcal{A}, \alpha \notin S, \exists \beta \in S$ s.t. $\beta \mathcal{R} \alpha$.*

Regarding the generalizations of Dung's AF, we recall the following frameworks. A Bipolar AF (BAF) [4] is an extension of Dung's AF in which two kinds of interactions between arguments are possible: the attack relation and the support relation. A Weighted AF (WAF) [7] is another extension of Dung's AF in which attacks between arguments are associated with a weight, indicating the relative strength of the attack. A Bipolar Weighted AF ($BWAF$) [10] incorporates both above generalizations of Dung-style AFs. The idea behind it is to allow not only weighted attack relations between abstract arguments, but also weighted support relations. This is achieved by assigning to each relation a weight which can be positive or negative. Finally, ranking-based semantics [1] methods determine, for any framework, a ranking of the available arguments in the form of a pre-order (reflexive and transitive relation). These semantics focus on the evaluation of individual arguments rather than sets of arguments.

2.1 The Argumentation Matrix

An argumentation graph can be represented with a slightly different version of its adjacency matrix.

Definition 3 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF. Let $|\mathcal{A}| = n$, then the* Argumentation Matrix *of F is a $n \times n$ matrix $M_F = [M_{ij}]$ such that for any two arguments $\alpha_i, \alpha_j \in \mathcal{A}$ it holds that*

$$M_{ij} = \begin{cases} -1 & \text{if } \langle \alpha_i, \alpha_j \rangle \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

The substantial difference from $[12, 8, 11, 2]$ is that we have the entry $M_{ij} = -1$ iff there is an attack from the argument in i^{th} row to the argument in j^{th} column. This representation choice fits more the concept of attack in the argumentation graph, to such an extent of representing a "negative" relation, which has a basic role in our novel study.

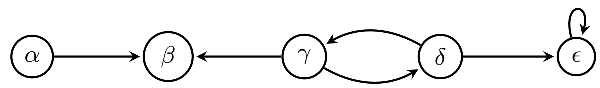

Fig. 1. F_1: Example of an AF representation

Example 1. The AF in Figure 1 has the following Argumentation Matrix:

$$\boldsymbol{M_{F_1}} = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{array} \begin{array}{c} \begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \epsilon \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

In the Argumentation Matrix, each row i represents the attacks launched towards the j^{th} argument. While, each column j represents the attacks received from the i^{th} argument. In this setting, our purpose is to exploit the Argumentation Matrix in order to devise a strategy to represent the defenses for each argument.

We notice that the notion of defense for an argument is based on the concept of transitivity which stipulates that relations between any two nodes in the graph can be described by even-length paths between the two nodes. With matrix theory, we can explore the argumentation graph uniformly with computing subsequent powers of the Argumentation Matrix. Interesting things happen when we multiply the adjacency matrix by itself. By multiplying the adjacency matrix by itself, we retrieve the number of walks of length 2 such that there is an edge from i to j. And this is exactly the A_{ij} entry in $\boldsymbol{A}^2$, by the definition of matrix multiplication. Generally, the powers of the adjacency matrix counts the number of walks such that the entry A_{ij} in $\boldsymbol{A}^k$ gives the number of walks from i to j of length k. Similarly, with increasing exponent, the powers of the Argumentation Matrix retrieve the indirect attack (respectively, defense) for an existing odd-length (respectively, even-length) path between two nodes.

Definition 4 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an acyclic AF, $k \in \mathbb{N} > 0$, and let $\boldsymbol{M_F}$ be the Argumentation Matrix of F such that $\boldsymbol{M_F^{k+1}} = 0$ (i.e., F is connected with paths of max-length k), then the matrix $\boldsymbol{S_F} = \boldsymbol{M_F} + \boldsymbol{M_F^2} + \ldots + \boldsymbol{M_F^k}$ is the resulting sum of power series $\boldsymbol{M_F^i}$, with $i = 1, \ldots, k$, called* Summation Matrix, *and contains entries $[S_{ij}]$ of three types:*

- $S_{ij} \in \mathbb{Z} < 0$, *indicating that there is an overall (indirect) attack from argument of row i towards argument of column j;*
- $S_{ij} \in \mathbb{Z} > 0$, *indicating that there is an overall defense from argument of row i towards argument of column j;*
- $S_{ij} = 0$ *if does not exist any path between argument of row i towards argument of column j.*

Specifically, $S_{ij} \in \mathbb{Z} < 0$ means that:

– $S_{ij} = -1$ if there exists only one path between the two nodes (i.e., an attack);
– otherwise, S_{ij} quantifies negatively how many attacks there are more than the defenses, if there exists more than one path between the two nodes.

While, $S_{ij} \in \mathbb{Z} > 0$ indicates that:

– $S_{ij} = 1$ if there exists only one path between the two nodes (i.e., a defense);
– otherwise, S_{ij} quantifies positively how many defenses there are more than the attacks, if there exists more than one path between the two nodes.

Roughly, the matrix $\boldsymbol{S_F}$ represents a matrix version of justified and defeated arguments. We show that, under appropriate constraints, this matrix has many properties, which can be exploited to provide a way to select reasonable sub-blocks of this form (i.e., sets of arguments) among all the possible ones in order to determine not only the classical extension-based semantics, but also other new ranking-based semantics.

3 The Justification Matrix

The strategy to handle the evaluation process with the matrix $\boldsymbol{S_F}$ is subject to a constraint: we have to ensure that before the computation of the power series of the Argumentation Matrix, there must not exist non-zero entries on the main diagonal. We may have this configuration in two cases:

1. there exist self-loops (i.e., self-attacking arguments);
2. there exist *initializers*, i.e., arguments that do not receive any attack.

The purpose is to correctly evaluate arguments in a matrix standard form. Taking into account also self-loops in the power series computation of the Argumentation Matrix, will lead to unreliable results, since the path walk transition would declare a self-attacking argument as justified when the power of the Argumentation Matrix has an even exponent. Then, we choose to momentarily subtract from the starting Argumentation Matrix $\boldsymbol{M_F}$ the main diagonal, which we call $diag_s(\boldsymbol{M_F})$. In this way, we avoid the problem of self-loops.

Definition 5 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an acyclic AF, $\boldsymbol{M_F}$ be the Argumentation Matrix of F and $diag_s(\boldsymbol{M_F})$ the main diagonal of $\boldsymbol{M_F}$. The* Argumentation Walk Matrix *is the resulting matrix $\boldsymbol{W_F} = \boldsymbol{M_F} - diag_s(\boldsymbol{M_F})$ without non-zero entries on the main diagonal.*

With $\boldsymbol{W_F}$ we avoid self-loops in the Argumentation Matrix. Then, the process of power series computation and summation to determine the matrix $\boldsymbol{S_F}$ always converge to stationary point in which, for a $n \in \mathbb{N} > 0$, the Argumentation Walk Matrix raised to the power of n will result 0. Once the Summation Matrix $\boldsymbol{S_F}$ is achieved, we hold an argumentation matrix representation in which each node is compared with each other. If there were self-loops in the starting Argumentation Matrix, they are then re-added to the Summation Matrix in order to assess the acceptability of arguments, then we need to add the previously omitted main diagonal $diag_s(\boldsymbol{M_F})$ with self-loops.

The last thing that we have to take on aboard, is that initializer arguments are "justified" by default in the process of evaluation of a semantics and play a key role in the arguments' evaluation process. Then, we define another diagonal matrix with a positive value of 1 for each initializer argument, which we call $diag_u(\boldsymbol{M_F})$. The resulting matrix collects all the information useful to evaluate the acceptability of arguments, and it is called Justification Matrix.

Definition 6 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an acyclic AF, $\boldsymbol{M_F}$ be the Argumentation Matrix of F, $diag_s(\boldsymbol{M_F})$ the main diagonal of $\boldsymbol{M_F}$ with self-loops, and $diag_u(\boldsymbol{M_F})$ a diagonal matrix of $\boldsymbol{M_F}$ with the entry $M_{Fii} = 1$ for initializer arguments. Given the Argumentation Walk Matrix $\boldsymbol{W_F} = \boldsymbol{M_F} - diag_s(\boldsymbol{M_F})$, and $k \in \mathbb{N} > 0$ such that $\boldsymbol{W_F}^{k+1} = 0$, then the Summation Matrix $\boldsymbol{S_F} = \boldsymbol{W_F} + \boldsymbol{W_F}^2 + \ldots + \boldsymbol{W_F}^k$. The* Justification Matrix *is the resulting matrix*

$$\boldsymbol{J_F} = \boldsymbol{S_F} + diag_s(\boldsymbol{M_F}) + diag_u(\boldsymbol{M_F}).$$

Once the Justification Matrix $\boldsymbol{J_F}$ is obtained, we can exploit its algebraic properties in order to evaluate the acceptability of arguments. We have now a particular representation of the argumentation graph in which all the evaluations of attacks and defenses are made explicit. An entry J_{ij} is thus the accumulated sum of paths between argument i and argument j where paths of odd length contribute negatively and paths of even length contribute positively. In this representation, the j^{th} column in $\boldsymbol{J_F}$ gives an overview on how argument j is assessed by all arguments in the framework. With the Justification Matrix we can now extract different sub-blocks of it in the same way in which the extension-based semantics are extracted.

3.1 Power Series Stop Criterion: How to Handle Cycles

It is worth to clarify how the power series summation procedure stops. We envision this procedure as an iterative process that converges to a fixed point. It comes out that for acyclic AFs the problem of power series stop criterion is trivial, since it is pretty easy to check for that $k > 0$ such that its Argumentation Matrix representation $\boldsymbol{W_F}^{k+1} = 0$. In this respect, k indicates the length of longest directed path, which it has a linear time solution for directed acyclic graphs. Hence, we get a fixed point that stops our algorithm. It is indeed a necessary (but not sufficient) condition that, in order for $\sum \boldsymbol{W_F}^k$ to converge, the i, j entry of the matrix power $\boldsymbol{W_F}^k$ must converge to zero as $k \to \infty$. In the matter of converging matrices, it is significantly easier to consider a submultiplicative matrix norm. A particularly nice norm of this type is the *Frobenius norm*. What we can say then is that a *necessary condition* for the convergence of $\sum \boldsymbol{W_F}^k$ is that $||\boldsymbol{W_F}^k|| \to 0$ as $k \to \infty$. A more impressive result is that a *sufficient condition* for the convergence of $\sum \boldsymbol{W_F}^k$ is that $||\boldsymbol{W_F}^k|| < 1$.

The problem comes with cyclic graphs. Considering an argumentation graph that contains cycles, one has to determine a stop criterion to correctly evaluate arguments, otherwise the power series computation would not terminate. Moreover, in contrast to the shortest path problem, which can be solved in polynomial time in graphs without negative-weight cycles, the longest path problem is NP-hard, meaning that it cannot be solved in polynomial time for arbitrary cyclic graphs. One has therefore to determine a cut-off point for the computation of the Justification Matrix, which in our case we set to be the *diameter* of a graph, which formally is the greatest distance between any pair of nodes, i.e., the length of the *longest shortest path* between any two nodes in the graph. To find the diameter of a graph, first find the shortest path between each pair of nodes. The greatest length of any of these paths is the diameter of the graph. In this context, if the diameter is $k \in \mathbb{N} > 0$, it is enough to show that $\boldsymbol{M}_{\boldsymbol{F}}^0, \boldsymbol{M}_{\boldsymbol{F}}, \boldsymbol{M}_{\boldsymbol{F}}^2, \ldots, \boldsymbol{M}_{\boldsymbol{F}}^k$ are linearly independent. Therefore, in cyclic argumentation graphs, it suffices to compute the diameter of the AF to ensure that the Justification Matrix would compare each node to any other.

Example 2. In Example 1 there exist a self-loop for node ϵ and there is an initializer argument α. Hence, it is necessary to subtract the diagonal matrix $diag_s(\boldsymbol{M}_{\boldsymbol{F_1}})$ from $\boldsymbol{M}_{\boldsymbol{F_1}}$. Since there is also a cycle in F (γ attacks δ and vice versa), we set as exponent of the last Argumentation Walk Matrix to raise the diameter of F, which is 2. Then, it suffices to raise the matrix $\boldsymbol{W}_{\boldsymbol{F_1}}$ to the power of 2. We can now calculate the Justification matrix by re-adding the diagonal matrix $diag_s(\boldsymbol{M}_{\boldsymbol{F_1}})$ for self-loops and by adding the diagonal matrix $diag_u(\boldsymbol{M}_{\boldsymbol{F_1}})$ for initializer argument. Below, $\boldsymbol{W}_{\boldsymbol{F_1}}$ is the resulting Argumentation Walk Matrix, $\boldsymbol{W}_{\boldsymbol{F_1}}^{\boldsymbol{2}}$ is its 2nd power, $\boldsymbol{S}_{\boldsymbol{F_1}}$ is the related Summation Matrix, and $\boldsymbol{J}_{\boldsymbol{F_1}}$ is the resulting Justification Matrix.

$$\boldsymbol{W}_{\boldsymbol{F_1}} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{S}_{\boldsymbol{F_1}} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{W}_{\boldsymbol{F_1}}^{\boldsymbol{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{J}_{\boldsymbol{F_1}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

4 Characterizing Extension-based Semantics

By Definition 6, the matrix $\boldsymbol{J}_{\boldsymbol{F}}$ contains all the information of the AF F. In this section, we mainly focus on finding the characterization of various extensions in the matrix $\boldsymbol{J}_{\boldsymbol{F}}$ of F. The idea is to establish the relation between the extensions of F and the sub-blocks of $\boldsymbol{J}_{\boldsymbol{F}}$. Formally, we have first to define the sub-block of a matrix.

Definition 7 *Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a square matrix. A* sub-block *matrix (or partitioned matrix) is a matrix $\boldsymbol{B} \in \mathbb{R}^{k \times k}$, with $k \leq n$ whose arrays of elements* *are belonging to (not necessarily consecutive) rows* $i_1, i_2, \ldots, i_k$ *and columns* $j_1, j_2, \ldots, j_k$ *of* $\boldsymbol{A}$.

4.1 Characterizing the Grounded Extension

Since the j^{th} column in $\boldsymbol{J_F}$ gives an overview on how argument j is assessed by all arguments in the framework, it is sufficient to sum the entries in each column and check whether the resulting value is ≥ 1. In fact, having a column sum strictly positive would mean that the argument in column j is fully justified in the framework, and thus would be a member of the Grounded Extension. For the Grounded Semantics, we consider as sub-block of the Justification Matrix the whole matrix itself, since the Grounded Extension is a single-status semantics, and would at least be the empty set.

Axiom 1 *Let* $F = \langle \mathcal{A}, \mathcal{R} \rangle$ *be an AF, with* $|\mathcal{A}| = n$, *and* $\boldsymbol{J_F}$ *be the* $n \times n$ *Justification Matrix of* F. *If* $cs(\boldsymbol{J_F})^{1\times n}$ *is the resulting array of* $\boldsymbol{J_F}$ *column-wise sum, then the Grounded Extension of* F *consists of the* i^{th} *elements of* $cs(\boldsymbol{J_F})$ *with* $cs(\boldsymbol{J_F})_i \geq 1$, *unless there is a negative entry in the corresponding columns of* $\boldsymbol{J_F}$. *If none of the array elements satisfies this requirement, then the Grounded Extension of* $F = \{\emptyset\}$.

Example 3. In Example 2 the Justification Matrix of F_1 is the following:

$$\boldsymbol{J_{F_1}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Let's consider the sum of the entries for each column:

$$cs(\boldsymbol{J_{F_1}}) = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \end{bmatrix}$$

The only justified argument, with the corresponding column sum ≥ 1 is $\{\{\alpha\}\}$, which indeed is the Grounded Extension *gr(*F_1*)*.

Another interesting interpretation of the array $cs(\boldsymbol{J_F})$ is that for each argument it is possible to evaluate how much it is defended or defeated. In this sense, we go beyond the classical accepted/rejected evaluations, and we can therefore exhibit a ranking-based evaluation of arguments.

4.2 Characterizing Conflict-free Subsets

The first basic requirement for any extension is the conflict-free principle, i.e., if an argument i attacks another argument j then they can not be included together in an extension. So, we present the matrix condition which ensures that a subset of an AF is conflict-free.

Axiom 2 *Let* $F = \langle \mathcal{A}, \mathcal{R} \rangle$ *be an AF, with* $|\mathcal{A}| = n$, *and* $\boldsymbol{J_F}$ *be the* $n \times n$ *Justification Matrix of* F. *The* $k \times k$ *matrix* $\boldsymbol{C_F}$, *with* $k \leq n$, *is a* sub-block conflict-free matrix *of* F *iff each entry* $[C_{Fij}] \geq 0$.

4.3 Characterizing the Admissible Subsets

To determine the admissibility of a given subset of arguments, we start by collecting all the conflict-free sub-blocks of the Justification Matrix and check for those sub-blocks whose entries hold non-negative values.

Axiom 3 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF, with $|\mathcal{A}| = n$, and $\boldsymbol{J_F}$ be the $n \times n$ Justification Matrix of F. The $k \times k$ sub-block conflict-free matrix $\boldsymbol{D_F}$, with $k \leq n$, is a* sub-block admissible matrix *of F iff the sum of each row $[D_{Fi}] > 0$ and the sum of each column $[D_{Fj}] > 0$.*

Example 4. Given the AF F_1 in Example 1 and the Justification Matrix $\boldsymbol{J_{F_1}}$ in Example 2, we report all the conflict-free sub-blocks of $\boldsymbol{J_{F_1}}$:

conflict-free subset	$\{\ \}$	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\delta\}$	$\{\alpha, \gamma\}$	$\{\alpha, \delta\}$	$\{\beta, \delta\}$
sub-block	$[\ ]$	$[1]$	$[0]$	$[1]$	$[1]$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

The admissible sets are the following: $\{\{\}, \{\alpha\}, \{\gamma\}, \{\delta\}, \{\alpha, \gamma\}, \{\alpha, \delta\}\}$. As we can see, the corresponding sub-blocks of each subset have rows sum and columns sum > 0.

4.4 Characterizing the Stable Extensions

Every stable extension $S \subseteq \mathcal{A}$, if it exists, separates the set of arguments $\mathcal{A}$ into two disjoint parts: S and $\mathcal{A} \setminus S$. So, except for the conflict-freeness of S, we only need to concentrate on whether the arguments in $A \setminus S$ are attacked by S. Hence, for each conflict-free sub-block matrix, we need to consider the remaining outer sub-block of the Justification Matrix, containing information about the external arguments. If there exists at least a negative entry in each column of the outer sub-block, then the conflict-free sub-block matrix is a stable sub-block.

Axiom 4 *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF, with $|\mathcal{A}| = n$, and $\boldsymbol{J_F}$ be the $n \times n$ Justification Matrix of F. The $k \times k$ sub-block conflict-free matrix $\boldsymbol{E_F}$, with $k \leq n$ and $l = n - k$ is a* sub-block stable matrix *of F iff its corresponding outer $k \times l$ sub-block matrix contains at least one negative entry in each column.*

Example 5. Given the AF F_1 in Example 1 and the Justification Matrix $\boldsymbol{J_{F_1}}$ in Example 2, we report all the conflict-free sub-blocks of $\boldsymbol{J_{F_1}}$ together with its outer sub-blocks divided by a line, obtaining a $k \times n$ sub-block:

conflict-free subset	$\{\ \}$	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\delta\}$
sub-block	$[\ ]$	$\left[\begin{array}{c\|cccc} 1 & -1 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{c\|cccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{c\|cccc} 1 & 0 & -1 & -1 & 1 \end{array}\right]$	$\left[\begin{array}{c\|cccc} 1 & 0 & 1 & -1 & -1 \end{array}\right]$

conflict-free subset	$\{\alpha, \gamma\}$	$\{\alpha, \delta\}$	$\{\beta, \delta\}$
sub-block	$\left[\begin{array}{cc\|ccc} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{array}\right]$	$\left[\begin{array}{cc\|ccc} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right]$	$\left[\begin{array}{cc\|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \end{array}\right]$

The only stable extension is $\{\{\alpha, \delta\}\}$. As we can see, the corresponding outer sub-blocks of the sub-block for $\{\alpha, \delta\}$ holds at least one negative entry for each column.

5 Characterizing BAFs

In the following, we show that also a BAF can be mapped to a particular matrix representation. The intuition of representing attacks as a negative relation remains of actual interest, and finds its counterpart in supports, which can be represented as a "positive" relation.

Definition 8 *Let $B = \langle \mathcal{A}, \mathcal{R}_{att}, \mathcal{R}_{sup} \rangle$ be a BAF. Let $|\mathcal{A}| = n$, then the* Signed Argumentation Matrix *of B is a $n \times n$ matrix $\boldsymbol{M_B} = [M_{ij}]$ such that for any two arguments $\alpha_i, \alpha_j \in \mathcal{A}$ it holds that*

$$M_{ij} = \begin{cases} -1 & \text{if } \langle \alpha_i, \alpha_j \rangle \in \mathcal{R}_{att} \\ 1 & \text{if } \langle \alpha_i, \alpha_j \rangle \in \mathcal{R}_{sup} \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to note that the Justification Matrix of $\boldsymbol{M_B}$ (with the *power series stop criterion*) is still a valid representation to compute sub-blocks which ensure BAF conflict-freeness, admissibility, and stable semantics requirements. Anyway, it is worth mentioning that there are different interpretation of the support in BAFs that can lead to alternative semantics for which the approach proposed may not work.

6 Characterizing WAFs

Taking into account WAFs, we redefine the Argumentation Matrix to have the positive weights as its entries.

Definition 9 *Let $W = \langle \mathcal{A}, \mathcal{R}, w \rangle$ be a WAF, with $|\mathcal{A}| = n$. Then, the* Weighted Argumentation Matrix *of W is a $n \times n$ matrix $\boldsymbol{M_W} = [M_{ij}]$ such that for any two arguments $\alpha_i, \alpha_j \in \mathcal{A}$ it holds that*

$$M_{ij} = \begin{cases} w(\langle \alpha_i, \alpha_j \rangle) & \text{if } \langle \alpha_i, \alpha_j \rangle \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}$$

In this representation, we choose to set positive weights as entries of the Weighted Argumentation Matrix instead of negative ones. In WAFs weights are not propagated to determine a level of acceptability for arguments, but they are only used for deciding which attacks can be ignored when computing the extensions. This means that we don't need a Justification Matrix of $\boldsymbol{M_W}$. Some inconsistencies are tolerated in subsets of arguments, provided that the sum of the weights of attacks between arguments does not exceed a given inconsistency threshold $\beta \in \mathbb{R}^+$ [7]. Therefore, depending on the inconsistency threshold, we can delete all the column entries in which their column sum exceeds β.

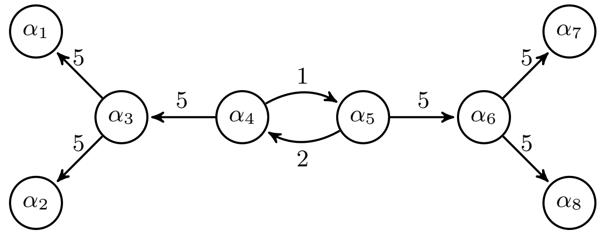

Fig. 2. W_1: WAF example

Definition 10 *Let* $W = \langle \mathcal{A}, \mathcal{R}, w \rangle$ *be a WAF, with* $|\mathcal{A}| = n$, $\boldsymbol{M_W} = [M_{ij}]$ *be the Weighted Argumentation Matrix of* W*, and* $\beta \in \mathbb{R}^+$ *be an inconsistency threshold. Then, the* β-consistent Argumentation Matrix *of* $\boldsymbol{M_W}$ *is a* $n \times n$ *matrix* $\boldsymbol{M_{W,\beta}} = [M_{ij,\beta}]$ *such that for any entry* M_{ij} *of* $\boldsymbol{M_W}$ *it holds that*

$$M_{ij,\beta} = \begin{cases} -1 & \text{if } \sum_{k=1}^{n} M_{kj} < \beta \\ 0 & \text{otherwise} \end{cases}$$

The β-consistent Argumentation Matrix is a matrix representation of the WAF in which all the attacks exceeding the inconsistency threshold are neglected. This resulting matrix is now in the standard form such that its Justification Matrix representation can now be computed. Admissibility is defined in the standard way, and standard semantics are considered leading to various notions of β-extensions which echo Dung's ones. Then, it will be still possible to compute sub-blocks of such matrix according to semantics defined above, that will correspond to β-admissible, β-grounded, β-stable subsets of arguments.

Example 6. Consider the WAF W_1 in Figure 2. Below, $\boldsymbol{M_{W_1}}$ is its matrix representation Let $\beta = 3$, then $\boldsymbol{M_{W_1,\beta}}$ is the corresponding β-consistent Argumentation Matrix.

$$\boldsymbol{M_{W_1}} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 5&5&0&0&0&0&0&0 \\ 0&0&5&0&1&0&0&0 \\ 0&0&0&2&0&5&0&0 \\ 0&0&0&0&0&0&5&5 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} \qquad \boldsymbol{M_{W_1,\beta}} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&-1&0&0&0&0&0&0 \\ 0&0&-1&0&0&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&-1&-1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

7 Characterizing BWAFs

Let's now consider the Argumentation Matrix representation for the BWAF. A BWAF is a further generalization of the Dung's AF, in which new features are introduced to deal with bipolar weighted relations [9]: negative attacks and positive supports, bounded in a specific interval.

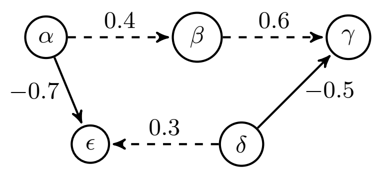

Fig. 3. G_2: Example to illustrate BWAF

Definition 11 *A **BWAF** is a triplet* $G = \langle \mathcal{A}, \hat{\mathcal{R}}, w_{\hat{\mathcal{R}}} \rangle$*, where* $\mathcal{A}$ *is a finite set of arguments,* $\hat{\mathcal{R}} \subseteq \mathcal{A} \times \mathcal{A}$ *and* $w_{\hat{\mathcal{R}}} \colon \hat{\mathcal{R}} \mapsto [-1, 0[\; \cup \;]0, 1]$ *is a function assigning a weight to each relation. Attack relations are defined as* $\hat{\mathcal{R}}_{att} = \{\langle \alpha, \beta \rangle \in \hat{\mathcal{R}} \mid w_{\hat{\mathcal{R}}}(\langle \alpha, \beta \rangle) \in [-1, 0[\; \}$ *and support relations as* $\hat{\mathcal{R}}_{sup} = \{\langle \alpha, \beta \rangle \in \hat{\mathcal{R}} \mid w_{\hat{\mathcal{R}}}(\langle \alpha, \beta \rangle) \in \;]0, 1] \; \}$.

Given two arguments $\alpha, \beta \in \mathcal{A}$ *and a path* $\langle \alpha, x_1, x_2, \ldots, x_n, \beta \rangle$ *from* α *towards* β*, then:*

- α bw-defends β *if the product of weights* $w_{\hat{\mathcal{R}}}(\langle \alpha, x_1 \rangle) \cdot w_{\hat{\mathcal{R}}}(\langle x_1, x_2 \rangle) \cdot \ldots \cdot w_{\hat{\mathcal{R}}}(\langle x_n, \beta \rangle)$ *is positive.*
- α bw-attacks β *if the product of weights* $w_{\hat{\mathcal{R}}}(\langle \alpha, x_1 \rangle) \cdot w_{\hat{\mathcal{R}}}(\langle x_1, x_2 \rangle) \cdot \ldots \cdot w_{\hat{\mathcal{R}}}(\langle x_n, \beta \rangle)$ *is negative.*

If we have positive and negative weights on the edges, then the assumptions made for representing AFs, BAFs and WAFs as adjacency matrices are still valid. Taking into account BWAFs, we redefine the Argumentation Matrix to have the negative and positive weights as its entries.

Definition 12 *Let* $G = \langle \mathcal{A}, \hat{\mathcal{R}}, w_{\hat{\mathcal{R}}} \rangle$ *be a BWAF with weights in the interval* $[-1, 0[\cup]0, 1]$*, and* $|A| = n$*. Then, the* Signed Weighted Argumentation Matrix *of* G *is a* $n \times n$ *matrix* $\boldsymbol{M_G} = [M_{ij}]$ *such that for any two arguments* $\alpha_i, \alpha_j \in \mathcal{A}$ *it holds that*

$$M_{ij} = \begin{cases} w_{\hat{\mathcal{R}}}(\langle \alpha_i, \alpha_j \rangle) & \text{if } \langle \alpha_i, \alpha_j \rangle \in \hat{\mathcal{R}} \\ 0 & \text{otherwise} \end{cases}$$

We define the weight of a walk as the product of the weights of the arcs. Then if we want to know the total sum of weights of i, j paths of given length, that is the entry in the appropriate power.

Example 7. Consider the BWAF G_2 depicted in Figure 3. Below, $\boldsymbol{M_{G_2}}$ is its matrix representation. We can compute its Justification Matrix $\boldsymbol{J_{G_2}}$ with the power series summation of $\boldsymbol{M_{G_2}}$. Below, $\boldsymbol{M^2_{G_2}}$ is the he 2nd power of G_2. Since there is no path of length 3 in G_2, $\boldsymbol{M^3_{G_2}}$ is the zero matrix. Arguments α and δ are initializers, so the diagonal matrix $diag_u(\boldsymbol{M_{G_2}})$ will have positive values of 1 in the corresponding entries. Then, $\boldsymbol{J_{G_2}}$ is its resulting Justification Matrix.

$$\boldsymbol{M_{G_2}} = \begin{bmatrix} 0 & 0.4 & 0 & 0 & -0.7 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{M^2_{G_2}} = \begin{bmatrix} 0 & 0 & 0.24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{J_{G_2}} = \begin{bmatrix} 1 & 0.4 & 0.24 & 0 & -0.7 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 1 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can now exploit this resulting matrix to compute semantics. For instance, the set of arguments with the corresponding column sum of entries ≥ 1 of $\boldsymbol{J_{G_2}}$ is the (*bw*-)grounded extension.

7.1 Laplacian Ranking Semantics for BWAFs

The *spectral graph theory* studies the properties of graphs via the eigenvalues and eigenvectors of their associated graph matrices: the adjacency matrix and the graph Laplacian and its variants. In the following we consider the possible benefits of adopting spectral linear algebra methods as a tool for analyzing argumentation structures, as [3] first started to study, and present a new ranking-based semantics, called *Laplacian ranking semantics*. With the justification matrix in place, we can provide the main definition:

Definition 13 *Let $G = \langle \mathcal{A}, \hat{\mathcal{R}}, w_{\hat{\mathcal{R}}} \rangle$ be a BWAF, with $|\mathcal{A}| = n$, and let $\boldsymbol{J^{\star}_{G}}$ be the Justification Matrix of G without positive values for initializers in the main diagonal. The* degree matrix *of G is the matrix $\boldsymbol{D_G} = diag(\deg(\alpha_1), \ldots, \deg(a_n))$, where $\alpha_1, \ldots, \alpha_n \in A$, and $\forall j = 1, \ldots, n$:*

$$\deg(\alpha_j) = \sum_{i=1}^{n} J^{\star}_{Gij}$$

Intuitively, the degree matrix of a BWAF is a diagonal matrix which contains information about the sum of weights of the edges connected to a node. The name 'Laplacian Ranking Semantics' derives from the fact that, in graph theory, the Laplacian matrix $\boldsymbol{L_G}$ of a graph G is given by the difference $\boldsymbol{D_G} - \boldsymbol{J^{\star}_{G}}$.

In yet other words, the degree matrix $\boldsymbol{D_G}$ collects in the main diagonal the column-wise sum of its entries. Hence, we argue, it captures natural information to compare the relative 'strength' of arguments in a BWAF, since its Justification Matrix collects all the attacks and defenses for each node in the graph.

So we assign an 'acceptability' degree to each argument in a BWAF G, which equals its degree in $\boldsymbol{D_G}$. It follows that the degree of an argument always lies in the interval $[-1, 1]$, so that the ranking of 0 will now *tip the scales*, meaning that rejected arguments will have a negative ranking, while accepted ones will have a positive ranking. Naturally, such degrees induce a total preorder, for each BWAF G and $\alpha, \beta \in \mathcal{A}$:

$$\alpha \succeq^{\deg}_{G} \beta \text{ iff } \deg(\alpha) \geq \deg(\beta).$$

It is worth to clarify why we do not consider the weight of initializer arguments in this setting. Generally, arguments that not receive any attack or support play a key role in the (classical) acceptability. Dealing with Laplacian matrix, we want to ensure that the algebraic properties about its eigenvalues, eigenvectors and connectivity are always satisfied. Furthermore, it will be of particular interest to verify which ranking semantics properties such new semantics will hold, considering also the particular case of weighted attacks and weighted supports.

In [1] several rationality postulates have been proposed which should be satisfied by any argumentation semantics based on ranking. We are now in a position to check which of the rationality postulates are satisfied, assuming that those postulates, suitably designed for AFs, are extended in the BWAF case, by considering simple attacks as bw-attacks (which include weighted notions of supported attack and indirect attack), and defenses as bw-defenses (which include a weighted notion of support and defense as a whole).

Axiom 5 *The Laplacian ranking semantics satisfies Independence, Void Precedence, Self Contradiction, Cardinality Precedence, Quality Precedence, Defense Precedence, Distributed-Defense Precedence, Strict Addition of Defense Branch, Addition of Defense Branch, Addition of Attack Branch, Total, Non-attacked Equivalence, and Attack vs Full Defense. Other properties are not satisfied.*

The proof of the above theorem is omitted due to space limitations but straightforward. For a detailed discussion of these postulates see [1]. Briefly, we state that Abstraction, Counter-transitivity, Strict Counter-transitivity, Increase of Attack Branch, and Increase of Defense Branch are not satisfied by our semantics: *Abstraction* is not satisfied because it relies only on attacks between argument, while in our semantics we have to deal with supports too; *Counter-transitivity*, and its strict form, are not satisfied because it does not matter how many bw-attacks a node can receive, Laplacian-ranking semantics accounts for the strength of bw-attackers, not for the quantity, and it may happen that an argument α, receiving more bw-attackers than the ones received by β, would be at least acceptable as β ; *Increase of Attack Branch* is not satisfied because increasing the length of a bw-attacker of an argument does not improve its ranking. The same observation holds for *Increase of Defense Branch* property.

Example 8. Let us consider the BWAF G_2 depicted in Figure 3. The Laplacian matrix of G_2 is $\boldsymbol{L_{G_2}}$. In particular, the degree matrix of G_2 is $\boldsymbol{D_{G_2}}$.

$$\boldsymbol{L_{G_2}} = \begin{bmatrix} 0 & -0.4 & -0.24 & 0 & 0.7 \\ 0 & 0.4 & -0.6 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & -0.3 \\ 0 & 0 & 0 & 0 & -0.4 \end{bmatrix} \qquad \boldsymbol{D_{G_2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4 \end{bmatrix}$$

where $\deg(\alpha) = 0, \deg(\beta) = 0.4, \deg(\gamma) = 0.34, \deg(\delta) = 0, \deg(\epsilon) = -0.4$. Therefore, the Laplacian Ranking Semantics of G_2 is:

$$\beta \succ^{\deg}_{G_2} \gamma \succ^{\deg}_{G_2} \alpha \succeq^{\deg}_{G_2} \delta \succ^{\deg}_{G_2} \epsilon.$$

8 Conclusion

An alternative representation of computational models of argument, based on the Matrix Theory, has been proposed. This line of research have demonstrated that it is possible to represent and evaluate arguments by devising a general strategy which led to the definition of the Justification Matrix. In this way, the computation of semantics relies no more on rules, but only on matrices products. The potential of the algebraic approach has been demonstrated to devise a new ranking-based semantics, i.e. the Laplacian semantics, which *naturally* satisfies a lot of well-known postulates of ranking semantics literature.

The matrix-approach may have some potential in efficiently processing abstract AFs since finding extensions can be a computationally hard procedure especially when we are dealing with several arguments and (complex) relations. Therefore, further developments on this field will be devoted to the characterization of other extension-based semantics and on the new perspectives that Linear Algebra opens on ranking semantics, with a deeper understanding of the properties that the Laplacian matrix representation of an Argument Graph holds.

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