Reliability of Adaptive Traffic Lights Ensured by Warm Standby With Estimation of its Use

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Abstract. For the case when quantitative characteristics of traffic flows and the failure rates of traffic lights' elements are known, quantitative relationships for the feasibility analysis of warm standby of the adaptive traffic lights at intersections have been found. The equations and recurrence relations of their probability distributions are obtained as the Laplace transform of sequences of times of limited permissive phases of alternative movement directions. The intelligent system of traffic control with the adaptive traffic lights as the main control unit is considered as a single-line queuing system with the FIFO (first in, first out) service discipline for the Markov flow of recoverable components' failures.

As a result, two-way estimate for the failure probability of the adaptive traffic lights is obtained and the effect of warm standby use for the traffic lights at two-way stop-controlled intersection is assessed. In the presented example waiting time for the vehicles at the intersection with adaptive controlled traffic appeared to be considerably less than in the case when there is no adaptive traffic control. It allows to highlight advantages of warm standby use as a mean to ensure the reliability of the adaptive traffic lights.

Keywords: Warm Standby, Reliability, Markov Flow, Adaptive Traffic Lights, Laplace Transform.

1 Introduction

Adaptive traffic lights are the most common and most powerful components of control in intelligent traffic management systems (ITMS). It allows actuating of time phases depending on actual traffic demand and, compared to pre-timed control, it significantly reduces traffic delays at intersections when properly configured. Though the advantages of ITMSs are obvious, ensuring the reliability of their work is not explored enough due to the absence of critical consequences when traffic lights fail, since this situation is foreseen by the traffic codes and is not a direct cause of traffic accidents. The failure of the traffic lights leads to a deterioration in the passage conditions at the intersection and an increase in vehicle delays. Pedestrians do not suffer from the failure as they receive a preferential right to cross the street via unregulated crosswalks.

Ensuring the reliability of light signaling systems is of considerable attention in railway transport, because railway signaling systems are used to ensure the safe operation of railway traffic [1, 2]. As a result, a lot of effort has gone towards ensuring the reliable operation of railway signaling systems that have to be designed to avoid single-point failures [1].

A new impetus to ensure reliability of traffic lights is added by LED (Light-Emitting Diode) arrays as a color signal emitter. LED emitters have allowed to minimize power requirements for the functioning of traffic lights, while expanding the list of standbys for their functioning with the ideal switching arrangement (warm standby), which makes it possible to increase the reliability of adaptive traffic lights in urban ITMSs.

A redundant warm standby is a very promising means of ensuring the reliability of ITMS comparing to a cold standby since traffic lights are always dispersed over fairly large territories. This leads to randomness of the recovery time of failed items, which leads to high expenditures due to the need of the repair crew to get to the failed traffic lights. Police and signal maintenance crews can often be stretched too thin when responding to power failure situations, especially in cases of concurrent failures at a large amount of intersections [3]. Throughout this time, the ITMS does not work, that causes undue delays of vehicles. At the same time, the warm reservation of traffic lights' elements allows to postpone the process of restoring its failed items until a off-peak traffic period, when it does not lead to negative consequences for road users.

By now two warm standbys for traffic lights are known: redundant LED groups to allow indication in degraded mode [4, 5] and alternative power sources as backup power to maintain normal signal operations during power outages [3]. The list can be expanded if a metric to quantify results of warm standby is defined since a cost of new units is always known and both variants of reservation are expensive enough. Transportation agencies, facing limited budgets, need decision-making support when solving the question whether the traffic operation can be improved by installing warm standbys for traffic lights – and if so, by how much.

Statistical models of road accidents [3] and a point system to score each site [6] are not a sufficient basis for making decisions in this area. The number of reported road accidents is relatively low and when traffic lights are fault there is no way to statistically quantify the expected number of accidents [6]. The point system includes traffic volume, frequency of injury accidents, proximity to a school zone, speed of approach traffic, and availability of pedestrian pre-emption controls and provides prioritization sites only [6]. The Markov technique is the most suitable for modelling road signaling systems in which the level of redundancy varies with time due to component failure and repair [1].

2 Laplace transform of the distribution function of the intersection busy period

It is possible to evaluate the feasibility of warm standby use to ensure the reliability of adaptive traffic lights through obtaining quantitative estimates for both the probability of failure of adaptive traffic lights with a different number of standbys for one operating item, and for the negative consequences of failure from the road user's point of view.

The first quantitative estimates should be obtained for the most common intersection of two nearly perpendicular roads. When traffic is controlled by traffic lights, the times of permissive phases in alternative intersecting directions are limited. Adaptive traffic control at the intersection allows switching of the permissive signal between alternative directions depending on current circumstances at the intersection. That is, when, with the permissive signal turned on in one direction I, there are no vehicles for moving in it, and the queue of vehicles is on the other direction II. Vehicles can pass through the intersection as long as they are at a given spacing or if there are no vehicles coming from the other direction, but the permissive phase should not exceed the time limit for the corresponding direction.

We have:

– $B_I^{(\nu(I))}(t)$ is the distribution function for the time of passing over the intersection from the start of the permissive signal in direction I; $\beta_I^{(\nu(I))}(s)=\int\limits_{t>0}\exp(-st)dB_I^{(\nu(I))}(t)$ is the Laplace transform of this variable;
– $\Pi_I^{(\nu(I))}(t)$ is the distribution function for the intersection's busy period as a single-line queuing system, which begins with the $\nu(I)$ vehicles' departure and the flow rate Λ_I of the vehicles arriving in the direction I of the intersection;
– $\pi_I^{(\nu(I))}(s)=\int\limits_{t>0}\exp(-st)d\Pi_I^{(\nu(I))}(t)$ is the Laplace transform of this variable.

We use the additional event method according to the total probability rule. Provided that passing-over $\nu(I)$ vehicles from direction I take the time t, and during this time t exactly n cars have arrived at the intersection in direction II, the conditional probability that the additional event, the rate of which equals s, does not occur during this busy period t and during the n busy periods, generated by the arrival of these n cars, is equal to $\left[\pi_I^{(1)}(s)\right]^n\exp(-st)$. Then we multiply it by the probability of the arrival of exactly n cars in the direction I during this busy period t $\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)$.

The sum of all these multiplications $\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)\left[\pi_I^{(1)}(s)\right]^n\exp(-st)$ for all non-negative integer numbers of incoming cars $n\geq 0$ is equal to

$$\sum_{n\geq 0}\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)\left[\pi_I^{(1)}(s)\right]^n\exp(-st) \tag{1}$$

Then we integrate the obtained expression over all $t>0$ values with the distribution $dB_I^{(\nu(I))}(t)$ for the time of passing over $\nu(I)$ vehicles from the moment of the enabling signal in the direction I :

$$\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)\left[\pi_I^{(1)}(s)\right]^n\exp(-st)dB_I^{(\nu(I))}(t). \tag{2}$$

As a result, we obtain the equation

$$\pi_I^{(\nu(I))}(s)=\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)\left[\pi_I^{(1)}(s)\right]^n\exp(-st)dB_I^{(\nu(I))}(t). \tag{3}$$

After identical transformations we come to the following:

$$\begin{aligned}
&\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_I t)^n}{n!}\exp(-\Lambda_I t)\left[\pi_I^{(1)}(s)\right]^n\exp(-st)dB_I^{(\nu(I))}(t)=\\
&=\int_{t>0}\sum_{n\geq 0}\frac{\left[\Lambda_I t\pi_I^{(1)}(s)\right]^n}{n!}\exp(-\Lambda_I t)\exp(-st)dB_I^{(\nu(I))}(t)=\\
&=\int_{t>0}\exp(-\Lambda_I t)\exp(-st)\sum_{n\geq 0}\frac{\left[\Lambda_I t\pi_I^{(1)}(s)\right]^n}{n!}dB_I^{(\nu(I))}(t)=\\
&=\int_{t>0}\exp(-\Lambda_I t)\exp(-st)\exp\left[\Lambda_I t\pi_I^{(1)}(s)\right]dB_I^{(\nu(I))}(t)=\\
&=\int_{t>0}\exp\left\{-\left[s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right]\right\}dB_I^{(\nu(I))}(t).
\end{aligned} \tag{4}$$

By definition, the integral $\int_{t>0}\exp\left\{-\left[s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right]\right\}dB_I^{(\nu(I))}(t)$ is equal to the Laplace transform $\beta_I^{(\nu(I))}\left(s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right)$ at the point $s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)$:

$$\int_{t>0}\exp\left\{-\left[s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right]\right\}dB_I^{(\nu(I))}(t)=\beta_I^{(\nu(I))}\left(s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right). \tag{5}$$

Then we have a functional equation for the Laplace transform as a result:

$$\pi_I^{(\nu(I))}(s)=\beta_I^{(\nu(I))}\left(s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right). \tag{6}$$

In particular, with $\nu(I)=1$,

$$\pi_I^{(1)}(s)=\beta_I^{(1)}\left(s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right). \tag{7}$$

By definition, it is considered that the time of the permissive phase in the direction I is limited by the value T_I. Therefore, a random variable with a distribution $\Pi_I^{(\nu(I))}(t)$ of the intersection busy period limited by the value T_I to the segment $[0;T_I]$, which began with vehicles passing over the intersection in the direction I with the flow rate Λ_I, will have distribution function as follows:

$$\Pi_I^{(\nu(I))}[T_I](t) = \frac{\Pi_I^{(\nu(I))}(t)}{\Pi_I^{(\nu(I))}(T_I)} \cdot I(0 \le t \le T_I) + I(t > T_I) \tag{8}$$

and the Laplace transform

$$\pi_I^{(\nu(I))}[T_I](s) = \int_0^{T_I} \exp(-st) d\Pi_I^{(\nu(I))}[T_I](t) = \frac{\int_0^{T_I} \exp(-st) d\Pi_I^{(\nu(I))}(t)}{\Pi_I^{(\nu(I))}(T_I)} . \tag{9}$$

During the intersection busy period limited by the value T_I, which began with $\nu(I)$ vehicles passing over the intersection in the direction I with the flow rate Λ_I, that is, during this permissive phase for vehicle direction I, some vehicles may arrive from the perpendicular direction II.

Next, we find the Laplace transform of the intersection busy period, which is formed by movement in direction II generated at the intersection by the vehicles arriving in this direction with flow rate Λ_{II} during the previous permissive phase for direction I, using the total probability rule according to the method of the additional event, the rate of which equals s.

Provided that passing-over $\nu(I)$ vehicles from direction I take the time t, and during this time t exactly n cars have arrived at the intersection in direction II, the conditional probability that the additional event, the rate of which equals s, does not occur during this busy period t and during the n busy periods, generated by the arrival of these n cars, is equal to $\left[\pi_I^{(1)}(s)\right]^n \exp(-st)$. Then we multiply it by the probability of arrival of exactly n cars in direction II during this busy period t $\frac{(\Lambda_{II} t)^n}{n!} \exp(-\Lambda_{II} t)$.

The sum of all these multiplications $\frac{(\Lambda_{II} t)^n}{n!} \exp(-\Lambda_{II} t)\left[\pi_{II}^{(1)}(s)\right]^n \exp(-st)$ for all non-negative integer numbers of incoming cars $n \ge 0$ is equal to

$$\sum_{n \ge 0} \frac{(\Lambda_{II} t)^n}{n!} \exp(-\Lambda_{II} t)\left[\pi_{II}^{(1)}(s)\right]^n \exp(-st) . \tag{10}$$

Then we integrate the obtained expression over all $t > 0$ with the distribution $d\Pi_I^{(\nu(I))}[T_I](t)$ for the time of passing over $\nu(I)$ vehicles from the moment of the enabling signal in direction I .

After that we integrate the obtained expression over all $d\Pi_I^{(\nu(I))}[T_I](t)$ of the busy period, which began with the $\nu(I)$ vehicles' departure and the flow rate Λ_I of arriving vehicles from the direction *I* of the intersection and limited by the value T_I :

$$\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_{II}t)^n}{n!}\exp(-\Lambda_{II}t)\left[\pi_{II}^{(1)}(s)\right]^n\exp(-st)d\Pi_I^{(\nu(I))}[T_I](t)\,. \tag{11}$$

As a result, we obtain the equation

$$\pi_{II}^{(\nu(I))}(s)=\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_{II}t)^n}{n!}\exp(-\Lambda_{II}t)\left[\pi_{II}^{(1)}(s)\right]^n\exp(-st)d\Pi_I^{(\nu(I))}[T_I](t) \tag{12}$$

for the Laplace transform $\pi_{II}^{(\nu(I))}(s)=\int_{t>0}\exp(-st)d\Pi_{II}^{(\nu(I))}(t)$ of the distribution of the intersection busy period in direction II , generated by vehicles arriving at the intersection in this direction with the flow rate Λ_{II} during the previous permissive phase for direction I .

After similar identity transformations, we have

$$\begin{gathered}\int_{t>0}\sum_{n\geq 0}\frac{(\Lambda_{II}t)^n}{n!}\exp(-\Lambda_{II}t)\left[\pi_{II}^{(1)}(s)\right]^n\exp(-st)d\Pi_I^{(\nu(I))}[T_I](t)=\\=\pi_I^{(\nu(I))}[T_I]\left(s+\Lambda_{II}\left(1-\pi_{II}^{(1)}(s)\right)\right).\end{gathered} \tag{13}$$

We obtain an expression of the Laplace transform:

$$\pi_{II}^{(\nu(I))}(s)=\pi_I^{(\nu(I))}[T_I]\left(s+\Lambda_{II}\left(1-\pi_{II}^{(1)}(s)\right)\right). \tag{14}$$

The equation for the function $\pi_{II}^{(1)}(s)$ will be written below when studying the distributions of the sequence of permissive times that begins with direction II .

By the definition, it is considered that the time of the permissive phase in direction II is limited by the value T_{II}. Therefore, a random variable with distribution $\Pi_{II}^{(\nu(I))}(t)$ of the intersection busy period is limited by the value T_{II} to the segment $[0;T_{II}]$, which begins with vehicles passing over the intersection in direction II with the flow rate Λ_{II} have following distribution function:

$$\Pi_{II}^{(\nu(I))}[T_{II}](t)=\left[\Pi_{II}^{(\nu(I))}(t)\big/\Pi_{II}^{(\nu(I))}(T_{II})\right]\cdot I(0\leq t\leq T_{II})+I(t>T_{II}) \tag{15}$$

and the Laplace transform equals

$$\pi_{II}^{(\nu(I))}\left[T_{II}\right](s)=\int_{0}^{T_{II}}\exp(-st)d\Pi_{II}^{(\nu(I))}\left[T_{II}\right](t)=\frac{\int_{0}^{T_{II}}\exp(-st)d\Pi_{II}^{(\nu(I))}(t)}{\Pi_{II}^{(\nu(I))}(T_{II})}. \qquad (16)$$

Further we use the same set of formulas to obtain other related values.

If the process of passing over the intersection begins with cars from direction II, then the entire sequence of the distributions of limited periods of permissive phases is constructed similarly, but this time starting from direction II.

We have:

- $B_{II}^{(\nu(II))}(t)$ is the distribution function for the time of passing over the intersection from the start of the permissive signal in direction II; $\beta_{II}^{(\nu(II))}(s)=\int_{t>0}\exp(-st)dB_{I}^{(\nu(II))}(t)$ is Laplace transform for this random variable;
- $\Pi_{II}^{(\nu(II))}(t)$ is the distribution function for the intersection busy period as a single-line queuing system, which began with the $\nu(II)$ vehicles' departure and the flow rate Λ_{II} of arriving vehicles in direction II of the intersection;
- $\pi_{II}^{(\nu(II))}(s)=\int_{t>0}\exp(-st)d\Pi_{II}^{(\nu(II))}(t)$ is Laplace transform for this random variable.

Using the method of the additional event according to the formula of full probability, we have a functional equation for the Laplace transform:

$$\pi_{II}^{(\nu(II))}(s)=\beta_{II}^{(\nu(II))}\left(s+\Lambda_{II}\left(1-\pi_{II}^{(1)}(s)\right)\right). \qquad (17)$$

In particular, with $\nu(II)=1$

$$\pi_{II}^{(1)}(s)=\beta_{II}^{(1)}\left(s+\Lambda_{II}\left(1-\pi_{II}^{(1)}(s)\right)\right). \qquad (18)$$

By the definition, it is considered that the time of the permissive phase in direction II is limited by the value T_{II}. Therefore, a random variable with distribution $\Pi_{II}^{(\nu(I))}(t)$ of the intersection busy period is limited by the value T_{II} to the segment $\left[0;T_{II}\right]$, which began with vehicles passing over the intersection in direction II with the flow rate

Λ_{II}, have following distribution function:

$$\Pi_{I}^{(\nu(II))}\left[T_{I}\right](t)=\frac{\Pi_{I}^{(\nu(II))}(t)}{\Pi_{I}^{(\nu(II))}(T_{I})}\cdot I\left(0\le t\le T_{I}\right)+I\left(t>T_{I}\right) \qquad (19)$$

and the Laplace transform

$$\pi_I^{(\nu(II))}\left[T_I\right](s)=\int_0^{T_I}\exp(-st)d\Pi_I^{(\nu(II))}\left[T_I\right](t)=\frac{\int_0^{T_I}\exp(-st)d\Pi_I^{(\nu(II))}(t)}{\Pi_I^{(\nu(II))}(T_I)}\,. \tag{20}$$

The obtained equations and recurrent expressions of Laplace transform of the sequences of durations of the permissive phases in alternative traffic directions ultimately allow paying attention to the direction of increased intensity of movement and speed of passing over the intersection, and thereby reconcile the restrictions on the times of the permission signals with the loads

$$\rho_I=\Lambda_I\int_{t>0}tdB_I^{(\nu(I))}(t)=\Lambda_I m^I \tag{21}$$

and

$$\rho_{II}=\Lambda_{II}\int_{t>0}tdB_{II}^{(\nu(II))}(t)=\Lambda_{II} m^{II} \tag{22}$$

of the corresponding directions I and II using equations (7) and (18). Differentiation of equation (7) with the opposite sign with respect to s at zero gives

$$-\frac{d}{ds}\pi_I^{(1)}(s)\Big|_{s=0}=-\frac{d}{ds}\beta_I^{(1)}\left(s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)\right)\Big|_{s=0} \tag{23}$$

using the derivative of a complex function for $u=s+\Lambda_I\left(1-\pi_I^{(1)}(s)\right)$ and expression of first moments

$$z^I=-\frac{d}{ds}\pi_I^{(1)}(s)\Big|_{s=0} \tag{24}$$

and

$$m^I=-\frac{d}{du}\beta_I^{(1)}(u)\Big|_{u=0} \tag{25}$$

we have

$$z^I=-\frac{d}{du}\beta_I^{(1)}(u)\Big|_{u=0}\cdot\left[1+\Lambda_I\left(-\frac{d}{ds}\pi_I^{(1)}(s)\Big|_{s=0}\right)\right], \tag{26}$$

or

$$z^I=m^I\left(1+\Lambda_I z^I\right). \tag{27}$$

From this equation the expectation z^I of the period of vehicle service in direction I in the absence of restrictions $(T_I = \infty)$ equals

$$z^I = \frac{m^I}{1-\rho_I}. \tag{28}$$

Similarly,

$$z^{II} = \frac{m^{II}}{1-\rho_{II}}. \tag{29}$$

The obtained expressions for the mathematical expectations of vehicles' passing times in the absence of restrictions can be used to obtain relations for constraints T_I and T_{II}. Namely,

$$\frac{T_I}{T_{II}} = \frac{z^I}{z^{II}}, \tag{30}$$

or

$$\frac{m^I}{(1-\rho_I)T_I} = \frac{m^{II}}{(1-\rho_{II})T_{II}}. \tag{31}$$

The obtained Laplace transforms for travel time (7) and (18), in the absence of restrictions on the duration of the permissive phases, make it possible to determine the ratio of the values of these restrictions from a practical point of view. The absence of such restrictions, with a high total intensity of traffic flows on competing directions, will lead to undue delays for a secondary direction. Failure of the adaptive traffic lights means the removal of restrictions T_I and T_{II}.

In this regard, the study of the reliability of the adaptive traffic lights is of great interest when they are considered as recoverable systems with redundancy, which is provided by warm standby for their components.

3 An estimate of the probability of system failure during the regeneration period

Some elements of adaptive traffic lights may fail over time. The restoration of system elements is provided by repair facility (RF), which is a single-line queue with a FIFO service discipline. The repair times for failed elements are independent and identically distributed with distribution function $G(x)$. The flow of system's element failures complies with Markov chains.

If all the elements are in order, that is, the random process of servicing the failed elements is in the state $\{0\}$, then the failure rate of at least one element in the system is $\lambda(0)$. If the system contains failed elements, then the failure rate of an element in the system is λ. After recovery, the element returns to where it came from. A random regeneration process of maintenance at a time t is defined by the number of serviced elements in the RF. The moments of regeneration are the times of transition of a random process to the state $\{0\}$ when there are no requirements in RF. At the moment of transition of this random process from the state $\{n\}$ to the state $\{n+1\}$, a failure occurs ($n=1,2,...$). Let the probability of failure on the regeneration period of this random maintenance process be denoted by q. Let

$$\overline{G}(x)=1-G(x) \tag{32}$$

and

$$b_{n-1}=\int_0^\infty \lambda \frac{(\lambda x)^{n-1}}{(n-1)!}\exp(-\lambda x)\overline{G}(x)dx . \tag{33}$$

Lemma 1. Let for numbers $a_{ij}\ge 0$, $b_i>0$, $x_j>0$, $i=1,2,...,n$, $j=1,2,...,n$ it is known that $x_i \le b_i+\sum_{j=1}^n a_{ij}x_j$ for all $i=1,2,...,n$. Then for all $i=1,2,...,n$ the inequality $x_i \le \frac{b_i}{1-\alpha}$, where $\alpha=\max_{1\le i\le n}\sum_{j=1}^n \frac{a_{ij}b_j}{b_i}$, is fair.

Proof. From the lemma's condition for all $j=1,2,...,n$ we have $0<x_j\le b_j$. From here, for all $j=1,2,...,n$ it can be established that $x_j=\frac{b_j}{1-\alpha_j}$, where $\alpha_j=1-\frac{b_j}{x_j}$. If we let $\alpha_k=\max_{1\le j\le n}\alpha_j$, the chain of relations from the condition and the last equality is fair:

$$\begin{gathered} b_k\left(1+\frac{\alpha_k}{1-\alpha_k}\right)=\frac{b_k}{1-\alpha_k}=x_k\le b_k+\sum_{j=1}^n a_{kj}x_j=b_k\left(1+\frac{\sum_{j=1}^n a_{kj}x_j}{b_k}\right)= \\ =b_k\left(1+\frac{\sum_{j=1}^n a_{kj}\frac{b_j}{(1-\alpha_j)}}{b_k}\right)\le b_k\left(1+\frac{\sum_{j=1}^n a_{kj}b_j}{b_k(1-\alpha_k)}\right). \end{gathered} \tag{34}$$

Comparing the left and right sides of these relations, we see that

$$b_k\left(1+\frac{\alpha_k}{1-\alpha_k}\right)\le b_k\left(1+\frac{\sum_{j=1}^{n}a_{kj}b_j}{b_k\left(1-\alpha_k\right)}\right),\qquad(35)$$

hence the inequalities

$$\alpha_j\le\alpha_k\le\sum_{j=1}^{n}\frac{a_{kj}b_j}{b_k}\le\alpha=\max_{1\le i\le n}\sum_{j=1}^{n}\frac{a_{ij}b_j}{b_i}.\qquad(36)$$

are fair.

Therefore, for all $j=1,2,...,n$ it is true that

$$x_j=\frac{b_j}{1-\alpha_j}\le\frac{b_j}{1-\alpha}.\qquad(37)$$

Lemma 2. For any non-negative integers i and j the inequality $b_ib_j\le C_{i+j}^{i}b_0b_{i+j}$ is fair.

Proof. We denote

$$f\left(x\right)=\frac{\lambda\exp\left(-\lambda x\right)\overline{G}\left(x\right)}{\int_0^\infty\lambda\exp\left(-\lambda x\right)\overline{G}\left(x\right)dx}\qquad(38)$$

and

$$M_i=\int_0^\infty x^i f\left(x\right)dx.\qquad(39)$$

It needs to be noted that $M_i=\frac{i!b_i}{\lambda^i b_0}$.

Hence, a chain of relationships follow from the inequality for the moments $M_iM_j\le M_{i+j}$ [7]:

$$\begin{aligned}b_ib_j&=\frac{b_0^2}{i!\,j!}\frac{i!b_i}{\lambda^i b_0}\frac{j!b_j}{\lambda^j b_0}\lambda^{i+j}=\lambda^{i+j}\frac{b_0^2}{i!\,j!}M_iM_j\le\lambda^{i+j}\frac{b_0^2}{i!\,j!}M_{i+j}=\\&=\lambda^{i+j}\frac{b_0^2}{i!\,j!}\frac{\left(i+j\right)!b_{i+j}}{\lambda^{i+j}b_0}=C_{i+j}^{i}b_0b_{i+j},\end{aligned}\qquad(40)$$

i.e. $b_ib_j\le C_{i+j}^{i}b_0b_{i+j}$.

We denote the conditional probability of failure during the regeneration period as $q_r(n+1)$, provided that at its beginning in RF there are exactly r complete requirements for the element repair, $r=1,2,...,n-1$.

Theorem 1. For all natural numbers n, the inequality $q=q_1(n+1)\le \dfrac{b_{n-1}}{1-b_0\left(2^{n-1}-1\right)}$ is true.

Proof. Let j denote the number of failed elements during the recovery of the first failed element in the RF busy period. Using the total probability rule, we have

$$q_1(n+1)=b_{n-1}+\sum_{j=1}^{n-1}a_j q_j(n+1). \tag{41}$$

According to the total probability rule we create the expression for the probability of failure $q_r(n+1)$, when at the beginning of the busy period in RF there are exactly r (at least two) full requirements:

$$q_r(n+1)=b_{n-r}+\sum_{j=0}^{n-r}a_j q_{r-1+j}(n+1),\quad 2\le r\le n. \tag{42}$$

Note that $a_0=1-b_0$ and $a_j=b_{j-1}-b_j$, $j\ge 1$.

These equalities and the Abel transform make it possible to write out and estimate, from the above formula, the second terms on the right-hand sides of the last two series of expressions, respectively, in the form

$$\begin{aligned}\sum_{j=1}^{n-1}a_j q_j(n+1)&=\sum_{j=1}^{n-1}\left[b_{j-1}-b_j\right]q_j(n+1)=\\ &=\sum_{j=1}^{n-1}b_{j-1}\left[q_j(n+1)-q_{j-1}(n+1)\right]-b_{n-1}q_{n-1}(n+1)\le\\ &\le\sum_{j=1}^{n-1}b_{j-1}\left[q_j(n+1)-q_{j-1}(n+1)\right]\end{aligned} \tag{43}$$

Here by definition we consider $q_0(n+1)=0$,

$$\begin{aligned}\sum_{j=0}^{n-r}a_j q_{r-1+j}(n+1)&=\sum_{j=1}^{n-r}\left[b_{j-1}-b_j\right]q_{r-1+j}(n+1)+\left[1-b_0\right]q_{r-1}(n+1)=\\ &=\sum_{j=1}^{n-r}b_{j-1}\left[q_{r-1+j}(n+1)-q_{r-2+j}(n+1)\right]-b_{n-r}q_{n-1}(n+1)+q_{r-1}(n+1)\le\\ &\le\sum_{j=1}^{n-r}b_{j-1}\left[q_{r-1+j}(n+1)-q_{r-2+j}(n+1)\right]+q_{r-1}(n+1)\end{aligned} \tag{44}$$

for $2\le r\le n$.

We denote $\gamma_{n-j+1}(n+1)=q_j(n+1)-q_{j-1}(n+1)$, $2\le j\le n-1$.

By definition $\gamma_n(n+1)=q_1(n+1)$. Substituting the obtained upper estimates instead of the secondary terms in the right-hand sides of the above expressions and transferring the last terms (for r equal to at least two) from right to left, we transform expressions of the probability of failures having the form of equalities into inequalities for them and their mathematical differences, respectively

$$\gamma_n(n+1)\le b_{n-1}+\sum_{j=1}^{n-1}b_{j-1}\gamma_{n-(j-1)}(n+1) \text{ for } r=1 \tag{45}$$

and

$$\gamma_{n+1-r}(n+1)\le b_{n-r}+\sum_{j=1}^{n-r}b_{j-1}\gamma_{n+1-r-(j-1)}(n+1) \text{ for } 2\le r\le n-1. \tag{46}$$

For this system of inequalities under the conditions of Lemma 1, a square matrix of $A=\{a_{ij}\}$ of $(n-1)$-th order is

$$A=\begin{pmatrix} b_{n-2}b_{n-3}b_{n-4}...b_1b_0 \\ b_{n-3}b_{n-4}b_{n-5}...b_0\,0 \\ .... \\ b_2b_1b_0...........00 \\ b_1b_0\,0.............00 \\ b_0\,00..............00 \end{pmatrix}. \tag{47}$$

For the above system of inequalities, we can use Lemma 1 with

$$x_i=\gamma_{n+1-i}(n+1),\quad i=1,2,...,n-1. \tag{48}$$

Than from this system of inequalities according to Lemma 1

$$\gamma_n(n+1)=q_1(n+1)\le\frac{b_{n-1}}{1-\alpha}, \tag{49}$$

where $\alpha=\max\limits_{1\le i\le n-1}\sum_{j=1}^{n-i}\frac{b_{j-1}b_{n-j}}{b_{n-1}}$.

According to Lemma 2 the inequality $1\le j\le n$ is true for all integers

$$b_{j-1}b_{n-j}\le C_{n-1}^{j-1}b_0b_{n-1} \tag{50}$$

and

$$\alpha = \max_{1\le i\le n-1} \sum_{j=1}^{n-i} \frac{b_{j-1} b_{n-j}}{b_{n-1}} \le \max_{1\le i\le n-1} \sum_{j=1}^{n-i} \frac{C_{n-1}^{j-1} b_0 b_{n-1}}{b_{n-1}} = \\ = b_0 \max_{1\le i\le n-1} \sum_{j=1}^{n-i} C_{n-1}^{j-1} = b_0 \sum_{j=1}^{n-1} C_{n-1}^{j-1} = b_0 \left(2^{n-1} - 1\right) \qquad (51)$$

Having obtained the upper bound for the value α ,

$$\alpha \le b_0 \left(2^{n-1} - 1\right). \qquad (52)$$

we have also estimated the probability of system failure during the regeneration period of a random process in the redundancy model with recovery

$$q = q_1 \left(n+1\right) \le \frac{b_{n-1}}{1 - b_0 \left(2^{n-1} - 1\right)} .$$

Let $m_k = \int\limits_{t>0} t^k dG\left(t\right) - k$ is the moment of service time and $\rho = \lambda m_1 < 1$. At the initial moment of time $t = 0$ the system is in the state $\{0\}$ (all elements are in order). We denote by τ the time of the first failure of the ITMS from the moment when all of its elements are in order. Theorem 1 of this paper implies the following theorem.

Theorem 2. Let there be a finite moment $m_2 < \infty$. Then in the process $\frac{\lambda m_2}{m_1 \left(1-\rho\right)} q \to 0$ probability

$$P\left\{ \frac{\lambda\left(0\right)\left(1-\rho\right)}{1-\rho+\lambda\left(0\right) m_1} q\tau > x \right\} \to \exp\left(-x\right). \qquad (53)$$

where the two-way estimate q is true for the failure probability

$$b_{n-1} \le q = q_1 \left(n+1\right) \le \frac{b_{n-1}}{1 - b_0 \left(2^{n-1} - 1\right)}, \quad n = 1, 2, \dots , \qquad (54)$$

that is the time until the first failure of the ITMS which has asymptotically exponential distribution. This means a higher frequency of small periods of time before the first fail and, given the large number of adaptive traffic lights in ITMSs, it indicates the importance of finding new kinds of warm standby for them.

4 Results

The main result of the failure probability estimation presented above is the fairness of two-sided estimate (54). For simple duplication, when $n = 1$, this estimate gives the exact value of the probability

$$q_1(2) = b_0 . \tag{55}$$

We can compare this result with result according to another method – with a similar upper estimate obtained by A.D. Solovyev in [8], where there is no subtracted unit in brackets in the denominator:

$$b_{n-1} \le q = q_1(n+1) \le \frac{b_{n-1}}{1-2^{n-1}b_0} . \tag{56}$$

So, the estimate of the failure probability (54) is more narrow than the estimate (56) and can provide more exact predicted reliability of traffic lights with and without warm standby. In case when the traffic lights fail with probability q_1, it is possible to estimate the effect of warm standby use for adaptive traffic lights at two-way intersection. It can be done by means of determining the difference between vehicle service time at the intersection with and without adaptive traffic control.

Comparing to the intersection with adaptive traffic control, the uncontrolled intersection with one major direction (street) has the next differences:

- vehicles on the major road have no delays when passing the intersection;
- vehicles on the minor road are obliged to slow down or even stop before entering the major road and make sure of ability to move forward without traffic hindrances on the main road.

It increases the time of passing the intersection and for vehicles on the minor road it can be estimated as $m^{II} = 2m^{I}$ [9]. Hence, an average waiting time for the vehicles on the minor (critical) movement direction can be estimated as the consequence of adaptive traffic lights failure.

For example, let the traffic volume on one lane of the major road equals to $\lambda^{I} = 0.3\,\mathrm{s}^{-1}$. Time of passing the intersection when traffic is permitted (for the major road) equals to $m^{I} = 2$ s. Analogous time for the vehicles on minor road when there is no adaptive traffic control at intersection equals to $m^{II} = 2m^{I} = 4$ s. Also let the traffic volume on the minor road equals to $\lambda^{II} = 0.075\,\mathrm{s}^{-1}$. Then, when the traffic lights fail, total intersection load equals to $\rho = \rho^{I} + \rho^{II} = \lambda^{I} m^{I} + \lambda^{II} m^{II} = 0.6 + 0.3 = 0.9$. Then average waiting time in a queue on the minor road [7] equals to

$$W^{II} = \left(\rho^{I} m^{I} / (1-\rho) + \rho\right) \Big/ (1-\rho) . \tag{57}$$

After example data substitution $W^{II} = \left[(0.6\cdot 2)/(1-0.9) + 0.9\right] \big/ (1-0.9) = 129$ s. The upper estimate of analogous time expenses on the minor road when there is an adaptive traffic control at intersection can be obtained regarding to restrictions from equation (31) and denoting that $T = T_I + T_{II} = 80$ s. Then

$$T_{II} = \left[T_I(1-\rho_I)m^{II}\right] \Big/ \left[m^{I}(1-\lambda m^{II})\right] = T_I \cdot 0.4/0.85 , \tag{58}$$

wherefrom $T_{II} = 25.6$ s, $T_I = 54.4$ s.

The upper estimate of waiting time is obtained upon the Smith theorem [10]: $W^{II} = T_I / 2 = 27.2$ s. It means that waiting time is upper bounded by the value that is almost 5 times less than in the case when there is no adaptive traffic control at intersection. At that, average waiting time for the vehicles on the major road does not exceed the value of $W^I = T_{II} / 2 = 12.8$ s that allows to highlight advantages increasing the reliability of adaptive traffic lights by warm standby use.

5 Conclusions

1. The Laplace transforms (7) and (18), which were obtained for the time interval of passing over the intersection in a given direction when there are no restrictions on the duration of the permissive signal, allow the determination of the ratio of values of such restrictions from a practical point of view.
2. An upper estimate of the probability of failure of the system with one standby during the regeneration period has been found. It leads to an exact value of the probability of failure, which coincides with a lower estimate.
3. This article defines the quantitative ratios that create the opportunity to assess the feasibility of using warm standby in adaptive traffic lights, when quantitative characteristics of traffic flows and the failure rates of TMS' elements are known.

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