Computational Modeling and Structural Stability

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Abstract. The structural stability of a mathematical model with respect to small changes is a necessary condition for its correctness. The same condition is also necessary for the applicability of numerical methods, a computational experiment. But after S. Smale's works it became clear that in smooth dynamics the system of a general form is not structurally stable, therefore there is no strict mathematical basis for modeling and computational analysis of systems. The contradiction appeared in science: according to physicists dynamics is simple and universal. The paper proposes a solution to this problem based on the construction of dynamic quantum models (DQM). DQM is a perturbation of a smooth dynamical system by a Markov cascade (time is discrete). The dynamics obtained in this way are simpler than smooth dynamics: the structurally stable DQM realizations are everywhere dense and open on the set of all DQM realizations. This dynamics in contrast to the classical one has a clear structural theory, which makes it possible to construct effective algorithms for study of concrete systems. For example this paper shows the use of computer simulation for rigorous proof of hyperbolicity of the Henon system attractor. On the other hand, when fluctuations tend to zero, i.e. in the semiclassical limit, the dynamics of the DQM goes into the initial smooth dynamics. In this paper the equivalence of structural stability and hyperbolicity for smooth discrete dynamical systems is established along this path.

Keywords: modeling, computer simulation, structural stability, dynamical system, dynamic quantum model, Markov cascade, attractor.

1 Introduction

Computational modeling derives from two steps: (i) modeling, i.e. finding a model description of a real system, and (ii) solving the resulting model equations using computational methods [1]. Computational modeling has been used in physics, chemistry and related engineering for many decades because in practice hardly any model equations of systems of interest can be solved analytically, and this is where the computer comes in [2].

However, if an arbitrarily small perturbation of the model leads to a qualitatively different picture of the dynamics, then such a model is not applicable to the real process: strictly speaking, perturbations are included in the definition of a model. Therefore traditionally the stability of a mathematical model with respect to relatively small changes is a necessary condition for its correctness [3]. The same stability condition is necessary for applicability of numerical methods, computational experiments since they inevitably lead to errors of discretization and rounding in calculations [4].

The qualitative invariance of a mathematical model under small perturbations is usually called structural stability. This formally means equivalence, in some exact sense, between the model and its small enough perturbation. For the smooth dynamical systems (sets of differential or difference equations) this equivalence is usually a homeomorphism between the phase portraits of these systems. Such theory of a structural stability going back to H. Poincare, has been developed by A.Andronov and L. Pontrjagin in the case of small dimension of the phase space (1 or 2) [5]. However, the optimism generated by the successes of this theory disappeared after S. Smale's works [6]. It was shown in [7] that when the phase space has larger dimension, then there exist smooth dynamic systems whose neighborhoods do not contain any structurally stable system. For the theory of smooth dynamical systems (its old name is the qualitative theory of differential equations) this result has the same value as Liouville's theorem on insolvability of the differential equations in quadratures has for the theory of their integration. Namely, it shows that the problem of full topological classification of smooth dynamical systems is hopeless. This means also that there is no strict mathematical basis for modeling and computational analysis. The contradiction has appeared in science, because physicists believe that the dynamics is simple and universal [8].

The paper proposes solution to this problem, based on the construction of dynamic quantum models (DQM). It turns out that taking into account random fluctuations, necessary for the transition to the quantum model of reality, allows us to return in fact to the simple picture of A. Poincare's dynamics: a dense set of structurally stable systems.

DQM is so named because for Hamiltonian systems it is simply related to the corresponding Schrödinger equation, and its construction is the basis of the method for solving spectral problems [9]. But the definition of DQM is not formally related to Hamiltonian systems; it is defined for any ordinary differential equation or any diffeomorphism on any smooth Riemannian manifold.

The structural stability of the general form DQM opens the way to a mathematically grounded numerical analysis of the dynamics. As an example, this paper shows the use of computer simulation for rigorous proof of hyperbolicity of the Henon system attractor [10] at certain values of parameters. DQM is the natural basis for solving the traditional problems of machine learning [11].

On the other hand, when fluctuations tend to zero, i.e. in the semiclassical limit, the dynamics of the DQM goes into a more complex initial smooth dynamics. The old problem – the equivalence of structural stability and hyperbolicity for smooth discrete dynamical systems [12] is established by this way in this paper.

The **paper goal** is 1) to build the foundations of the theory of dynamic quantum models (DQM); 2) to demonstrate the application of this theory for computer research of concrete systems and for solving traditional problems of the theory of smooth dynamical systems.

The paper is organized as follows: in part 2 we synthesize the dynamic quantum models (DQM), in section 2.2 we define the DQM attractor, show the uniqueness of this definition and establish properties of the DQM attractor; in part 3 we show that structurally stable realizations of DQM are dense and open on the set of all its realizations; in part 4 we demonstrate the use of computer modeling for rigorous proof of hyperbolicity of the attractor of Henon system; part 5 concludes.

We had to omit proofs of some theorems in order to fit the paper format.

2 The Dynamic Quantum Model: Basic Definitions

2.1 DQM Definition

Let $p(x)$ be an n-dimensional smooth vector field on an n-dimensional smooth Riemannian manifold M, where $x(x_1, x_2, ..., x_n)$ are local Euclidean coordinates on M, $p_i(x) \in C^\infty(R^n)$ ($i = 1,...,n$). On each phase curve $x(t) \in M$ of the dynamical system generated by this vector field

$$\frac{dx_i}{dt} = p_i(x), \quad (i = 1,...,n) \tag{1}$$

consider the integral of the "shortened action" $s(t) = \int\limits_{x(t)} p(x)dx = \int\limits_0^t \|p(\tau)\|^2 d\tau$,

where $\|p(\tau)\|^2 = \sum\limits_{i=1}^n p_i^2(\tau)$. The value of $s(t)$ on each curve $x(t)$, which is different from a fixed point, is diffeomorphically expressed in t and is called "optical time". Let ρ be a metric such that $s(t) = \int\limits_{x(t)} d\rho$: $d\rho = \|p(t)\|^2 dt$. The following is the heuristic derivation or explanation of the definition of dynamic quantum model (Definition 1).

So, the distance d traveled by a point along the path of (1) during the time Δt is equal to $d = \int\limits_0^{\Delta t} \|p(\tau)\| d\tau = \|p(t_c)\| \cdot \Delta t$, where $p_c = p(t_0)$ is the average value $(0 \le t_0 \le \Delta t)$. (Of course this is with a single bypass of trajectory during Δt: turning points are the special case). Further, we assume that the fluctuations generate "white noise" $\xi(t)$, acting on the configuration space with the dispersion $D\xi(t) = \sigma^2 t$, where the diffusion coefficient σ^2 is constant over the considered time interval. It will take some time Δt, until the point moves to a distance d from the initial position, which exceeds the mean square error caused by $\xi(t)$ during the time Δt, i.e. $\|p_c\|\Delta t$ will exceed $\sqrt{\sigma^2 \Delta t}$. With such a minimal Δt $\|p_c\|\Delta t = \sigma\sqrt{\Delta t}$, whence $\sigma^2 = \|p_c\|^2 \Delta t$ and therefore

$$\Delta t = \frac{\sigma^2}{\|p_c\|^2}, \qquad d = \|p_c\|\Delta t = \frac{\sigma^2}{\|p_c\|} \tag{2}$$

Here by assumption Δt is the minimal time interval after which it becomes possible to make a new measurement, the difference from which will exceed the error, i.e. get a significantly different measurement. Owing to (2) $\sigma^2 = \|p_c\|^2 \Delta t \approx \int_0^{\Delta t} \|p(\tau)\|^2 d\tau = s(\Delta t)$. Thus 1) the time interval between the nearest significant measurements is unchanged on the optical time scale and is equal to σ^2. (In other words, the distance between them in the metric ρ is equal to σ^2). 2) During this time "white noise" $\xi(t)$ generates an irremovable random error, the standard deviation of which is equal to the distance d between the nearest significant measurements along the trajectory.

So, a dynamic quantum model first shifts each point along the phase curve of a given dynamic system over the optical time σ^2 (or ρ – length σ^2), and then randomly shifts on a distance not exceeding the length of the trajectory from the original to the new point. The following rigorous definition summarizes this description. The definition of a dynamic quantum model is given for an arbitrary dynamic system (1) on an arbitrary compact Riemannian manifold M.

Let G be the shift map along the phase trajectories of (1) during the lag time Δt. Consider a smooth function $q(y,z) \geq 0$ ($y,z \in M$) such that

$$\|z - Gy\| \leq d(y), \quad \int_M q(y,z)dz = 1, \quad \left\| \int_M zq(y,z)dz - Gy \right\| \leq d(y), \tag{3}$$

where $d(y) > 0$ is a continuous function on M. Here $q(y,z)$ defines the density of "local random dissipation caused by white noise," the numbers $d(y)$ are assumed to be small. Of course, the function $q(y,z)$ can also be assumed continuous, approximating it on M with a smooth function for any given accuracy. Then

Definition 1. The Markov process with the transition function

$$P(y,A) = \int_A q(y,z)dz \quad (A \subset M) \tag{4}$$

is called the dynamic quantum model (DQM) for the dynamic system (1). Given the initial distribution, we obtain a Markov process P with this initial distribution and the transition function $P(y,A)$: if μ_t is the distribution at time t, Δt is the lag between the two nearest measurements, then the DQM sets new distribution $P(\mu_t) = \mu_{t+\Delta t}$ at time $t + \Delta t$.

Thus, based on the differential equations (1), we arrive at difference equations with a lag of at least σ^2 on the optical time scale. At first glance, the DQM may surprise with the discreteness of time: in the traditional model of quantum mechanics errors are explicitly taken into account only for spatial variables. But, as can be seen from the deduction, the discreteness of the measurement process is an inevitable consequence of the unavoidable errors of coordinates and pulses. Indeed, to measure time ultimately requires a clock or other device in which readings on a scale are measured in proportion to time at a certain speed. But if these readings and speed are determined inaccurately, then the time is also known only with some error.

Definition 2. Let Δ_i be cells with a diameter ε of some partition of the phase space of a dynamical system and μ_0 is the initial state. Then the Markov chain with transition probabilities from Δ_i to Δ_j equal to $p_{ij} = \dfrac{1}{\mu_0(\Delta_i)} \int\limits_{y \in \Delta_i} P(y, \Delta_j)\, d\mu_0$ will be called the ε - discretization of DQM with transition function $P(y, A)$ and initial state μ_0.

2.2 DQM Attractor

Attractor is the key concept of the theory of dynamical systems; its physical meaning is that it is "the space of steady-state regimes". The point of the phase space is contained in the attractor if it belongs to the carrier of the "stationary state of the system", i.e. to a measure not changing over time.

Let M be a compact phase space, P is some DQM on M.

Definition 3. The probability measure μ on M will be called the stationary (equilibrium) state of DQM if $P(\mu) = \mu$. The DQM attractor is the union of the carriers of all stationary states.

Theorem 1. (Perron-Frobenius theorem for DQM). Let $\Lambda \subset M$ be an invariant closed set of DQM P that does not contain its own invariant closed subsets (that is minimal with respect to P). Then

1. there is a unique stationary state μ, whose carrier is Λ. The state μ is ergodic (that is the flow P is ergodic with respect to measure μ).
2. For any other state (probability measure) ν on Λ $\lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} P^k \nu = \mu$.
3. If $\overline{\mu}_\varepsilon$ is a probabilistic stationary measure of some ε - discretization of the given DQM on Λ then $\lim\limits_{\varepsilon \to 0} \overline{\mu}_\varepsilon = \mu$.

Proof. Let $\Lambda \subset M$ be an invariant closed set of DQM that does not contain its own invariant closed subsets. Let $\overline{\mu}_\varepsilon$ be a stationary measure of some discretization of the given DQM on Λ with cells of diameter ε (that is, a probability invariant measure of a Markov chain defined by Definition 3). On a compact subset Λ of the phase space the set of probability measures $R = R(\Lambda)$ forms a convex metrizable compact in the weak topology. Therefore in any sequence of measures $\overline{\mu}_{\varepsilon_k}$ one can find a subsequence $\overline{\mu}_{\varepsilon_n}$, converging to some measure $\overline{\mu}$ from R: $\lim_{n\to\infty} \overline{\mu}_{\varepsilon_n} = \overline{\mu} \in R$ in the sense of the weak topology on R. Since $P\overline{\mu}_{\varepsilon_n} - \overline{\mu}_{\varepsilon_n} \xrightarrow[\varepsilon_n \to 0]{} 0$ (in the sense of the weak topology) by virtue of definition 3, then $P\overline{\mu} = \overline{\mu}$ i.e. $\overline{\mu}$ is a stationary state of DQM. Since by the condition Λ does not contain non-empty proper invariant subsets of DQM (i.e. it is metrically transitive), then for any P-invariant measure on Λ the ergodic Neumann theorem holds: for any continuous function f on Λ

$$L^2 - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(P^k) = \int f d\overline{\mu} \qquad (5)$$

Since left side of this equality does not depend on the choice of a sequence of measures $\overline{\mu}_{\varepsilon_k}$, then any weakly convergent sequence $\overline{\mu}_{\varepsilon_n}$ converges to the same measure $\overline{\mu}$. Therefore $\lim_{\varepsilon\to 0} \mu_\varepsilon = \mu$ and it proves 3). Since (5) holds for any stationary state on Λ, then from (5) the uniqueness of an invariant measure $\overline{\mu}$ also follows, which establishes 1). Finally, since for any other probability measure ν on Λ $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k \nu$ exists by virtue of (5) and is an invariant measure, then it coincides with $\overline{\mu}$, which proves 2), QED.

Obviously, there are only finitely many components of the DQM attractor Λ_k on M, that is such invariant subsets of the attractor that do not contain proper invariant non-empty subsets. On each component Λ_k of the DQM attractor there is a unique probability invariant measure $\overline{\mu}_k$: $P\overline{\mu}_k = \overline{\mu}_k$. The density of $\overline{\mu}_k$ is positive on the interior of Λ_k by the definition of DQM. Any stationary state on M is a convex combination of stationary states $\overline{\mu}_k$ on Λ_k.

Let G be shift map along the phase trajectories of (1) during the lag time of DQM.

Definition 4. For the DQM trajectory ω for the time t_n: $y_0, y_1, \dots, y_n$ its differential is $D_n(\omega) = DG(y_n) \cdot \dots \cdot DG(y_1) \cdot DG(y_0)$ where $DG(y_k)$ is the differential of G at the point $y_k \in \Lambda$, $k \le n = 0,1,\dots$. For the DQM trajectory ω: $y_{-n}, \dots, y_{-1}, y_0$ differential $D_n(\omega) = DG(y_n) \cdot \dots \cdot DG(y_1) \cdot DG(y_0)$ at $n = 0,1,\dots$.

The measure $\overline{\mu}$, induced by the measure μ in accordance with the Kolmogorov theorem [13], is defined on the space Ω of DQM trajectories on the component Λ of the DQM attractor.

Theorem 2. Let $\Lambda \subset M$ is a component of the DQM attractor of dimension $m = \dim M$. Then for DQM with sufficiently small $d = \min_{y \in \Lambda} d(y)$ (where $d(y) > 0$ are constants from (3))

1. for almost all under measure $\overline{\mu}$ DQM trajectories ω at any nonzero vector $u \in R^m$ ($\|u\| = 1$) there are limits

$$\lim_{n \to \pm\infty} \frac{1}{n} \ln \|D_n(\omega) u\| = \pm\lambda_r ,$$

where $r = 1,2,..., s \leq m = \dim M$.

2. At each point of each such trajectory ω, the filtering of subspaces is uniquely defined:

forward $L_1^+(y) \subset L_2^+(y) \subset ... \subset L_s^+(y) = R^m$

and back $L_s^-(y) \subset ... \subset L_2^-(y) \subset L_1^-(y) = R^m$,

associated with the numbers $\lambda_1 < \lambda_2 < ... < \lambda_s$ so that

$$\lim_{n \to \infty} \frac{1}{n} \ln \|D_n(\omega) u\| = \lambda_r \Leftrightarrow u \in L_r^+(y) \ and \ u \notin L_{r-1}^+(y),$$

$$\lim_{n \to -\infty} \frac{1}{n} \ln \|D_n(\omega) u\| = -\lambda_r \Leftrightarrow u \in L_r^-(y) \ and \ u \notin L_{r+1}^-(y).$$

These filtrations are invariant with respect to the DQM differential. Exactly if y_n and y_{n+1} are consecutive points of the trajectory ω at times t_n and t_{n+1} respectively then the differential $DG(y_n)$ translates the filtering at the point y_n in the filtering at the point y_{n+1}.

Proof. Consider DQM on Λ as a random process $X(t, \omega)$, where t is discrete time, $t = t_k, k = 0, \pm 1, \pm 2, ...,$ ω is a DQM trajectory. Namely, for $\eta_k \in M$ let $\boldsymbol{\eta}$ = $= (..., \eta_{-k}, ..., \eta_{-1}, \eta_0, \eta_1, ..., \eta_k, ...)$. Then the DQM trajectory $\omega = \omega(t, y_0)$ with an initial point $y_0 \in M$ is the sequence $X(t_0, \omega) = y_0$, $X(t_1, \omega) = y_1 = Gy_0 + \eta_0$, $X(t_2, \omega) = y_2 = Gy_1 + \eta_1$, ..., $X(t_k, \omega) = y_k = Gy_{k-1} + \eta_{k-1}, ...$. (Here d is assumed to be so small that the addition of $Gy_{k-1} + \eta_{k-1}$ when $\|\eta_k\| \leq d$ performed on the local map of the manifold M in R^m). Thus, the DQM trajectory ω is defined uniquely by a sequence of vectors $\boldsymbol{\eta}$ and an initial point $y_0 : \omega = \omega(y_0, \ \boldsymbol{\eta}_0)$.

On the set Ω of DQM trajectories $X(t,\omega)$ on Λ induces the dynamic process T – the trajectory of the trajectories: $T\omega_0 = \omega_1$, $T\omega_1 = \omega_2$, …, $T\omega_{k-1} = \omega_k$, … . Namely if $\omega_0 = \omega(y_0,\ \boldsymbol{\eta}_0)$, where $y_0 = X(t_0,\omega_0)$, $\boldsymbol{\eta}_0 = (...,\eta_{-k},...,\eta_{-1},\eta_0,\eta_1,...,\eta_k,...)$ and $\omega_1 = \omega(y_1,\ \boldsymbol{\eta}_1)$, then $y_1 = Gy_0 + \eta_0 = X(t_1,\omega_0)$, $\boldsymbol{\eta}_1 = R\,\boldsymbol{\eta}_0$, where R is shift operator to the right. If $\omega_2 = \omega(y_2,\ \boldsymbol{\eta}_2)$ then $y_2 = Gy_1 + \eta_1 = X(t_2,\omega_0)$, $\boldsymbol{\eta}_2 = R\,\boldsymbol{\eta}_1$; in the general case for $\omega_k = \omega(y_k,\ \boldsymbol{\eta}_k)$ we get $y_k = Gy_{k-1} + \eta_{k-1} = X(t_k,\omega_0)$, $\boldsymbol{\eta}_k =$ $= R\,\boldsymbol{\eta}_{k-1}$. By the Kolmogorov theorem on the set Ω of DQM trajectories on Λ the probability measure $\overline{\mu}$ was determined, induced there by a stationary state μ on Λ. By construction measure $\overline{\mu}$ inherit from measure μ the invariance with respect to T ($\overline{\mu}(T) = \overline{\mu}$) and ergodicity of T (i.e. its metric transitivity) under measure $\overline{\mu}$.

Let $a(n,\omega) = D_n(\omega)$ for $\omega = \omega(y_0,\ \boldsymbol{\eta})$. Then $a(n,\omega)$ are measurable functions on a probability space Ω with measure $\overline{\mu}$ and $a(n+k,\omega) = a(k,T^k\omega)$. This means that the square matrices $a(n,\omega)$ of order m are a multiplicative cocycle on the space of trajectories Ω with respect to its automorphism T by the definition of the cocycle [11]. Since G is a diffeomorphism, then $\|DG(y)\| \neq 0$ for all $y \in \Lambda$, whence $\ln(\|DG(y)\|)$ is continuous function on compact Λ and $\int\limits_{y\in\Lambda} \ln\|DG(y)\| d\mu < \infty$. On the other hand by definition a measure for any open subset $C \subseteq \Lambda$ with the characteristic function χ_C

$$\int\limits_{\Omega} \chi_C\, d\overline{\mu} = \overline{\mu}(\{\omega = (y,\eta) | y \in C\}) = \mu(\{y | y \in C\}) = \int\limits_{M} \chi_C\, d\mu .$$

Therefore for any piecewise continuous function g on M $\int\limits_{\Omega} g\, d\overline{\mu} = \int\limits_{M} g\, d\mu$. In particular since $a(0,\omega) = DG(y)$ on each trajectory $\omega = \omega(y,\ \boldsymbol{\eta})$, then $\int\limits_{\omega\in\Omega} \ln\|a(0,\omega)\| d\overline{\mu} = \int\limits_{y\in\Lambda} \ln\|DG(y)\| d\mu < \infty$. This inequality means that the cocycle $a(n,\omega)$ is Lyapunov and this is the condition under which the multiplicative ergodic theorem for this cocycle holds.

This theorem asserts that almost all trajectories $\omega \in \Omega$ under measure $\overline{\mu}$ are Lyapunov correct. This means, in particular, that

1. for such ω with $u \in R^m$ $(\|u\| = 1)$ there are limits

$$\lim_{n\to\pm\infty}\frac{1}{n}\ln\|a(n,\omega)u\| = \pm\lambda_r(\omega),$$

where $r = 1,2,...,s = s(\omega) \le m$.

2. On each such trajectory ω, the filtering of subspaces is uniquely defined:

   forward $L_1^+(y) \subset L_2^+(y) \subset ... \subset L_s^+(y) = R^m$

   and back $L_s^-(y) \subset ... \subset L_2^-(y) \subset L_1^-(y) = R^m$,

   associated with the numbers $\lambda_1 < \lambda_2 < ... < \lambda_s$ ($s = s(\omega)$) so that

$$\lim_{n\to\infty}\frac{1}{n}\ln\|a(n,\omega)u\| = \lambda_r \Leftrightarrow u \in L_r^+(y) \text{ and } u \notin L_{r-1}^+(y),$$

$$\lim_{n\to-\infty}\frac{1}{n}\ln\|a(n,\omega)u\| = -\lambda_r \Leftrightarrow u \in L_r^-(y) \text{ and } u \notin L_{r+1}^-(y).$$

These filtrations are invariant with respect to the automorphism T: if $T\omega_n = \omega_{n+1}$, then the cocycle $a(n,\omega)$ takes the filtration ω_n to the filtration ω_{n+1}.

Since by the Kolmogorov theorem flow T on a probability space Ω with a measure $\overline{\mu}$ inherits ergodicity from ergodicity P on Λ with a measure μ, which was established in Theorem 1. Then the values of $\lambda_r(\omega) \equiv \lambda_r$, $s(\omega) \equiv s$ coincide for almost all DQM trajectories $\omega \in \Omega$ under measure $\overline{\mu}$. In view of the correspondence $a(n,\omega) = D_n(\omega)$ the theorem immediately follows from here, QED.

By analogy with the theory of smooth dynamical systems the numbers λ_r we will call the Lyapunov characteristic exponents of the component Λ of the DQM attractor.

3 Structural Stability in DQM

Definition 5. The DQM realization is a sequence of smooth mappings $G_k(y)$ on Λ in $\Lambda \subseteq M$ ($k = 0,\pm1,\pm2,...$) if $\|G_k(y) - G(y)\|_{C^1} \le d(y) \le d$, where $d(y)$ are the constants from (3).

Here all the maps $G_k(y)$ are diffeomorphisms on Λ in Λ for sufficiently small d. In terms of content $\eta(t_k, y) = G_k(y) - G(y)$ are small random deviations caused by “white noise” at the point y at time t_k. By definition, any DQM trajectory $\omega = \omega(y_0,\ \boldsymbol{\eta})$: $y_k = Gy_{k-1} + \eta_k$ ($k = 0,\pm1,\pm2,...$) is given by the initial point $y_0 \in M$ and sequence of deviations $\boldsymbol{\eta} = (...,\eta_{-k},...,\eta_{-1},\eta_0,\eta_1,...,\eta_k,...)$. But on the DQM realization the function of deviations $\eta(t_k, y)$ is fixed; therefore, on it the DQM trajectory ω with the initial point y_0 is uniquely determined: $\omega = \omega(y_0)$.

Definition 6. A DQM realization $G_k(y)$ ($k = 0, \pm 1, \pm 2, \ldots$) on a compact set $K \subseteq \Lambda$ will be called a hyperbolic realization of DQM if at each point $y \in K_k \subseteq \Lambda$, where $K_0 = K, G_k(K_{k-1}) = K_k$ there exists a decomposition of the tangent bundle TK_k into the Whitney sum of the subbundles $E_k^s(y)$ and $E_k^u(y)$: $TK_k = E_k^s(y) + E_k^u(y)$, satisfying the following conditions:

1. the tangent map DG_k preserves the subbundles:

$$DG_k(E_k^s) \subseteq E_{k+1}^s, \quad DG_k(E_k^u) \subseteq E_{k+1}^u;$$

2. DG_k compresses E_k^s: on every trajectory ω with an initial point $y \in K_k$ at the time moment t_k there are such constants $b > 0$ and $\lambda\, (0 < \lambda < 1)$ that for any $u \in E_k^s$ and any natural n

$$\|D_n(\omega)u\| \le b\lambda^n \|u\| \quad (u \in E_k^s(y)).$$

3. DG_k stretches $E_k^u(y)$, more precisely, on each trajectory ω with an initial point $y \in K_k$ at the time moment t_k for any $u \in E_k^u$ and a natural n

$$\|D_n(\omega)u\| \ge \frac{1}{b\lambda^n} \|u\| \quad (u \in E_k^u(y)).$$

Theorem 3. Hyperbolic realizations are everywhere dense on the set of DQM realizations. More precisely, for any DQM realization $G_k(y)$ ($k = 0, \pm 1, \pm 2, \ldots$) and for any sufficiently small $\varepsilon > 0$ there exists such hyperbolic realization $\widehat{G}_k(y)$ of this DQM on the compact $K \subseteq \Lambda$, that

1. $\mu(\Lambda / K \cup K / \Lambda) < \varepsilon$ for the probabilistic invariant DQM measure μ on Λ;
2. on K_k $\left\|G_k(y) - \widehat{G}_k(y)\right\|_{C^1} < \varepsilon$ ($k = 0, \pm 1, \pm 2, \ldots$).

Definition 7. The realization of the DQM $G_k(x)$ on a compact $K \subseteq \Lambda$ and the realization $\widetilde{G}_k(x)$ of this DQM on a compact $\widetilde{K} \subseteq \Lambda$ are topologically equivalent if they are conjugate by means of homeomorphisms H_k defined on some neighborhoods of the compacts $K_k \subseteq \Lambda$, where $G_k(K_{k-1}) = K_k$, $\tilde{G}_k(\tilde{K}_k) = \tilde{K}_{k+1}$, $K_0 = K$, $\tilde{K}_0 = \tilde{K}$: $\tilde{G}_k \circ H_k = H_{k+1} \circ G_k$ ($k = 0, \pm 1, \pm 2, \ldots$).

$$\begin{tikzcd}
\tilde{K}_k \arrow[r, "\tilde{G}_k"] & \tilde{K}_{k+1} \\
K_k \arrow[u, "H_k"] \arrow[r, "G_k"] & K_{k+1} \arrow[u, "H_{k+1}"']
\end{tikzcd}$$

Fig. 1. Commutative diagram of topological equivalence.

Definition 8. A DQM realization $G_k(x)$ on a compact $K \subseteq \Lambda$ is structurally stable if any realization of this DQM sufficiently close to $G_k(x)$ in C^1 topology for all $k = 0, \pm 1, \pm 2, \ldots$ is topologically equivalent to it.

In more detail: If for every point $x \in K_k$ there are numbers $d_k(x) > 0$ such that for any realization of this DQM $\tilde{G}_k(x)$ from $\left\| G_k(x) - \tilde{G}_k(x) \right\|_{C^1} \le d_k(x)$ for all k and $x \in K_k$ the topological equivalence of the realizations G_k and $\tilde{G}_k$ follows, then G_k is structurally stable.

If all $G_k(x)$ and all compacts K_k coincide for all k , then we obtain the definition of the structural stability of a diffeomorphism.

Theorem 4. Any hyperbolic realization of the DQM G_k on the compact set $K \subseteq \Lambda$ is structurally stable.

Corollary 1. A diffeomorphism G on a compact manifold M is structurally stable in the sense of Definition 8 if and only if it is (non-uniformly) hyperbolic.

Proof. Let a diffeomorphism G be structurally stable. By Definition 8, this means that for some $d(x) \le d = \varepsilon$ from $\left\| G(x) - \tilde{G}(x) \right\|_{C^1} \le d(x)$ ($x \in M$) it follows that the diffeomorphisms G and $\tilde{G}$ are topologically equivalent, that is, are conjugate on M by means of a homeomorphism. Consider DQM for G with the same d (x) in (3) for all $x \in M$. By Theorem 1, this DQM has an attractor, let Λ be a component of this attractor. By Theorem 3 there is a realization $\tilde{G}_k(x)$ of this DQM, hyperbolic in the sense of Definition 6 on a compact set K , which differs from Λ only by the order $d = \varepsilon$ on the measure μ of the stationary state on Λ . Then by virtue of the structural stability of G every diffeomorphism $\tilde{G}_k$ of this realization is conjugate to G in a neighborhood of Λ. Therefore the realization with zero deviations, i.e. coinciding with G for each k, is also hyperbolic in Λ in the sense of Definition 6, and the component Λ itself is invariant with respect to G with accuracy ε. But then the complement $M \setminus \Lambda$ is invariant with respect to G with accuracy ε too. Unless it turns out that with this accuracy $M = \Lambda$, then on $M \setminus \Lambda$ we can similarly consider DQM for G with perhaps smaller than the earlier $d(y) \le \varepsilon$. You can find there its component Λ_1 and establish for realization $\overline{G}_k$ with zero deviations hyperbolicity in neighborhood of in the sense of Definition 6 Λ_1 as we did it early; and so on. In general let Δ be the greatest G-invariant with accuracy ε subset in M, in which the realization with zero deviations $\overline{G}_k(x)$ is hyperbolic in the sense of Definition 6. If $\Delta \ne M$ with accuracy ε, then on $M \setminus \Delta$ as above we can obtain a new component, in which the realization with zero deviations $\overline{G}_k$ is hyperbolic contrary to the assumption about Δ. Tending ε to zero, we obtain the hyperbolicity of the realization $\overline{G}_k$ with zero deviations at almost all points $x \in M$. In this case, generally speaking, we have $\inf d(x) = 0$. This means the non-uniform hyperbolicity of the diffeomorphism G onto M.

Conversely, the fact that the non-uniform hyperbolicity of a diffeomorphism G onto M implies its structural stability in the sense of Definition 8 directly follows from Theorem 4, QED.

Corollary 2. If for some $\varepsilon > 0$ from $\left\| G(x) - \tilde{G}(x) \right\|_{C^1} \le \varepsilon$ ($x \in M$) follows that diffeomorphisms G and $\tilde{G}$ are topologically equivalent, then G is (uniformly) hyperbolic diffeomorphism on M.

Proof. It follows from Corollary 1 that G is hyperbolic, in general, non-uniform. If G is hyperbolic exactly non-uniformly and so $\inf d(y) = 0$ for $d(y)$ from definition 8, then we can get some diffeomorphism $\tilde{G}$ inequivalent to G with arbitrarily small perturbations. However from $\left\| G(x) - \tilde{G}(x) \right\|_{C^1} \le \varepsilon$ ($x \in M$) follows that G and $\tilde{G}$ are topologically equivalent by the condition of the corollary. Therefore G is uniformly hyperbolic on M, QED.

4 Example of DQM Application: Henon System Attractor

For the two-dimensional M. Henon system [10]: $(x, y) \to (1 + y - ax^2, bx)$ values of parameters $a = 1.7,\ b = 0.5$ are chosen such that the hyperbolicity of dynamics on the attractor with this parameters is rigorously proved for R. Lozi system [14] $(x, y) \to (1 + y - a|x|, bx)$. The proof of the hyperbolic dynamics here is based on the following statement, specifically focused on the study of concrete dynamical systems. For ease of application to the Henon system in the formulation we restrict ourselves to the two-dimensional case, although the multidimensional generalization is also true.

Corollary 3. Let Δ_i be the cells of ε-discretization of the DQM attractor for the system given by the diffeomorphism G, $x_i \in \Delta_i$ ($1 \le i \le N$). Let the eigenvalues $\lambda_1(x_i)$ and $\lambda_2(x_i)$ of the differential DG at each point $x_i \in \Delta_i$ ($1 \le i \le N$) satisfy the conditions $\lambda_1(x_i) < \mu$, $\lambda_2(x_i) > \dfrac{1}{\mu}$ for some μ ($0 < \mu < 1$) and

$$\varepsilon \le \frac{(1-\mu)^2}{4(4\|G\|_2 + 1)}, \tag{6}$$

where $\|G\|_2$ is the norm of G in C^2. Then

1. the initial system given by the diffeomorphism G is hyperbolic on its attractor;
2. any DQM ε - realization of this system is hyperbolic on the DQM attractor and is topologically equivalent to the initial system;
3. the support of the attractor of the initial hyperbolic system and the attractor of its DQM ε - realization coincide with an accuracy of order ε.

The proof of this statement essentially reproduces the proof of Theorem 4, estimate (6) is actually obtained there. The verification of the conditions of Corollary 3 for the Henon system uses 4 *Maple* procedures.

1. The *Animate* procedure visualizes system behavior using animation technologies in *Maple*. This allows you to localize the region of the phase space in which the system attractor is hypothetically contained. In the graph of next fig.2 for each iteration t shows the point in phase space of Henon system.

   On the basis of outcomes of the numerical researches, visually presented in Figure 2, we choose a rectangle $\Omega = \{(x, y) \mid -1 \le x \le 1.5; -0.1 \le y \le 0.1\}$. In next Figure 3 for each iteration $t = 1,2,...,500$ of Henon system corresponds its coordinate $x(t)$ on the ordinate axis.

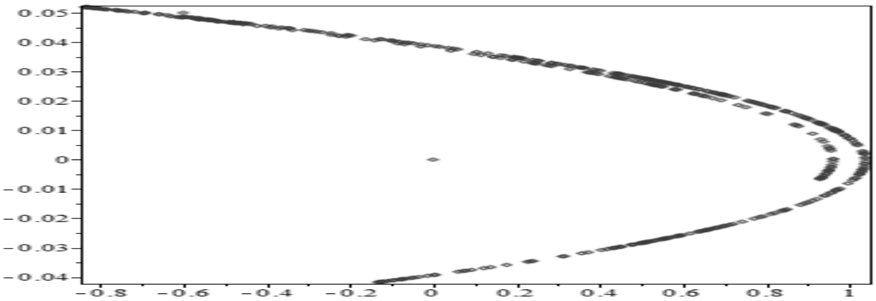

Fig. 2. Phase curve of Henon system.

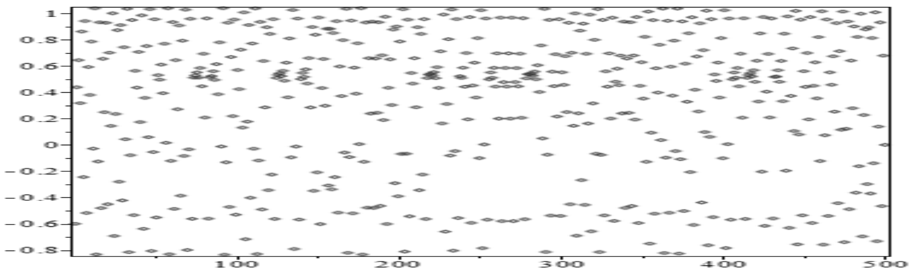

Fig. 3. Trajectory of Henon system.

The animation in Figure 3 suggests that the system is hyperbolic.

2. The *Prestep* procedure splits the rectangle Ω into cells Δ_i - squares with sides of length 0.01 parallel to the axes of coordinates ($1 \le i \le N$).Then each cell Δ_i *Prestep* associates a set of cells into which points from Δ_i can fall into one step of the dynamics of the Henon system. In this case it is formally verified that the domain Ω is indeed invariant with respect to the discretization of the DQM, given by the constructed partition of Ω. In other words *Prestep* defines a topological Markov chain H, the state space of which is the set of cells $\Delta_i \subset \Omega$.
3. The *Findattr* procedure finds in Ω the attractor of a topological Markov chain H defined in *Prestep*. Its algorithm is based on the following consideration. On the state space $\Omega = \{\Delta_i\}$ consider a transitive quasi-order relation: $\Delta_i \prec \Delta_j$ if there exists a trajectory H from Δ_i to Δ_j. The state Δ_i is recurrent if $\Delta_i \prec \Delta_i$. Recurrent states are divided into equivalence classes: $\Delta_i \sim \Delta_j \Leftrightarrow \Delta_i \prec \Delta_j \prec \Delta_i$. On $\Omega = \{\Delta_i\}$ $H(\Omega) \supseteq H^2(\Omega) \supseteq H^3(\Omega) \supseteq \ldots \supseteq H^n(\Omega)$. If $H^n(\Omega) = H^{n+1}(\Omega)$ then $H^n(\Omega)$ is the DQM attractor. In the case under consideration, the attractor turns out to be connected, which corresponds to Fig. 2, obtained by the *Animate* procedure.

4. The *Hyperproc* procedure performs a main check: do the conditions of Corollary 3 be satisfied on the attractor found by *Findattr*? For the Henon system under consideration on a rectangle $\Omega = \{(x, y) \mid -1 \le x \le 1.5; -0.1 \le y \le 0.1\}$ we obtain $\|G\|_2 = \max\limits_{\Omega}\{\sqrt{(2ax)^2 + b^2} + 1, 2a\} \approx 6.1$. The *Hyperproc* procedure establishes that for the differential DG eigenvalues $\lambda_1(x_i) < 0.4$ and $\lambda_2(x_i) > 1.7$ for all $x_i \in \Delta_i$. The value $1/1{,}7 \approx 0.59$. Thus $\mu \ge 0.59$; however, we choose the value $\mu = 0.7$ with a margin. Then, in accordance with (6), it is necessary that $\varepsilon \le 0.00089$.

Now the cell length of Δ_i (the length of a square with sides parallel to the axes of coordinates) is chosen equal to 0.0005 ($1 \le i \le N$) and already for such a small partition of the rectangle Ω we repeat the *Prestep* $\to$ *Findattr* $\to$ *Hyperproc* cycle described above. Now the other smaller cells are Δ_i, the other $x_i \in \Delta_i$ and the other eigenvalues $\lambda_1(x_i)$ and $\lambda_2(x_i)$ respectively ($1 \le i \le N$). If now again $\lambda_1(x_i) < 0.4$ and $\lambda_2(x_i) > 1.7$ holds for all i, then (6) holds for such a partition and therefore Corollary 3 holds. In our case, the test was successful, which proves the hyperbolicity of the dynamics on the attractor of the Henon system for the values of the parameters $a = 1.7,\ b = 0.5$. As a result, the structure of a topological Markov chain obtained in the course of computer calculations, by virtue of 2) and 3) of Corollary 3, gives detailed and rigorously proved data on the dynamics of this system.

The selected values of the parameters $a = 1.7,\ b = 0.5$ are not the only ones. For example, similar results are obtained for $a = 1.4,\ b = 0.35$.

5 Conclusion

The structural stability of a mathematical model is a necessary condition for its correctness. It is also necessary for applicability of numerical methods, computational experiments since they inevitably lead to errors.

But after S. Smale's works it became clear that in smooth dynamics the system of a general form is not structurally stable and therefore there is no strict mathematical basis for modeling and computational analysis of systems. The contradiction appeared in science: according to physicists dynamics is simple and universal.

The paper proposes a solution to this problem based on the construction of dynamic quantum models (DQM). DQM is a perturbation of a smooth dynamical system by a Markov cascade (time is discrete). The dynamics obtained in this way are simpler than the classical smooth dynamics: the structurally stable realizations of DQM are everywhere dense (Theorem 3) and open (Theorem 4) on the set of all DQM realizations. This dynamics has a clear structural theory: unlike the classical systems, the DQM attractor is uniquely defined (Theorem 1), Lyapunov exponents exist for any DQM (Theorem 2).

As a Markov cascade, the DQM is approximated by a Markov chain and on a compact set by a finite Markov chain arbitrarily exactly (Theorem 1). This allows you to clearly understand the DQM dynamics and build effective algorithms for the study of concrete systems that are always oriented towards parallel computing and do not require stable (according to Hadamard) solutions. For example, in part 4 we demonstrate the use of computer simulation for rigorous proof of hyperbolicity of the attractor of Henon system.

On the other hand, when fluctuations tend to zero, i.e. in the semiclassical limit, the dynamics of the DQM goes into the initial smooth dynamics. In part 3 the equivalence of structural stability and hyperbolicity for smooth discrete dynamical systems is established along this path (Corollaries 1 and 2).

In the future, we intend to apply the DQM algorithms, that oriented towards parallel computing and do not require stable solutions, to traditional problems of computational methods.

We also intend to generalize dynamic quantum models on dynamical systems that using logical operations: proofs of theorems, software applications, information and network systems, etc. A natural and even obvious implementation tools for such a generalization are the specialized neural network. This will allow the use of DQM methods for problems of artificial intelligence: identification, prediction, filtering, etc.

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