On the Semantic Equivalence of Language Syntax Formalisms

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Abstract. Several formalisms for language syntax specification exist in literature. In this paper, we prove that longstanding syntactical transformations between *context-free grammars* and *algebraic signatures* give rise to *adjoint equivalences* that preserve the *abstract syntax* of the generated terms. The main result is a *categorical equivalence* between the *categories of algebras* (i.e., all the possible semantics) over the objects in these formalisms up to the provided syntactical transformation, namely that all these frameworks are essentially the same from a semantic perspective.

Keywords: Syntax specification · Algebraic signatures · Context-free grammars · Equivalence

1 Introduction

Several formalisms for language syntax specification exist in literature [19]. Among them, *formal grammars* [3,5,12] and *algebraic signatures* [4,10,7] have played and still play a pivotal role. The former are widely used to define syntax of programming languages [17], notably due to compelling results on context-free parsing techniques [5,18,13]. The latter provide an algebraic approach to syntax specification, and they are ubiquitous in the fields of universal algebra [4], model theory [2], and logics in general.

In this paper, we narrow the focus to three different syntax formalisms: *context-free grammars* (**Grm**), *many-sorted signatures* (**Sig**), and *order-sorted signatures* ($\mathbf{Sig}^{\leq}$). The aim is to provide mappings between these frameworks (see Figure 1) able to translate language syntax specifications from one formalism to another without altering their classes of semantics. Put differently, if $\mathbb{A}$ is a semantics for an object $\mathscr{X}$ and $\Upsilon\mathscr{X}$ is its conversion to another formalism, we shall find a semantics $\mathbb{B}$ for $\Upsilon\mathscr{X}$ such that for each term t in $\mathscr{X}$, $[\![t]\!]_{\mathbb{A}} = [\![\Upsilon(t)]\!]_{\mathbb{B}}$ holds, where $\Upsilon(t)$ is the conversion of the term t from $\mathscr{X}$ to $\Upsilon\mathscr{X}$.

Formally, this requires two constraints: (1) each syntactical transformation Υ shall preserve the abstract syntax of terms, and (2) it must exist a *categorical equivalence* between the *categories of algebras* $\mathbf{Alg}(\mathscr{X})$ and $\mathbf{Alg}(\Upsilon\mathscr{X})$.

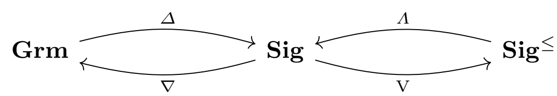


Fig. 1. An informal overview of the mappings between the different syntax formalisms.

The mathematical links between these different frameworks have already been partially studied in literature. Goguen et al. [8] provide a definition of $\Delta\colon \mathbf{Grm} \to \mathbf{Sig}$ that yields an isomorphism between the sets of terms (i.e., the *term algebras*) over G and its conversion to many-sorted signature ΔG, and conversely the definition of $\nabla\colon \mathbf{Sig} \to \mathbf{Grm}$ that makes the term algebras over $\mathfrak{G}$ and $\nabla\mathfrak{G}$ isomorphic (the proofs are outlined in detail in [20]). Other results on the subject are given in [7]. The authors provide a definition of $\Lambda\colon \mathbf{Sig}^{\leq} \to \mathbf{Sig}$ that gives rise to an equivalence between the categories of algebras over an order-sorted signature $\mathcal{S}$ and its many-sorted conversion $\Lambda\mathcal{S}$. Both these results of [8,7] are an instance of the aim of this paper, as we will prove later.

In the following sections, we unify and broaden such results in a more general settings. We model **Grm**, **Sig**, and $\mathbf{Sig}^{\leq}$ as the *categories* whose objects are grammars, many- and order-sorted signatures, respectively (Sections 2.1 and 2.2). Arrows between objects in the same category are morphisms preserving the *abstract syntax* [14]. This is a fundamental point: According to [6], "the essential syntactical structure of programming languages is not that given by their concrete or surface syntax [...]. Rather, the deep structure of a phrase should reflect its semantic import". This viewpoint is also made explicit in [8,15] where the semantics of a language is defined by the unique homomorphism from the initial algebra (i.e., the abstract syntax) to another algebra in the same category.

The mappings from one formalism to another are therefore defined in terms of *functors* between the respective categories. Since the naturality of such constructions, the *adjoint* nature of these functors is then investigated, discussing their semantic implications over the categories of algebras (Sections 3 and 4).

Contributions. The first contribution of this paper is the *categorical equivalence* of three different formalisms for syntax specification. In particular, we prove that some longstanding syntactical transformations between context-free grammars and many-sorted signatures and between many-sorted signatures and order-sorted signatures give rise to adjoint equivalences that preserve the *abstract syntax* of the generated terms (Theorems 1 and 4). Moreover, we broaden some already known results of [8,7,20] and show that the aforementioned syntactical transformations preserve — up to an equivalence[1] — the categories of algebras over the objects in their respective formalisms (Theorems 2, 3, and 5). The conclusion is twofold: Every categorical property and construction can be shifted between these frameworks; and all these formalisms are essentially the same from a semantic perspective.

[1] Some of these equivalences are presented as *isomorphisms of categories*. It is well-known that an isomorphism of categories is a strong notion of categorical equivalence where functors compose to the identity.

2 Formalisms for Language Syntax Specification

In this section, we provide a brief presentation of the three syntax formalisms discussed in the rest of the article. Their technical aspects are deferred to the next subsections.

The most popular formalism to specify languages are *context-free grammars*. They enable language designers to easily handle both abstract and concrete aspects of the syntax by combining terminal symbols with syntactic constituents of the language through production rules. Several definitions of context-free grammars exist in literature [20,16]. Here, we are following [8] (or, the so-called *algebraic grammars* in [16]) and, for the sake of succintness, we sometimes refer to them simply as grammars.

Although grammars are an easy-to-use tool for syntax specification, *signatures* provide a more algebraic approach to language definition. The concept of *many-sorted signature* arose in [10] in order to lift the theory of (full) abstract algebras in case of partially defined operations. From the language syntax perspective, signatures allow the specification of sorted operators, which in turn provide a basis for an algebraic construction of the language semantics. In the rest of the paper, we follow the exposition of [8] and [1] on this subject.

The last formalism considered here are *order-sorted signatures* [7]. They are built upon many-sorted signatures to which they add an explicit treatment of polymorphic operators. Their main aim is to provide a basis on which to develop an algebraic theory to handle several types of polymorphism, multiple inheritance, left inverses of subsort inclusion (retracts), and complete equational deduction.

Basic Notions and Notations. If $f\colon A \to B$ is a function defined by cases, we sometimes use the *conditional operator* $f(a) = \llbracket\, P(a) \mathbin{?} b_1 \mathbin{\S} b_0 \,\rrbracket$ as a shorthand for $f(a) = b_1$ if the predicate P holds for a and b_0 otherwise. If A and B are two sets and $f\colon A \to B$ is a function, we denote by $f^*\colon A^* \to B^*$ the unique *monoid homomorphism* induced by the Kleene closure on the sets A and B extending the function f, i.e., $f^*(a_1 \dots a_n) = f(a_1) \dots f(a_n)$. Given a *set of sorts* S, an *S-sorted set* A is a family of sets indexed by S, i.e., $A = \{A_s | s \in S\}$.[2] If A is an S-sorted set, we denote by $\bigcup A$ the union of all its sorted components, i.e., $\bigcup_{s\in S} A_s$. Similarly, an *S-sorted function* $f\colon A \to B$ is a family of functions $f = \{f_s\colon A_s \to B_s | s \in S\}$ In addition, if A is an S-sorted set and $w = s_1 \dots s_n \in S^+$, we denote by A_w the cartesian product $A_{s_1} \times \cdots \times A_{s_n}$. Likewise, if f is an S-sorted function and $a_i \in A_{s_i}$ for $i \in \{1, \dots, n\}$, then the function $f_w\colon A_w \to B_w$ is such that $f_w(a_1, \dots, a_n) = (f_{s_1}(a_1), \dots, f_{s_n}(a_n))$. Moreover, if $g\colon A \to B$ is a function, we still use the symbol g to denote the *direct image map of* g (also called the *additive lift* of g), i.e., the function $g\colon \wp(A) \to \wp(B)$ such that $g(X) = \{g(a) \in B | a \in X\}$. Analogously, if $\leq$ is a binary relation on a set A (with elements $a \in A$), we use the same relation symbol to denote its *pointwise extension*, i.e., we write $a_1 \dots a_n \leq a'_1 \dots a'_n$ for $a_1 \leq a'_1, \dots, a_n \leq a'_n$.

[2] If the name of an S-sorted set contains a subscript, we shift it to a superscript when denoting its sorted components. For instance, if A_n is an S-sorted set, its elements are denoted by $A^n_{s_1}$, $A^n_{s_2}$, etc.

2.1 Context-Free Grammars

A *context-free grammar* [8] (or, a *CF grammar*) is a triple $G = \langle N, T, P\rangle$, where N is the set of *non-terminal symbols* (or, *non-terminals*), T is the set of *terminal symbols* (or, *terminals*) disjoint from N, and $P \subseteq N \times (N \cup T)^*$ is the set of *production rules* (or, *productions*). If (A, β) is a production in P, we stick to the standard notation $A \to \beta$ (although some authors [20] reverse the order and write $\beta \to A$ to match the signature formalism). If $\alpha, \gamma \in (N \cup T)^*$, $B \in N$, and $B \to \beta \in P$, we define $\alpha B \gamma \Rightarrow \alpha\beta\gamma$ the *one-step reduction relation* on the set $(N \cup T)^*$. The language $\mathcal{L}(G)$ generated by G is the union of the N-sorted family $\mathcal{L}_N(G) = \{\mathcal{L}_A(G) \mid A \in N\}$, i.e., $\mathcal{L}(G) = \bigcup \mathcal{L}_N(G)$, where $\mathcal{L}_A(G) = \{t \in T^* \mid A \Rightarrow^* t\}$ and $\Rightarrow^*$ is the reflexive transitive closure of $\Rightarrow$. The *non-terminals projection* $\mathrm{nt}\colon N \cup T \to N \cup \{\varepsilon\}$ on G is defined by $\mathrm{nt}(x) = (\!| x \in N \mathbin{?} x \mathbin{:} \varepsilon |\!)$. In the following, we implicitly characterize the function nt according to the subscript/superscript of G, namely, if G', G_1, etc. are grammars, we denote by nt', nt_1, etc. their non-terminals projections, respectively.

An *abstract grammar morphism* (henceforth *morphism*, when this terminology does not lead to ambiguities) $f\colon G_1 \to G_2$ is a map between two grammars $G_1 = \langle N_1, T_1, P_1\rangle$ and $G_2 = \langle N_2, T_2, P_2\rangle$ that preserves the abstract structure of the generated strings. Formally, f is a pair of functions $f_0\colon N_1 \to N_2$ and $f_1\colon P_1 \to P_2$ such that $f_1(A \to \beta) = f_0(A) \to \beta' \in P_2$, where $\mathrm{nt}_2^*(\beta') = (f_0 \circ \mathrm{nt}_1)^*(\beta)$.

The identity morphism on an object $G = \langle N, T, P\rangle$ is denoted by $\mathbf{1}_G$ and is such that $(\mathbf{1}_G)_0 = \mathbf{1}_N$ and $(\mathbf{1}_G)_1 = \mathbf{1}_P$. The composition of two grammar morphism $f\colon G_1 \to G_2$ and $g\colon G_2 \to G_3$ is obtained by defining $(g \circ f)_0 = g_0 \circ f_0$ and $(g \circ f)_1 = g_1 \circ f_1$.

Proposition 1. *The class of all grammars and the class of all abstract grammar morphisms form the category* **Grm**.

The following section makes clear the semantic implications that a grammar morphism $f\colon G_1 \to G_2$ induces on the categories of algebras over G_1 and G_2. The insight is that preserving the abstract syntax of G_1 into G_2 ensures the possibility to employ G_2-algebras in order to provide meaning to G_1-terms.

Algebras over a Context-Free Grammar The algebraic approach applied to context-free languages is introduced in [9,15]. The authors exploit the theory of *heterogeneous algebras* [1] to provide semantics for context-free grammars (see also [8]). The algebraic notions that lead to the category of algebras over a context-free grammar are here summarized.

Let $G = \langle N, T, P\rangle$ be a grammar. A *G-algebra* [9,8] is a pair $\mathbb{A} = \langle A, F_\mathbb{A}\rangle$, where A is an N-sorted set of *semantic domains* (or, *carrier sets*) and $F_\mathbb{A} = \big\{[\![C \to \delta]\!]_\mathbb{A}\colon A_{\mathrm{nt}^*(\delta)} \to A_C \mid C \to \delta \in P\big\}$ is a set of *interpretation functions*. A *G-homomorphism* [9,8] $h\colon \mathbb{A} \to \mathbb{B}$ between two G-algebras $\mathbb{A} = \langle A, F_\mathbb{A}\rangle$ and $\mathbb{B} = \langle B, F_\mathbb{B}\rangle$ is an N-sorted function $h\colon A \to B$ such that $[\![C \to \delta]\!]_\mathbb{B} \circ h_{\mathrm{nt}^*(\delta)} = h_C \circ [\![C \to \delta]\!]_\mathbb{A}$ for each production $C \to \delta \in P$.

It is well-known [8,9] that the class of all G-algebras and the class of all G-homomorphisms form a category, denoted by $\mathbf{Alg}(G)$. The *initial object* in

$\mathbf{Alg}(G)$ is the *term algebra* (or, *initial algebra*) and it is denoted by $\mathbb{T}$. Specifically, the carrier sets T_C of $\mathbb{T}$ are inductively defined as the smallest sets such that, if $C \to \delta \in P$ and $\mathrm{nt}^*(\delta) = \varepsilon$, then $C \to \delta \in T_C$, and, if $\mathrm{nt}^*(\delta) = C_1 \dots C_n$ and $t_i \in C_i$ for $i \in \{1, \dots, n\}$, then $C \to \delta\,\mathtt{(}t_1, \dots, t_n\mathtt{)} \in T_C$.[3] Then, the interpretation functions are obtained by defining $[\![C \to \delta]\!]_{\mathbb{T}} = C \to \delta$, if $\mathrm{nt}^*(\delta) = \varepsilon$, and $[\![C \to \delta]\!]_{\mathbb{T}}(t_1, \dots, t_n) = C \to \delta\,\mathtt{(}t_1, \dots, t_n\mathtt{)}$, if $\mathrm{nt}^*(\delta) = C_1 \dots C_n$ and $t_i \in C_i$ for $i \in \{1, \dots, n\}$.

Intuitively, the initial algebra $\mathbb{T}$ carries the terms over G (the programs), and the semantic function $h \colon \mathbb{T} \to \mathbb{A}$ provides the unique meaning of each term t in $\mathbb{T}$ in the algebra $\mathbb{A}$ (the semantics).

We now show the semantic effects that grammar morphisms induce on the respective categories of algebras. Let $G_1 = \langle N_1, T_1, P_1 \rangle$ and $G_2 = \langle N_2, T_2, P_2 \rangle$ be two context-free grammars. Suppose that $f \colon G_1 \to G_2$ is a grammar morphism and let $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ be a G_2-algebra. We can make $\mathbb{A}$ into a G_1-algebra $\xi_f \mathbb{A} = \langle \xi_f A, \xi_f F_{\mathbb{A}} \rangle$ by defining $(\xi_f A)_C = A_{f_0(C)}$ for each $C \in N_1$, and $[\![C \to \delta]\!]_{\xi_f \mathbb{A}} = [\![f_1(C \to \delta)]\!]_{\mathbb{A}}$ for each $C \to \delta \in P_1$. Moreover, if $h \colon \mathbb{A} \to \mathbb{B}$ is a G_2-homomorphism, then $(\xi_f h)_C = h_{f_0(C)}$ is G_1-homomorphism from $\xi_f \mathbb{A}$ to $\xi_f \mathbb{B}$.

Proposition 2. *The map $\xi_f \colon \mathbf{Alg}(G_2) \to \mathbf{Alg}(G_1)$ induced by the abstract grammar morphism $f \colon G_1 \to G_2$ is a functor.*

The next proposition provides an isomorphism between the categories of algebras under a grammar isomorphism.

Proposition 3. *If $f \colon G_1 \to G_2$ is an abstract grammar isomorphism, then*

$$\xi_{f^{-1}} \circ \xi_f = \mathbf{1}_{\mathbf{Alg}(G_1)} \qquad \textit{and} \qquad \xi_f \circ \xi_{f^{-1}} = \mathbf{1}_{\mathbf{Alg}(G_2)}$$

Therefore, $\xi_{f^{-1}} = \xi_f^{-1}$ and hence $\mathbf{Alg}(G_1)$ and $\mathbf{Alg}(G_2)$ are isomorphic.

In other words, isomorphic grammars give rise to isomorphic categories of algebras, implying that f does not lose any (semantic relevant) information.

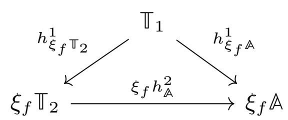
Example 1 (Deriving a Compiler). In this example, we show how a grammar morphism $f \colon G_1 \to G_2$ induces a compiler w.r.t. the semantic functions in $\mathbf{Alg}(G_2)$. Consider the following grammar specifications $G_1 = \langle N_1, T_1, P_1 \rangle$ (left) and $G_2 = \langle N_2, T_2, P_2 \rangle$ (right) in the Backus-Naur form:

$$\mathbb{n} ::= \mathtt{+}\,\mathbb{n}\,\mathbb{n} \mid \mathtt{0} \mid \mathtt{1} \mid \mathtt{2} \mid \cdots \qquad\qquad \mathbb{p} ::= \mathtt{(}\,\mathbb{p}\,\mathtt{+}\,\mathbb{p}\,\mathtt{)} \mid \mathtt{even} \mid \mathtt{odd}$$

(Here, we have just specified the productions; terminals and non-terminals can be easily recovered from such specifications assuming no useless symbols in both sets). Let $f \colon G_1 \to G_2$ be the grammar morphism that maps $\mathbb{n}$ to $\mathbb{p}$, $\mathbb{n} \to \mathtt{+}\,\mathbb{n}\,\mathbb{n}$ to $\mathbb{p} \to \mathtt{(}\,\mathbb{p}\,\mathtt{+}\,\mathbb{p}\,\mathtt{)}$, and each production $\mathbb{n} \to \bar{n}$ to $\mathbb{p} \to \bar{p}$, where $\bar{n} \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \dots\}$ and $\bar{p} = \mathtt{even}$ if $\bar{n}$ represents an even natural number, and $\bar{p} = \mathtt{odd}$ otherwise. Suppose that $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ is the G_2-algebra such that $A_{\mathbb{p}} = \{0, 1\}$, $[\![\mathbb{p} \to \mathtt{even}]\!]_{\mathbb{A}} = 0$, $[\![\mathbb{p} \to \mathtt{odd}]\!]_{\mathbb{A}} = 1$, and

[3] The parentheses that occur in terms definition are not to be intended as those for the function application. For this reason, we use the monospaced font to disambiguate these two different situations.

$[\![\mathbb{p} \to \texttt{(}\,\mathbb{p}\,\texttt{+}\,\mathbb{p}\,\texttt{)}]\!]_{\mathbb{A}}(p_1, p_2) = (p_1 + p_2) \mod 2$. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote the G_1- and G_2-term algebras, respectively. Thanks to the initiality of $\mathbb{T}_2$, there exists a unique homomorphism $h^2_{\mathbb{A}}\colon \mathbb{T}_2 \to \mathbb{A}$, i.e., the language semantics over G_2 to $\mathbb{A}$. Applying the functor ξ_f to $h^2_{\mathbb{A}}$ yields the following commutative diagram† (due to the initiality of $\mathbb{T}_1$) in $\mathbf{Alg}(G_1)$:



† $h^1_{\xi_f \mathbb{T}_2}$ and $h^1_{\xi_f \mathbb{A}}$ are the unique homomorphisms leaving $\mathbb{T}_1$.

In this case, the commutativity has an interesting meaning: $h^1_{\xi_f \mathbb{T}_2}$ is the compiler w.r.t. the semantic function $h^2_{\mathbb{A}}$ under the morphism f. Indeed, for instance, let `+ 5 3` denotes the $\mathbb{T}_1$-term $\mathbb{n} \to \texttt{+}\,\mathbb{n}\,\mathbb{n}(\mathbb{n} \to \texttt{3}, \mathbb{n} \to \texttt{5})$. If we apply the compiler $h^1_{\xi_f \mathbb{T}_2}$ to `+ 5 3`, we obtain a $\mathbb{T}_2$-term which $h^2_{\mathbb{A}}$-semantics agrees with $h^1_{\xi_f \mathbb{A}}$, i.e., $\left(h^1_{\xi_f \mathbb{T}_2}\right)_{\mathbb{n}}(\texttt{+ 5 3}) = \texttt{(odd + odd)}$ where `(odd + odd)` denotes the $\mathbb{T}_2$-term $\mathbb{p} \to \texttt{(}\,\mathbb{p}\,\texttt{+}\,\mathbb{p}\,\texttt{)}(\mathbb{p} \to \texttt{odd}, \mathbb{p} \to \texttt{odd})$, and

$$\left(h^2_{\mathbb{A}}\right)_{\mathbb{p}}(\texttt{(odd + odd)}) = 0 = \left(h^1_{\xi_f \mathbb{A}}\right)_{\mathbb{n}}(\texttt{+ 5 3})$$

2.2 Many-Sorted and Order-Sorted Signatures

A *many-sorted signature* [8] (or, an *MS signature*) is a pair $\mathfrak{S} = \langle S, \Sigma \rangle$, where S is a *set of sorts* and Σ is a disjoint family of sets $\Sigma_{w,s}$ such that $w \in S^*$ and $s \in S$. As in the case of context-free grammars, we suppose that $S \cap \bigcup \Sigma = \varnothing$. If $\sigma \in \Sigma_{w,s}$, we call σ an *operator symbol* (or simply, an *operator*), and we write $\sigma\colon w \to s$ as a shorthand. Moreover, if $w = \varepsilon$, we say that σ is a *constant symbol* (or simply, a *constant*) and we write $\sigma\colon s$ instead of $\sigma\colon \varepsilon \to s$. Finally, given $\sigma\colon w \to s$, we define $\mathrm{ar}(\sigma) = w$ the *arity*, $\mathrm{srt}(\sigma) = s$ the *sort*, $\mathrm{rnk}(\sigma) = (w, s)$ the *rank* of σ.

A *many-sorted signature morphism* $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ is a map between two many-sorted signatures $\mathfrak{S}_1 = \langle S_1, \Sigma_1 \rangle$ and $\mathfrak{S}_2 = \langle S_2, \Sigma_2 \rangle$ that preserves the underlying graph structure[4] of $\mathfrak{S}_1$ in $\mathfrak{S}_2$, in the following sense: f is a pair of functions $f_0\colon S_1 \to S_2$ and $f_1\colon \bigcup \Sigma_1 \to \bigcup \Sigma_2$ such that $f_1(\sigma)\colon f_0^*(w) \to f_0(s)$ in $\mathfrak{S}_2$ for each $\sigma\colon w \to s$ in $\mathfrak{S}_1$.

The identity arrow on $\mathfrak{S} = \langle S, \Sigma \rangle$ is denoted by $\mathbf{1}_{\mathfrak{S}}$ and is such that $(\mathbf{1}_{\mathfrak{S}})_0$ and $(\mathbf{1}_{\mathfrak{S}})_1$ are the set identity functions on their domains, and the composition of two morphisms $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ and $g\colon \mathfrak{S}_2 \to \mathfrak{S}_3$ is obtained by defining $(g \circ f)_0 = g_0 \circ f_0$ and $(g \circ f)_1 = g_1 \circ f_1$, which is trivially a morphism from $\mathfrak{S}_1$ to $\mathfrak{S}_3$.

Proposition 4. *The class of all many-sorted signatures and the class of all many-sorted signature morphisms form the category* **Sig**.

Similarly, we introduce the theory of order-sorted signatures. An *order-sorted signature* [7] (or, an *OS signature*) is a triple $\mathcal{S} = \langle S, \leq, \Sigma \rangle$, where $\langle S, \leq \rangle$ is a poset of sorts and Σ is an $(S^* \times S)$-sorted family of sets $\Sigma_{w,s}$ such that satisfies the following condition: If $\sigma \in \Sigma_{w_1,s_1} \cap \Sigma_{w_2,s_2}$ and $w_1 \leq w_2$, then $s_1 \leq s_2$.

[4] The graph similarity is obtained by considering an operator $\sigma\colon w \to s$ as a σ-labeled edge from w to s.

Note that S and Σ play the same role as before, except for the fact that Σ is no more required to be a disjoint family, thus enabling the definition of polymorphic operators. Furthermore, we extend to the order-sorted signatures the terminology that was introduced for the many-sorted case.

An *order-sorted signature morphism* $f\colon \mathcal{S}_1 \to \mathcal{S}_2$, where $\mathcal{S}_1 = \langle S_1, \leq_1, \Sigma_1\rangle$ and $\mathcal{S}_2 = \langle S_2, \leq_2, \Sigma_2\rangle$, is formed by the two components f_0 and f_1. The former component f_0 is a function between S_1 and S_2. The latter is a family of functions $f_1 = \{f^1_{w,s}\colon \Sigma^1_{w,s} \to \Sigma^2_{f_0^*(w),f_0(s)} \mid w \in S_1^* \wedge s \in S_1\}$ that preserves the sorted structure of the signature.[5] The ordering on the sorts is not preserved by f_0 for the simple reason that it does not play any role in the abstract syntax of the terms (a discussion on this is given when discussing the future works).

The identity morphism $\mathbf{1}_{\mathcal{S}}$ over an order-sorted signature $\mathcal{S}$ is defined by taking $(\mathbf{1}_{\mathcal{S}})_0$ and each component $(\mathbf{1}_{\mathcal{S}})^1_{w,s}$ of $(\mathbf{1}_{\mathcal{S}})_1$ the set-theoretic identities on their domains. The composition $g \circ f$ of two order-sorted signature morphisms $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ and $g\colon \mathcal{S}_2 \to \mathcal{S}_3$ is obtained by defining $(g \circ f)_0 = g_0 \circ f_0$ and $(g \circ f)^1_{w,s} = g^1_{f_0^*(w),f_0(s)} \circ f^1_{w,s}$ for each $w \in S^*$ and $s \in S$.

Proposition 5. *The class of all order-sorted signatures and the class of all order-sorted signature morphisms form the category* $\mathbf{Sig}^{\leq}$.

Algebras over a Signature In this section, we prove the same results developed in Section 2.1 for the classes of algebras over a many-sorted and order-sorted signature. Again, we provide the basic algebraic notions required to build the category of algebras over a given signature, and we redirect the reader to [7] for a thorough exposition of the following concepts.

Many-Sorted Algebra. Let $\mathfrak{S} = \langle S, \Sigma\rangle$ be a many-sorted signature. A *many-sorted $\mathfrak{S}$-algebra* [7] is a pair $\mathbb{A} = \langle A, F_{\mathbb{A}}\rangle$, where A is an S-sorted set of *semantic domains* (or, *carrier sets*) and $F_{\mathbb{A}} = \left\{[\![\sigma]\!]_{\mathbb{A}}\colon A_w \to A_s \mid \sigma \in \Sigma_{w,s}\right\}$ is the set of *interpretation functions* (we use the same terminology adopted for an algebra over a context-free grammar). A *many-sorted $\mathfrak{S}$-homomorphism* [7] $h\colon \mathbb{A} \to \mathbb{B}$ between two many-sorted $\mathfrak{S}$-algebras $\mathbb{A} = \langle A, F_{\mathbb{A}}\rangle$ and $\mathbb{B} = \langle B, F_{\mathbb{B}}\rangle$ is an S-sorted function $h\colon A \to B$ such that $[\![\sigma]\!]_{\mathbb{B}} \circ h_w = h_s \circ [\![\sigma]\!]_{\mathbb{A}}$ for each $\sigma \in \Sigma_{w,s}$. The category of all $\mathfrak{S}$-algebras and $\mathfrak{S}$-homomorphisms is denoted by $\mathbf{Alg}(\mathfrak{S})$. The *many-sorted term $\mathfrak{S}$-algebra* $\mathbb{T}$ is the *initial algebra* in its category (i.e., the *initial object*) and it is obtained in an analogous way to the term algebra over a grammar [7].

Order-Sorted Algebra. If $\mathcal{S} = \langle S, \leq, \Sigma\rangle$ is an order-sorted signature, an *order-sorted $\mathcal{S}$-algebra* [7] is a pair $\mathbb{A} = \langle A, F_{\mathbb{A}}\rangle$ where A is an S-sorted set and $F_{\mathbb{A}} = \{[\![\sigma]\!]^{w,s}_{\mathbb{A}}\colon A_w \to A_s \mid \sigma \in \Sigma_{w,s}\}$. Moreover, the following monotonicity conditions must be satisfied:

(i) $\sigma \in \Sigma_{w_1,s_1} \cap \Sigma_{w_2,s_2}$ and $w_1 \leq w_2$ implies $[\![\sigma]\!]^{w_1,s_1}_{\mathbb{A}}(a) = [\![\sigma]\!]^{w_2,s_2}_{\mathbb{A}}(a)$ for each $a \in A_{w_1}$; and
(ii) $s_1 \leq s_2$ implies $A_{s_1} \subseteq A_{s_2}$.

[5] One can check this definition collapses to that in the many-sorted case when there are no polymorphic operators in Σ.

An *order-sorted $\mathcal{S}$-homomorphism* $h\colon \mathbb{A} \to \mathbb{B}$ between two order-sorted $\mathcal{S}$-algebras $\mathbb{A} = \langle A, F_{\mathbb{A}}\rangle$ and $\mathbb{B} = \langle B, F_{\mathbb{B}}\rangle$ is an S-sorted function $h\colon A \to B$ such that

(i) $[\![\sigma]\!]^{w,s}_{\mathbb{B}} \circ h_w = h_s \circ [\![\sigma]\!]^{w,s}_{\mathbb{A}}$ for each $\sigma \in \Sigma_{w,s}$; and

(ii) $s_1 \leq s_2$ implies $h_{s_1}(a) = h_{s_2}(a)$ for each $a \in A_{s_1}$.

The category formed by $\mathcal{S}$-algebras and $\mathcal{S}$-homomorphisms is denoted by $\mathbf{Alg}(\mathcal{S})$. The *order-sorted term $\mathcal{S}$-algebra* $\mathbb{T}$ is guaranteed to be initial only if $\mathcal{S}$ is *regular* (see [7] for the regularity definition and for details on the construction of $\mathbb{T}$ in the order-sorted case).

We now have all the elements to show the semantic effects induced by a many-sorted signature morphism $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$, where $\mathfrak{S}_1 = \langle S_1, \Sigma_1\rangle$ and $\mathfrak{S}_2 = \langle S_2, \Sigma_2\rangle$. As in the case of context-free grammars, we can build a mapping from the category of algebras $\mathbf{Alg}(\mathfrak{S}_2)$ to $\mathbf{Alg}(\mathfrak{S}_1)$, in order to employ $\mathfrak{S}_2$-algebras to provide meaning to $\mathfrak{S}_1$-terms: Let $\mathbb{A} = \langle A, F_{\mathbb{A}}\rangle$ be an $\mathfrak{S}_2$-algebra. We can make $\mathbb{A}$ to a $\mathfrak{S}_1$-algebra $\zeta_f\mathbb{A} = \langle \zeta_f A, \zeta_f F_{\mathbb{A}}\rangle$ by defining $(\zeta_f\mathbb{A})_s = A_{f_0(s)}$ for each $s \in S_1$ and $[\![\sigma]\!]_{\zeta_f\mathbb{A}} = [\![f_1(\sigma)]\!]_{\mathbb{A}}$ for each $\sigma \in \bigcup \Sigma_1$. Moreover, given a $\mathfrak{S}_2$-homomorphism $h\colon \mathbb{A} \to \mathbb{B}$, we can define the $\mathfrak{S}_1$-homomorphism $\zeta_f h\colon \zeta_f\mathbb{A} \to \zeta_f\mathbb{B}$ such that $(\zeta_f h)_s = h_{f_0(s)}$. The very same construction can be applied to the order-sorted case, namely, if $g\colon \mathcal{S}_1 \to \mathcal{S}_2$ is an order-sorted signature morphism, the map $\psi_g\colon \mathbf{Alg}(\mathcal{S}_2) \to \mathbf{Alg}(\mathcal{S}_1)$ is defined analogously to ζ_f.

Proposition 6. *The maps $\zeta_f\colon \mathbf{Alg}(\mathfrak{S}_2) \to \mathbf{Alg}(\mathfrak{S}_1)$ and $\psi_g\colon \mathbf{Alg}(\mathcal{S}_2) \to \mathbf{Alg}(\mathcal{S}_1)$ induced by the signature morphisms $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ and $g\colon \mathcal{S}_1 \to \mathcal{S}_2$, respectively, are functors.*

Again, we can prove that isomorphic signatures lead to isomorphic categories of algebras:

Proposition 7. *If $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ and $g\colon \mathcal{S}_1 \to \mathcal{S}_2$ are isomorphism, then*

$$\zeta_{f^{-1}} \circ \zeta_f = \mathbf{1}_{\mathbf{Alg}(\mathfrak{S}_1)} \qquad \psi_{g^{-1}} \circ \psi_g = \mathbf{1}_{\mathbf{Alg}(\mathcal{S}_1)}$$
$$\zeta_f \circ \zeta_{f^{-1}} = \mathbf{1}_{\mathbf{Alg}(\mathfrak{S}_2)} \qquad \psi_g \circ \psi_{g^{-1}} = \mathbf{1}_{\mathbf{Alg}(\mathcal{S}_2)}$$

Therefore, $\zeta_{f^{-1}} = \zeta_f^{-1}$ and $\psi_{g^{-1}} = \psi_g^{-1}$, and thus ζ_f and ψ_g are isomorphisms.

3 Equivalence between MS Signatures and CF Grammars

In this section, we generalize the results of [8] by proving the conversion of a grammar into a signature and vice versa can be extended to functors that give rise to an *adjoint equivalence* between **Grm** and **Sig**. The major benefit of such new development is the preservation of all the categorical properties (such as initiality, limits, colimits, ...) from **Grm** to **Sig**, and vice versa. A concrete example is provided at the end of the section.

The map $\Delta\colon \mathbf{Grm} \to \mathbf{Sig}$ transforms a grammar $G = \langle N, T, P\rangle$ to the signature $\Delta G = \langle S_G, \Sigma_G\rangle$, where $S_G = N$ and $\Sigma^G_{w,s} = \{A \to \beta \in P \mid A = s \wedge \mathrm{nt}^*(\beta) = w\}$, and a grammar morphism $f\colon G_1 \to G_2$ to the signature morphism Δf such that $(\Delta f)_0 = f_0$ and $(\Delta f)_1 = f_1$.

Proposition 8. $\varDelta\colon \mathbf{Grm} \to \mathbf{Sig}$ *is a functor.*

Similarly, we define $\nabla\colon \mathbf{Sig} \to \mathbf{Grm}$ that maps objects and arrows between the specified categories. The conversion of a signature $\mathfrak{S} = \langle S, \varSigma\rangle$ to a grammar $\nabla\mathfrak{S} = \langle N_{\mathfrak{S}}, T_{\mathfrak{S}}, P_{\mathfrak{S}}\rangle$ is obtained by defining $N_{\mathfrak{S}} = S$, $T_{\mathfrak{S}} = \bigcup \varSigma$, and $P_{\mathfrak{S}} = \{s \to \sigma w \mid \sigma \in \varSigma_{w,s}\}$, while a signature morphism $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ is mapped to the grammar morphism ∇f such that $(\nabla f)_0 = f_0$ and $(\nabla f)_1(s \to \sigma w) = f_0(s) \to f_1(\sigma) f_0^*(w)$.

Proposition 9. $\nabla\colon \mathbf{Sig} \to \mathbf{Grm}$ *is a functor.*

As underlined in [20], $\varDelta$ and ∇ are not isomorphisms. Indeed, in general, $\mathfrak{S} \neq \varDelta\nabla\mathfrak{S}$ and $G \neq \nabla\varDelta G$, and thus $\varDelta\nabla \neq \mathbf{1}_{\mathbf{Sig}}$ and $\nabla\varDelta \neq \mathbf{1}_{\mathbf{Grm}}$. However, as we prove in the next two propositions, there are natural isomorphisms η and ϵ^{-1} that transform the identity functors $\mathbf{1}_{\mathbf{Sig}}$ and $\mathbf{1}_{\mathbf{Grm}}$ to $\varDelta\nabla$ and $\nabla\varDelta$, respectively. It follows that $\mathfrak{S} \cong \varDelta\nabla\mathfrak{S}$ and $G \cong \nabla\varDelta G$ (where $\cong$ means *is isomorphic to*).

Let $\mathfrak{S} = \langle S, \varSigma\rangle$ be a many-sorted signature. We denote by $\eta_{\mathfrak{S}}\colon \mathfrak{S} \to \varDelta\nabla\mathfrak{S}$ the signature morphism defined by $(\eta_{\mathfrak{S}})_0 = \mathbf{1}_S$ and $(\eta_{\mathfrak{S}})_1(\sigma) = \mathrm{srt}(\sigma) \to \sigma\, \mathrm{ar}(\sigma)$. Since in the many-sorted case the arity and the rank are fully determined by the operator ($\varSigma$ is a disjoint family of sets) the previous function is well-defined.

Proposition 10. $\eta\colon \mathbf{1}_{\mathbf{Sig}} \Rightarrow \varDelta\nabla$ *is a natural isomorphism.*

Similarly, let $G = \langle N, T, P\rangle$ be a context-free grammar. We denote by $\epsilon_G\colon \nabla\varDelta G \to G$ the grammar morphism defined by $(\epsilon_G)_0 = \mathbf{1}_N$ and $(\epsilon_G)_1\big(A \to (A,\beta)\,\mathrm{nt}^*(\beta)\big) = A \to \beta$.[6]

Proposition 11. $\epsilon\colon \nabla\varDelta \Rightarrow \mathbf{1}_{\mathbf{Grm}}$ *is a natural isomorphism.*

The previous results suggest to study if ∇ and $\varDelta$ form an adjunction.

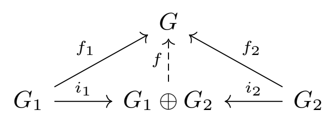
Theorem 1. ∇ *is left adjoint to* $\varDelta$ *and* (ϵ, η) *are the counit and the unit of the adjunction* $(\nabla, \varDelta, \epsilon, \eta)$.

Corollary 1. $(\nabla, \varDelta, \epsilon, \eta)$ *is an adjoint equivalence.*

Theorem 1 implies that **Grm** and **Sig** are identical except for the fact that each category may have different numbers of isomorphic copies of the same object. A particularly relevant consequence of this result is that we can move categorical limits between **Grm** and **Sig**. The next example provides a definition of *coproduct* in **Grm** able to recognize the union of two context-free languages. As a consequence of Theorem 1, we achieve for free the same concept in **Sig**.

[6] Note that the productions in $P_{\varDelta G}$ are formed from those in P, i.e., $P_{\varDelta G} = \{A \to (A,\beta)\,\mathrm{nt}^*(\beta) \mid A \to \beta \in P\}$. Therefore, when considering a general production in $P_{\varDelta G}$ derived from $A \to \beta$ in P, we write $A \to (A,\beta)\,\mathrm{nt}^*(\beta)$ instead of $A \to A \to \beta\,\mathrm{nt}^*(\beta)$ to avoid any confusion.

Example 2 (Coproduct Preservation). Suppose to have the following notion of categorical coproduct in **Grm**: Given two context-free grammars $G_1 = \langle N_1, T_1, P_1 \rangle$ and $G_2 = \langle N_2, T_2, P_2 \rangle$, the coproduct of G_1 and G_2 is defined by $G_1 \oplus G_2 = \langle N_1 \uplus N_2, T_1 \uplus T_2, P_1 \uplus P_2 \rangle$, where $\uplus$ is the disjoint union of sets. The inclusion morphism $i_k \colon G_k \to G_1 \oplus G_2$ for $k \in \{1, 2\}$ are defined by $(i_k)_0 = \mathbf{1}_{N_k}$ and $(i_k)_1 = \mathbf{1}_{P_k}$. Given two morphisms $f_1 \colon G_1 \to G$ and $f_2 \colon G_2 \to G$, where G is a context-free grammar, one can check that the unique morphism f that makes the following diagram commute



is obtained by defining $f_0(n) = \llparenthesis\, n \in N_1 \mathbin{?} (f_1)_0(n) \mathbin{:} (f_2)_0(n) \,\rrparenthesis$ and $f_1(A \to \beta) = \llparenthesis\, A \to \beta \in P_1 \mathbin{?} (f_1)_1(A \to \beta) \mathbin{:} (f_2)_1(A \to \beta) \,\rrparenthesis$. The term algebra over $G_1 \oplus G_2$ carries terms both in G_1 and G_2 and recognizes the (disjoint) union of the languages over G_1 and G_2. Since $(\nabla, \Delta, \epsilon, \eta)$ is an adjoint equivalence, then so is $(\Delta, \nabla, \eta^{-1}, \epsilon^{-1})$. Therefore, Δ is left adjoint to ∇ and hence it preserves colimits. Since a coproduct is a colimit, $\Delta(G_1 \oplus G_2)$ is the coproduct of ΔG_1 with ΔG_2 in **Sig**.

3.1 Semantic Equivalence

As mentioned in the introduction, [8] proves an equivalence between the many-sorted term ΔG-algebra $\mathbb{T}_{\Delta G}$ and the initial algebra $\mathbb{T}_G$ over each grammar G. We extend this result to the whole categories of algebras $\mathbf{Alg}(G)$ and $\mathbf{Alg}(\Delta G)$.

Let $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ be a G-algebra. Then, we map $\mathbb{A}$ to the many-sorted ΔG-algebra $\mathbb{A}^{\uparrow} = \langle A^{\uparrow}, F_{\mathbb{A}^{\uparrow}} \rangle$ such that $A^{\uparrow}_N = A_s$ for each $N \in S_G$ and $[\![C \to \delta]\!]_{\mathbb{A}^{\uparrow}} = [\![C \to \delta]\!]_{\mathbb{A}}$ for each $C \to \delta \in \bigcup \Sigma_G$ (we recall that operators in ΔG are productions in G). Furthermore, given a G-homomorphism $h \colon \mathbb{A} \to \mathbb{B}$, we define the ΔG-homomorphism $h^{\uparrow} \colon \mathbb{A}^{\uparrow} \to \mathbb{B}^{\uparrow}$ such that $h^{\uparrow}_N = h_N$.

Conversely, let $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ be a ΔG-algebra. Then, we define the inverse construction that maps $\mathbb{A}$ to the G-algebra $\mathbb{A}^{\downarrow} = \langle A^{\downarrow}, F_{\mathbb{A}^{\downarrow}} \rangle$ such that $A^{\downarrow}_s = A_s$ and $[\![C \to \delta]\!]_{\mathbb{A}^{\downarrow}} = [\![C \to \delta]\!]_{\mathbb{A}}$. Moreover, if $h \colon \mathbb{A} \to \mathbb{B}$ is a ΔG-homomorphism, then $h^{\downarrow} \colon \mathbb{A}^{\downarrow} \to \mathbb{B}^{\downarrow}$ such that $h^{\downarrow}_s = h_s$ is a proper G-homomorphism.

Theorem 2. *The inverse of $(\_)^{\uparrow}$ is $(\_)^{\downarrow}$, therefore they form an isomorphism of categories between $\mathbf{Alg}(G)$ and $\mathbf{Alg}(\Delta G)$.*

Since an isomorphism of categories is a strict notion of categorical equivalence, it preserves the initial objects, and thus we have exactly the result of [8] by applying $(\_)^{\uparrow}$ and $(\_)^{\downarrow}$ to the initial algebras, i.e., $\mathbb{T}^{\uparrow}_G = \mathbb{T}_{\Delta G}$ and $\mathbb{T}^{\downarrow}_{\Delta G} = \mathbb{T}_G$.

In a similar manner, we can extend the other result of [8], i.e., the equivalence between each many-sorted term $\mathfrak{S}$-algebra $\mathbb{T}_{\mathfrak{S}}$ and the initial algebra $\mathbb{T}_{\nabla \mathfrak{S}}$ over the context-free grammar $\nabla \mathfrak{S}$.

Let $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ be a many-sorted $\mathfrak{S}$-algebra. We define the $\nabla \mathfrak{S}$-algebra ${}^{\uparrow}\mathbb{A} = \langle {}^{\uparrow}\mathbb{A}, F_{{}^{\uparrow}\mathbb{A}} \rangle$ where ${}^{\uparrow}\mathbb{A}_s = A_s$ for each $s \in N_{\mathfrak{S}}$ and $[\![s \to \sigma w]\!]_{{}^{\uparrow}\mathbb{A}} = [\![\sigma]\!]_{\mathbb{A}}$ for each $s \to \sigma w \in P_{\mathfrak{S}}$. The conversion of a $\mathfrak{S}$-homomorphism $h \colon \mathbb{A} \to \mathbb{B}$ to a $\nabla \mathfrak{S}$-homomorphism ${}^{\uparrow}h \colon {}^{\uparrow}\mathbb{A} \to^{\uparrow} \mathbb{B}$ is analogous to the previous case.

On the contrary, if $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ is a $\nabla\mathfrak{S}$-algebra and $h\colon \mathbb{A} \to \mathbb{B}$ is a $\nabla\mathfrak{S}$-homomorphism, we can obtain a many-sorted $\mathfrak{S}$-algebra ${}^{\downarrow}\mathbb{A}$ and an $\mathfrak{S}$-homomorphism ${}^{\downarrow}h\colon {}^{\downarrow}\mathbb{A} \to^{\downarrow} \mathbb{B}$ by simply inverting the previous construction.

Theorem 3. *The inverse of* ${}^{\uparrow}(\_)$ *is* ${}^{\downarrow}(\_)$*, therefore they form an isomorphism of categories between* $\mathbf{Alg}(\mathfrak{S})$ *and* $\mathbf{Alg}(\nabla\mathfrak{S})$*.*

Again, the result of [8] is a special case of this last theorem by noting that ${}^{\uparrow}\mathbb{T}_{\mathfrak{S}} = \mathbb{T}_{\nabla\mathfrak{S}}$ and ${}^{\downarrow}\mathbb{T}_{\nabla\mathfrak{S}} = \mathbb{T}_{\mathfrak{S}}$.

Example 3 (Example 2 Continued). In the Example 2, we have shown how to preserve categorical constructions between **Grm** and **Sig**. Theorems 2 and 3 can be applied on the top of Theorem 1 to ensure the semantic equivalence of the achieved constructions. For instance, if the $(G_1 \oplus G_2)$-algebra $\mathbb{A}$ provides the semantics of the disjoint union of languages over G_1 and G_2, then $\mathbb{A}^{\uparrow}$ provides the equivalent semantics in the category $\mathbf{Alg}(\Delta(G_1 \oplus G_2))$, as a consequence of Theorem 2.

4 Equivalence between MS Signatures and OS Signatures

In this section, we show that analogous results of those in Section 3 hold for many-sorted and order-sorted signature transformations Λ and V.

The map $\Lambda\colon \mathbf{Sig}^{\leq} \to \mathbf{Sig}$ converts an order-sorted signature $\mathcal{S} = \langle S, \leq, \Sigma \rangle$ to the many-sorted signature $\mathfrak{S}_{\mathcal{S}} = \langle S_{\mathcal{S}}, \Sigma_{\mathcal{S}} \rangle$ defined by $S_{\mathcal{S}} = S$ and $\Sigma^{\mathcal{S}}_{w,s} = \{\sigma_{w,s} \mid \sigma \in \Sigma_{w,s}\}$ (such a construction is provided in [7]). The transformation of an order-sorted signature morphism $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ to a many-sorted signature morphism $\Lambda f\colon \Lambda\mathcal{S}_1 \to \Lambda\mathcal{S}_2$ is obtained by defining $(\Lambda f)_0 = f_0$ and $(\Lambda f)_1(\sigma_{w,s}) = \left(f^1_{w,s}(\sigma)\right)_{f_0^*(w), f_0(s)}$.

Proposition 12. $\Lambda\colon \mathbf{Sig}^{\leq} \to \mathbf{Sig}$ *is a functor.*

Similarly, the map $\mathrm{V}\colon \mathbf{Sig} \to \mathbf{Sig}^{\leq}$ maps the many-sorted signature $\mathfrak{S} = \langle S, \Sigma \rangle$ to the order-sorted signature $\mathcal{S}_{\mathfrak{S}} = \langle S_{\mathfrak{S}}, \leq_{\mathfrak{S}}, \Sigma_{\mathfrak{S}} \rangle$, where $S_{\mathfrak{S}} = S$, $\leq_{\mathfrak{S}}$ is the reflexive binary relation on S, and $\Sigma^{\mathfrak{S}}_{w,s} = \Sigma_{w,s}$. Moreover, if $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ is a many-sorted signature morphism, then $\mathrm{V}f\colon \mathrm{V}\mathfrak{S}_1 \to \mathrm{V}\mathfrak{S}_2$ defined by $(\mathrm{V}f)_0 = f_0$ and $(\mathrm{V}f)^1_{w,s} = f_1|_{\Sigma_{w,s}}$ is an order-sorted signature morphism.

Proposition 13. $\mathrm{V}\colon \mathbf{Sig} \to \mathbf{Sig}^{\leq}$ *is a functor.*

As before, we can provide natural isomorphisms $\varphi\colon\ \mathbf{1_{Sig}} \to \Lambda\mathrm{V}$ and $\vartheta\colon \mathrm{V}\Lambda \to \mathbf{1_{Sig^{\leq}}}$. Let $\mathfrak{S} = \langle S, \Sigma \rangle$ be a many-sorted signature. Then, the $\mathfrak{S}$-component $\varphi_{\mathfrak{S}}$ of φ is defined by $(\varphi_{\mathfrak{S}})_0 = \mathbf{1}_S$ and $(\varphi_{\mathfrak{S}})_1(\sigma) = \sigma_{\mathrm{ar}(\sigma),\mathrm{srt}(\sigma)}$.

Proposition 14. $\varphi\colon\ \mathbf{1_{Sig}} \Rightarrow \Lambda\mathrm{V}$ *is a natural isomorphism.*

Conversely, if $\mathcal{S} = \langle S, \leq, \Sigma \rangle$ is an order-sorted signature, then the $\mathcal{S}$-component $\vartheta_{\mathcal{S}}$ of ϑ is obtained by defining $(\vartheta_{\mathcal{S}})_0 = \mathbf{1}_S$ and $(\vartheta_{\mathcal{S}})^1_{w,s}(\sigma_{w,s}) = \sigma$.

Proposition 15. $\vartheta\colon \mathrm{V}\Lambda \Rightarrow \mathbf{1_{Sig^{\leq}}}$ *is a natural isomorphism.*

Theorem 4. V *is left adjoint to* Λ *and* (ϑ, φ) *are the counit and the unit of the adjunction* $(\mathrm{V}, \Lambda, \vartheta, \varphi)$.

Corollary 2. $(\mathrm{V}, \Lambda, \vartheta, \varphi)$ *is an adjoint equivalence.*

The functor Λ gives rise to an equivalence between the categories of algebras $\mathbf{Alg}(\mathcal{S})$ and $\mathbf{Alg}(\Lambda\mathcal{S})$ [7]. We now extend such a result to its left adjoint V.

Let $\mathbb{A} = \langle A, F_{\mathbb{A}} \rangle$ be a many-sorted $\mathfrak{S}$-algebra. We define the order-sorted $\mathrm{V}\mathfrak{S}$-algebra $\mathbb{A}_{\uparrow} = \langle A_{\uparrow}, F_{\mathbb{A}_{\uparrow}} \rangle$ such that $(A_{\uparrow})_s = A_s$ and $[\![\sigma]\!]^{w,s}_{\mathbb{A}_{\uparrow}} = [\![\sigma]\!]_{\mathbb{A}}$. Moreover, if $h\colon \mathbb{A} \to \mathbb{B}$ is an $\mathfrak{S}$-homomorphism, then $h_{\uparrow}\colon \mathbb{A}_{\uparrow} \to \mathbb{B}_{\uparrow}$ is the $\Lambda\mathfrak{S}$-homomorphism defined by $(h_{\uparrow})_s = h_s$. Furthermore, we denote by $(\_)_{\downarrow}$ the inverse functor that maps $\Lambda\mathfrak{S}$-algebras and $\Lambda\mathfrak{S}$-homomorphism to the category $\mathbf{Alg}(\mathfrak{S})$.

Theorem 5. *The inverse of* $(\_)_{\uparrow}$ *is* $(\_)_{\downarrow}$*, therefore they form an isomorphism of categories between* $\mathbf{Alg}(\mathfrak{S})$ *and* $\mathbf{Alg}(\mathrm{V}\mathfrak{S})$.

5 Discussion and Concluding Remarks

We briefly discuss the obtained results with a view to providing future works.

On the Compositionality of the Results. An immediate but important consequence of the underlying categorical model is the compositional nature of the proved results. Indeed, we can get a free equivalence between the category of grammars **Grm** and the category of order-sorted signatures $\mathbf{Sig}^{\leq}$ by composing $\mathrm{V}\Delta$ and $\nabla\Lambda$. The algebraic counterpart of the same observation allows us to claim that the composition of the functors $(\_)_{\downarrow} \circ (\_)^{\uparrow}$ gives rise to an isomorphism between $\mathbf{Alg}(G)$ and $\mathbf{Alg}(\mathrm{V}\Delta G)$ (and, of course, the dual result holds).

Future Works. Future works concern refinements of the syntactical transformations between the formalisms in order to preserve specific properties of the concrete syntax [11]. Among them, *polymorphism* seems the most interesting. Unfortunately, the composition of functors $\mathrm{V}\Delta$ and $\nabla\Lambda$ yields non-polymorphic set of operators. Another future work goes in the direction of providing syntactical transformation from **Grm** to $\mathbf{Sig}^{\leq}$ that yields only *regular* (see [7]) order-sorted signatures. Then, studying the adjoint of such a transformation could provide an interesting notion of regularity in the category of grammars that may be employed to weaken the notion of *ambiguity*.

Conclusion. In this paper, we have provided a categorical model of three different syntax formalisms (context-free grammars, many-sorted signatures and order-sorted signatures). We have shown how the extension to functors of already existing syntactical transformations gives rise to categorical equivalences that preserve the abstract syntax of the generated terms. Moreover, we have proved that the categories of algebras over the objects in these formalisms are isomorphic up to the provided transformations.

References

1. Birkhoff, G., Lipson, J.: Heterogeneous algebras. Journal of Combinatorial Theory **8**(1), 115 – 133 (1970)
2. Chang, C.C., Keisler, H.J.: Model theory, vol. 73. Elsevier (1990)
3. Chomsky, N.: Three models for the description of language. IRE Transactions on Information Theory **2**(3), 113–124 (1956)
4. Cohn, P.M.: Universal algebra, vol. 159. Reidel Dordrecht (1981)
5. Earley, J.: An efficient context-free parsing algorithm. Communications of the ACM **13**(2), 94–102 (1970)
6. Fiore, M., Plotkin, G., Turi, D.: Abstract syntax and variable binding. In: Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science. pp. 193–202. LICS, IEEE Computer Society, Washington, DC, USA (1999)
7. Goguen, J.A., Meseguer, J.: Order-sorted algebra i: Equational deduction for multiple inheritance, overloading, exceptions and partial operations. Theoretical Computer Science **105**(2), 217–273 (1992)
8. Goguen, J.A., Thatcher, J.W., Wagner, E.G., Wright, J.B.: Initial algebra semantics and continuous algebras. Journal of the ACM **24**(1), 68–95 (1977)
9. Hatcher, W.S., Rus, T.: Context-free algebras. Cybernetics and System **6**(1-2), 65–77 (1976)
10. Higgins, P.J.: Algebras with a scheme of operators. Mathematische Nachrichten **27**(12), 115–132 (1963)
11. Hopcroft, J.E., Motwani, R., Ullman, J.D.: Introduction to Automata Theory, Languages, and Computation (3rd Edition). Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA (2006)
12. Knuth, D.E.: Semantics of context-free languages. Mathematical systems theory **2**(2), 127–145 (1968)
13. Lee, L.: Fast context-free grammar parsing requires fast boolean matrix multiplication. Journal of the ACM **49**(1), 1–15 (2002)
14. McCarthy, J.: Towards a Mathematical Science of Computation, pp. 35–56. Springer Netherlands, Dordrecht (1993)
15. Rus, T.: Context-free algebra: a mathematical device for compiler specification. In: International Symposium on Mathematical Foundations of Computer Science. pp. 488–494. Springer (1976)
16. Rus, T., Jones, J.S.: Multi-layered pipeline parsing from multi-axiom grammars. Algebraic Methods in Language Processing **95**, 65–81 (1995)
17. Scott, M.L.: Programming Language Pragmatics. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA (2000)
18. Valiant, L.G.: General context-free recognition in less than cubic time. Journal of Computer and System Sciences **10**(2), 308–315 (1975)
19. Visser, E.: Syntax definition for language prototyping. Ponsen & Looijen (1997)
20. Visser, E.: Polymorphic syntax definition. Theoretical Computer Science **199**(1-2), 57–86 (1998)

A Proofs

In this appendix, we provide the proofs only for the non-trivial theorems and propositions.

Proposition 8. $\Delta\colon \mathbf{Grm} \to \mathbf{Sig}$ *is a functor.*

Proof. The only non-trivial fact in the proof is checking that Δf satisfies the signature morphism condition: Let $G_1 = \langle N_1, T_1, P_1\rangle$ and $G_2 = \langle N_2, T_2, P_2\rangle$ be two context-free grammars, and let $f\colon G_1 \to G_2$ be a grammar morphism. If $\Delta G_1 = \langle S_{G_1}, \Sigma_{G_1}\rangle$ and $\Delta G_2 = \langle S_{G_2}, \Sigma_{G_2}\rangle$, then given $A \to \beta\colon\ \mathrm{nt}_1^*(\beta) \to A$ in Σ_{G_1} holds that

$$(\Delta f)_1(A \to \beta) = f_1(A \to \beta) = f_0(A) \to \beta' \quad \text{where} \quad (f_0 \circ \mathrm{nt}_1)^* = \mathrm{nt}_2^*(\beta')$$

and therefore

$$\begin{aligned}(\Delta f)_1(A \to \beta)\colon\ & \mathrm{nt}_2^*(\beta') \to f_0(A) && \triangleright \text{ by definition of } \Sigma_{G_2}\\ :\ & (f_0 \circ \mathrm{nt}_1)^*(\beta) \to f_0(A) && \triangleright\ f \text{ is a grammar morphism}\\ :\ & (\Delta f)_0^*(\mathrm{nt}_1^*(\beta)) \to (\Delta f)_0(A) && \triangleright \text{ by definition of } (\Delta f)_0\end{aligned}$$

Hence Δf is a proper signature morphism from ΔG_1 to ΔG_2. Moreover, it is easy to prove that Δ satisfies the functor axioms.

Proposition 9. $\nabla\colon \mathbf{Sig} \to \mathbf{Grm}$ *is a functor.*

Proof. We show that ∇f yields a proper grammar morphism: Let $\mathfrak{S}_1 = \langle S_1, \Sigma_1\rangle$ and $\mathfrak{S}_2 = \langle S_2, \Sigma_2\rangle$ be two many-sorted signatures, and let $f\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ be a signature morphism. If $\nabla\mathfrak{S}_1 = \langle N_{\mathfrak{S}_1}, T_{\mathfrak{S}_1}, P_{\mathfrak{S}_1}\rangle$ and $\nabla\mathfrak{S}_2 = \langle N_{\mathfrak{S}_2}, T_{\mathfrak{S}_2}, P_{\mathfrak{S}_2}\rangle$, and if nt_{∇_1} and nt_{∇_2} denote the non-terminals projections on $\nabla\mathfrak{S}_1$ and $\nabla\mathfrak{S}_2$, respectively, then $(\nabla f)_1(s \to \sigma w) = f^*(s \to \sigma w) = f_0(s) \to f_1(\sigma) f_0^*(w)$ for each $s \to \sigma w \in P_{\mathfrak{S}_1}$. Since the following chain of equalities holds

$$\mathrm{nt}_{\nabla_2}^*(f_1(\sigma) f_0^*(w)) = f_0^*(w) = f_0^*(\mathrm{nt}_{\nabla_1}^*(\sigma w)) = (f_0 \circ \mathrm{nt}_{\nabla_1})^*(\sigma w)$$

then ∇f is a grammar morphism from $\nabla\mathfrak{S}_1$ to $\nabla\mathfrak{S}_2$. Proving that ∇ satisfies the functor axioms is easy and omitted.

Proposition 10. $\eta\colon\ \mathbf{1}_{\mathbf{Sig}} \Rightarrow \Delta\nabla$ *is a natural isomorphism.*

Proof. Let $\mathfrak{S} = \langle S, \Sigma\rangle$ be a many-sorted signature and let $\sigma\colon w \to s$ in $\mathfrak{S}$. Thus, $(\eta_{\mathfrak{S}})_1(\sigma) = s \to \sigma w$ has the same rank of σ. Since $(\eta_{\mathfrak{S}})_0$ is the identity on the set of sorts, $\eta_{\mathfrak{S}}$ satisfies the signature morphism condition. Moreover, it is easy to prove that each component $\eta_{\mathfrak{S}}$ is an isomorphism in **Sig** by defining its inverse $\eta_{\mathfrak{S}}^{-1}$ as $(\eta_{\mathfrak{S}}^{-1})_0 = \mathbf{1}_S$ and $(\eta_{\mathfrak{S}}^{-1})_1(s \to \sigma w) = \sigma$. We complete the proof by showing that the following diagram commutes for each signature morphism $f\colon \mathfrak{S} \to \mathfrak{S}'$:

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}' \\ \Big\downarrow{\scriptstyle \eta_{\mathfrak{S}}} & & \Big\downarrow{\scriptstyle \eta_{\mathfrak{S}'}} \\ \Delta\nabla\mathfrak{S} & \xrightarrow[\Delta\nabla f]{} & \Delta\nabla\mathfrak{S}' \end{array}$$

The 0-th components of the morphisms in the diagram trivially commute. As regards the 1-th components, they commute if and only if $(\eta_{\mathfrak{S}'})_1(f_1(\sigma)) = (\Delta\nabla f)_1\big((\eta_{\mathfrak{S}})_1(\sigma)\big)$ for each $\sigma \in \Sigma_{w,s}$:

$$\begin{aligned} (\eta_{\mathfrak{S}'})_1(f_1(\sigma)) &= f_0(s) \to f_1(\sigma) f_0^*(w) && \triangleright\ f_1(\sigma)\colon f_0(w) \to f_0(s) \\ &= (\Delta\nabla f)_1(s \to \sigma w) && \triangleright\ (\Delta\nabla f)_1 = (\nabla f)_1 \\ &= (\Delta\nabla f)_1\big((\eta_{\mathfrak{S}})_1(\sigma)\big) && \triangleright\ \sigma\colon w \to s \end{aligned}$$

and hence the thesis.

Proposition 11. *$\epsilon\colon \nabla\Delta \Rightarrow \mathbf{1}_{\mathbf{Grm}}$ is a natural isomorphism.*

Proof. Let $G = \langle N, T, P\rangle$ be a context-free grammar and let $A \to (A,\beta)\,\mathrm{nt}^*(\beta) \in P_{\Delta G}$. Then,

$$(\epsilon_G)_1(A \to (A,\beta)\,\mathrm{nt}^*(\beta)) = A \to \beta \qquad \text{and} \qquad (\epsilon_G)_0(A) = A$$

and

$$((\epsilon_G)_0 \circ \mathrm{nt}_{\nabla\Delta})^*((A,\beta)\,\mathrm{nt}^*(\beta)) = \mathrm{nt}^*(\beta)$$

where $\mathrm{nt}_{\nabla\Delta}$ is the non-terminals mapping on $\nabla\Delta G$. Thus, ϵ_G is a proper grammar morphism. Moreover, ϵ_G is an isomorphism in **Grm**: Let ϵ_G^{-1} denotes its inverse defined by

$$(\epsilon_G^{-1})_0 = \mathbf{1}_N \qquad \text{and} \qquad (\epsilon_G^{-1})_1(A \to \beta) = A \to (A,\beta)\,\mathrm{nt}^*(\beta)$$

Now one can check that $\epsilon_G \circ \epsilon_G^{-1} = \mathbf{1}_G$ and $\epsilon_G^{-1} \circ \epsilon_G = \mathbf{1}_{\nabla\Delta G}$. In order to prove the thesis, we show the commutativity of the following diagram for each grammar morphism $f\colon G \to G'$:

$$\begin{array}{ccc} \nabla\Delta G & \xrightarrow{\nabla\Delta f} & \nabla\Delta G' \\ \Big\downarrow{\scriptstyle \epsilon_G} & & \Big\downarrow{\scriptstyle \epsilon_{G'}} \\ G & \xrightarrow{f} & G' \end{array}$$

Since $\nabla\Delta f = f$ and $(\epsilon_G)_0$ and $(\epsilon_{G'})_0$ are the identity functions, we can conclude the commutativity of the 0-th components of the diagram. Moreover,

$$\begin{aligned} &(\epsilon_{G'})_1\big((\nabla\Delta f)_1(A \to (A,\beta)\,\mathrm{nt}^*(\beta))\big) \\ &\quad = (\epsilon_{G'})_1\big(f_0(A) \to f_1(A \to \beta)(f_0 \circ \mathrm{nt})^*(\beta)\big) \end{aligned}$$

for each production rule $A \to (A, \beta)\,\mathrm{nt}^*(\beta)$ in $P_{\Delta G}$. Let $G' = \langle N', T', P' \rangle$. Since f is a grammar morphism and $A \to \beta \in P$, then $f_0(A) \to \beta' \in P'$ for some β' where $(\mathrm{nt}')^*(\beta') = (f_0 \circ \mathrm{nt})(\beta)$. Therefore, we can continue the previous chain of equalities:

$$
\begin{aligned}
&= (\epsilon_{G'})_1\big(f_0(A) \to f_1(A \to \beta)(\mathrm{nt}')^*(\beta')\big) && \triangleright\ (\mathrm{nt}')^*(\beta') = (f_0 \circ \mathrm{nt})(\beta) \\
&= f_0(A) \to \beta' \\
&= f_1(A \to \beta) && \triangleright\ f \text{ is a grammar morphism} \\
&= f_1\big((\epsilon_G)_1(A \to (A, \beta)\,\mathrm{nt}^*(\beta))\big) && \triangleright\ \text{by definition of } (\epsilon_G)_1
\end{aligned}
$$

and the proof is complete.

Theorem 1. ∇ *is left adjoint to* Δ *and* (ϵ, η) *are the counit and the unit of the adjunction* $(\nabla, \Delta, \epsilon, \eta)$.

Proof. We prove the following triangle equalities:

The 0-th components of both diagrams trivially commutes. We prove only the commutativity of the 1-th components.

For each $s \to \sigma w \in P_{\mathfrak{S}}$

$$
\begin{aligned}
&(\epsilon_{\nabla\mathfrak{S}})_1\big((\nabla\eta_{\mathfrak{S}})_1(s \to \sigma w)\big) \\
&\quad = (\epsilon_{\nabla\mathfrak{S}})_1\big((\eta_{\mathfrak{S}})_0(s) \to (\eta_{\mathfrak{S}})_1(\sigma)(\eta_{\mathfrak{S}})_0^*(w)\big) \\
&\quad = (\epsilon_{\nabla\mathfrak{S}})_1(s \to (s, \sigma w)w) \\
&\quad = s \to \sigma w
\end{aligned}
$$

For each $A \to \beta \in \Sigma^G_{\mathrm{nt}^*(\beta), A}$

$$
\begin{aligned}
&(\Delta\epsilon_G)_1\big((\eta_{\Delta G})_1(A \to \beta)\big) \\
&\quad = (\Delta\epsilon_G)_1(A \to (A, \beta)\,\mathrm{nt}^*(\beta)) \\
&\quad = (\epsilon_G)_1(A \to (A, \beta)\,\mathrm{nt}^*(\beta)) \\
&\quad = A \to \beta
\end{aligned}
$$

Corollary 1. $(\nabla, \Delta, \epsilon, \eta)$ *is an adjoint equivalence.*

Proof. ∇ is left adjoint to Δ (Theorem 1) and η and ϵ are natural isomorphisms (Propositions 10 and 11).

Proposition 14. $\varphi\colon \mathbf{1_{Sig}} \Rightarrow \Lambda\mathrm{V}$ *is a natural isomorphism.*

Proof. For each many-sorted signature $\mathfrak{S} = \langle S, \Sigma \rangle$, the component $\varphi_{\mathfrak{S}}$ at $\mathfrak{S}$ of φ is trivially an invertible many-sorted signature morphism. Thus, we only prove the naturality, i.e., that

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}' \\ {\scriptstyle \varphi_{\mathfrak{S}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathfrak{S}'}} \\ \Lambda\mathrm{V}\mathfrak{S} & \xrightarrow[\Lambda\mathrm{V}f]{} & \Lambda\mathrm{V}\mathfrak{S}' \end{array}$$

commutes for each many-sorted signature morphism $f\colon \mathfrak{S} \to \mathfrak{S}'$. The 0-th component of the diagram commutes because $(\Lambda\mathrm{V}f)_0 = f$. As regards the 1-th component, we have that

$$(\Lambda\mathrm{V}f)_1(\sigma_{w,s}) = \big((\mathrm{V}f)_1(\sigma)\big)_{(\mathrm{V}f)_0^*(w),(\mathrm{V}f)_0(s)} = (f_1(\sigma))_{f_0^*(w),f_0(s)} = (\varphi_{\mathfrak{S}'})_1(f_1(\sigma))$$

and hence the thesis.

Proposition 15. *$\vartheta\colon \mathrm{V}\Lambda \Rightarrow \mathbf{1}_{\mathbf{Sig}^{\leq}}$ is a natural isomorphism.*

Proof. For each order-sorted signature $\mathcal{S} = \langle S, \leq, \Sigma\rangle$, the component $\vartheta_{\mathcal{S}}$ at $\mathcal{S}$ of ϑ is trivially an invertible order-sorted signature morphism. Thus, we only prove the naturality, i.e., that

$$\begin{array}{ccc} \mathrm{V}\Lambda\mathcal{S} & \xrightarrow{\nabla\Delta f} & \mathrm{V}\Lambda\mathcal{S}' \\ {\scriptstyle \vartheta_{\mathfrak{S}}}\downarrow & & \downarrow{\scriptstyle \vartheta_{\mathcal{S}'}} \\ \mathcal{S} & \xrightarrow{f} & \mathcal{S}' \end{array}$$

commutes for each order-sorted signature morphism $f\colon \mathcal{S} \to \mathcal{S}'$. The 0-th component of the diagram commutes because $(\mathrm{V}\Lambda f)_0 = f$. As regards the 1-th component, we have that

$$\begin{aligned} (\vartheta_{\mathcal{S}'})^1_{f_0^*(w),f_0(s)}\big((\mathrm{V}\Lambda f)^1_{w,s}(\sigma_{w,s})\big) &= (\vartheta_{\mathcal{S}'})^1_{f_0^*(w),f_0(s)}\big((\Lambda f)_1(\sigma_{w,s})\big) \\ &= (\vartheta_{\mathcal{S}'})^1_{f_0^*(w),f_0(s)}\big(f^1_{w,s}(\sigma)_{f_0^*(w),f_0(s)}\big) \\ &= f^1_{w,s}(\sigma) \\ &= f^1_{w,s}\big((\vartheta_{\mathcal{S}})^1_{w,s}(\sigma_{w,s})\big) \end{aligned}$$

and hence the thesis.

Theorem 4. *V is left adjoint to Λ and (ϑ, φ) are the counit and the unit of the adjunction $(\mathrm{V}, \Lambda, \vartheta, \varphi)$.*

Proof. We prove the following triangle equalities (0-th component trivially commutes):

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi\Lambda} & \Lambda\mathrm{V}\Lambda \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \Lambda\vartheta} \\ & & \Lambda \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{V}\Lambda\mathrm{V} & \xleftarrow{\mathrm{V}\varphi} & \mathrm{V} \\ {\scriptstyle \vartheta\mathrm{V}}\downarrow & =\!\!\!\!\swarrow & \\ \mathrm{V} & & \end{array}$$

For each $\sigma \in \Sigma^{\mathfrak{S}}_{w,s}$

$$
\begin{aligned}
&(\vartheta_{\mathrm{V}\mathfrak{S}})^1_{w,s}\big((\mathrm{V}\varphi_{\mathfrak{S}})^1_{w,s}(\sigma)\big)\\
&\quad= (\vartheta_{\mathrm{V}\mathfrak{S}})^1_{w,s}\big((\varphi_{\mathfrak{S}})_1(\sigma)\big)\\
&\quad= (\vartheta_{\mathrm{V}\mathfrak{S}})^1_{w,s}(\sigma_{w,s})\\
&\quad= \sigma
\end{aligned}
$$

For each $\sigma_{w,s} \in \Sigma^{\mathcal{S}}_{w,s}$

$$
\begin{aligned}
&(\Lambda\vartheta_{\mathcal{S}})_1\big((\varphi_{\Lambda\mathcal{S}})_1(\sigma_{w,s})\big)\\
&\quad= (\Lambda\vartheta_{\mathcal{S}})_1((\sigma_{w,s})_{w,s})\\
&\quad= \big((\vartheta_{\mathcal{S}})^1_{w,s}(\sigma_{w,s})\big)_{w,s}\\
&\quad= \sigma_{w,s}
\end{aligned}
$$

Corollary 2. $(\mathrm{V}, \Lambda, \vartheta, \varphi)$ *is an adjoint equivalence.*

Proof. V is left adjoint to Λ (Theorem 4) and ϑ and φ are natural isomorphisms (Propositions 14 and 15).