A Spherical Cutting-plane Method With Applications In Multimedia Flow Management

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Abstract. An important problem in Multimedia Flow Management of scheduling jobs on identical parallel machines aiming to minimize total completion time is studied. Based on the problem peculiarities, the prominence of applying cutting-plane approaches is justified. A new such approach called a spherical cutting-plane method (SCPM) is developed for solving linear permutation-based problems. It uses a fundamentally new way to construct cutting planes for sets inscribed into a hypersphere, and it is superior to existing methods of the optimization problems' class. The generic SCPM is adapted to the scheduling problem under consideration. For that, the problem's statement as a linear partially combinatorial permutation-based problem is built, and the SCPM is generalized for solving partially combinatorial problems.

Keywords: Scheduling, parallel machines, cutting plane method, permutation-based problem, Euclidean combinatorial problem, spherical-located set, well-described set.

1 Introduction

In the Smart Multimedia field, minimizing the completion time of jobs on parallel machines and solving other scheduling problems are inevitable components of efficient Multimedia Flow Management [1]. Most of the problems belong to a class of combinatorial or partially combinatorial optimization problems resulting in their higher computational complexity [1-3]. As a field of Optimization Theory, Combinatorial Optimization offers a variety of methods and algorithms to solve the problems of lower dimension exactly or get an approximate solution of the ones of high dimension in a reasonable time [4-7]. Nevertheless, steadily increasing demands to the solutions' accuracy along with a reduction in the time of their obtaining results in a need to develop new optimization approaches for both generic and special statements of the problems [1-3,8,9].

In this paper, a new method of partial combinatorial optimization, called a spherical cutting-plane method (SCPM), is offered for solving a scheduling problem for

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jobs with non-identical job sizes processed on parallel machines having the same capacity, which aims to minimize total completion time. For that, a new mathematical model of the problem as a linear partial permutation-based program is built, the generic SCPM is justified and then is adapted to solving the scheduling problem.

2 Problem Statement

The problem of scheduling jobs on identical parallel machines to minimize total completion time may be stated as follows [8-10]. Each of n jobs (numbered $A_1,\dots,A_n$) is to be processed on one of m identical parallel machines (numbered $M_1,\dots,M_m$). No machine can handle more than one job at a time. Each job A_i is available for processing at time zero and requires a positive integer processing time t_j on the machine to which it is assigned ($j \in J_n = \{1,\dots,n\}$). The objective is to find a schedule that minimizes the completion time of all these jobs.

Let $P = \left\{ \{ j \}_{j \in I_i'} \right\}_{i \in J_m}$, $\left| I_i' \right| = n_i$, $i \in J_m$ be a partition of the jobs induced the schedule, where $\{ A_j \}_{j \in I_i''}$ are jobs completed at the machine M_i ($i \in J_m$).

The completion time T is a maximum of completion time in each of the machines. Thus,

$$T = \max_{i \in J_m} T_i, \text{ where } T_i = \sum_{j \in I_i'} t_i,\ i \in J_m. \tag{1}$$

It is required to determine such a partition, where T is minimized. Thus, an issue is to find the partition:

$$T \to min, \tag{2}$$

where additional constraints

$$P \in \mathbf{P}, \tag{3}$$

are satisfied, where $\mathbf{P}$ is a set of admissible partitions.

Due to the presence of max(.) in the formulation of the problem (1)-(3) (further referred to as **Problem 1**), this one is a nonlinear partially combinatorial problem given in the form of its combinatorial statement. Here, the real-valued variable is T. The statement is suitable for algorithmization if (3) is missing, and heuristics based on generating the partitions are applied [11-15]. The situation gets worse if additional constraints are present, which is a typical case in practical applications [11,16,17]. Therefore, it is often not so easy to find the required number of feasible partitions to apply these heuristics and metaheuristics. Moreover, in many cases, a feasibility problem needs to be solved to get any of the feasible partitions [18].

In this paper, we propose a technique for solving Problem 1, provided that an approximate solution yielding $T = T^{**}$ has been found. The approach is based on embedding Problem 1 in Euclidean space and then treating it as a linear partially discrete optimization problem.

First, examine Problem 1 in order to reformulate it as a linear permutation-based optimization problem [12,13]. For that, at first, define a dimension of Euclidean space for the embedding. Let n^{min}, n^{max} be a minimal and maximal number of the jobs scheduled on a single machine.

Without loss of generality, assume that $t_1 \leq \ldots \leq t_n$, then

$$n^{min}, n^{max}: \sum_{i=1}^{n^{min}} t_{n-i+1} \leq T^{**}, \sum_{i=1}^{n^{min}+1} t_{n-i+1} > T^{**}; \sum_{i=1}^{n^{max}} t_i \leq T^{**}, \sum_{i=1}^{n^{min}+1} t_i > T^{**}.$$

Now, set the dimension of Euclidean space as follows – $N = m \cdot n^{max}$. After, we complement the multiset $\{t_i\}_{i \in J_n}$ by $N-n$ dummy zeros and form a multiset

$$G = \{g_i\}_{i \in J_N} = \{t_i\}_{i \in J_n} \cup \left\{0^{N-n}\right\}: g_1 \leq \ldots \leq g_n, \tag{4}$$

with exactly k different values $S(G) = \{e_i\}_{i \in J_k}: 0 = e_1 < \ldots < e_k$.

Introduce a vector of variables

$$x = \left(x_{11}, \ldots, x_{1n^{max}}, \ldots, x_{m1}, \ldots, x_{mn^{max}}\right).$$

In these denotations, Problem 1 can be formulated as finding $x \in R^N$:

$$z = \max_{i \in J_m} \sum_{j=1}^{n^{max}} x_{ij} \to min, \tag{5}$$

$$x \in E_{Nk}(G), \tag{6}$$

where $E_{Nk}(G)$ – is a basic generalized set of Euclidean permutation configurations (the generalized permutation $\mathcal{C}_b$-set) induced by G [19,20]. The problem (5), (6) – is a nonlinear nondifferentiable Euclidean combinatorial problem [19,21], which becomes much easier for dealing with by its lifting into space R^{N+1}. For that, let us introduce an additional variable $y \in R^1_{>0}$ such that $y = \max_{i \in J_m} \sum_{j=1}^{N} x_{ij}$. Now, (5) is rewritten as – find $\langle (x, y), z \rangle$:

$$z = y \to min, \tag{7}$$

subject to (6) and constraints

$$\sum_{j=1}^{N} x_{ij} - y \le 0,\ i \in J_m . \tag{8}$$

Assume that, if constraints (3) are present, then after the embedding, they become linear and are of the form:

$$A'' x - b'' \le 0,\ A'' \in R^{m' \times N},\ b'' \in R^{m'} . \tag{9}$$

The obtained problem (6)-(9), further referred to as **Problem 2**, is a linear constrained partially permutation-based problem with a single real-valued variable y .

3 The relevance of developing a cutting-plain approach to Problem 2

Problem 2 belongs to such a class of Euclidean linear partially combinatorial problems:

$$f(x, x') = cx + c' x' \to min , \tag{10}$$

subject to $c \in R^N,\ c' \in R^{n'}$,

$$Ax + A' x' \le b,\ \text{where } A \in R^{M \times N},\ A' \in R^{M \times n'},\ b \in R^M, M = m' + m , \tag{11}$$

$$x \in E \subset R^N,\ |E| < \infty , \tag{12}$$

where $E = E_{Nk}(G),\ n' \ge 1$.

Numerous features of set $E_{Nk}(G)$ underlie various optimization methods of solving problems such as (10)-(12).

One of the properties is that $E_{Nk}(G)$ lies on a hyperplane and hyperspheres centered at $\mathbf{a} = a\mathbf{e}$, where $a \in R^1 \setminus \{\varnothing\}$ is a parameter, $\mathbf{e}$ is a vector of units [14]. In the family is a circumsphere of minimal radius corresponding to the parameter $a = \frac{1}{N} \sum_{i=1}^{N} g_i$. It results in another peculiarity of crucial importance for us that $E_{Nk}(G)$ coincides with a vertex set of a polytope $P_{Nk}(G) = conv\, E_{Nk}(G)$. Such a set is called vertex-located (VLS) [22].

The class $E_{Nk}(G)$ is intensively studied the last couple of decades [4,11-16,19-25] in various directions, most of which concern optimization. Here, we outline the main approaches to solving linear permutation-based problems based on utilizing the above properties. First, there is a method of tightening constraints presented in [21] for solving linear combinatorial programs. Let us formulate its generalization for partially combinatorial linear programs. First, additional constraints (11) need to be replaced by $Ax + A'x' \le b - \delta$, where $\delta \in R_+^M$ is chosen in a specific way. Then the original partially combinatorial problem is replaced by a polyhedral relaxation of the new problem. Finally, the relaxation' solution $\left(x^0, x'^0\right)$ is rounded combinatorially thus yielding a point $\left(y^0, y'^0\right)$, where $y^0 \in E$. δ depends on E and is chosen such, that $\left(y^0, y'^0\right)$ is an admissible solution, i.e., $\left(x^{**}, x'^{**}\right) = \left(y^0, y'^0\right)$. The method of tightening constraints is approximate, which can be effectively combined with exact approaches. One of the exact techniques is a polyhedral-spherical method [23], which is a Branch&Bound approach exploiting simultaneously polyhedral-sphericity of E, its decomposition into generalized permutation $\mathcal{C}_b$-sets of lower dimension lying in parallel hyperplanes [23]. In [26], some graph-theoretic approaches to solving such optimization problems, both exact and approximate, are offered. They explore an equivalent statement of these problems as optimization ones on a node-set of graphs extracted from a transposition graph [27]. One more important group of methods is cutting-plane ones [28-31]. Among them are a combinatorial cutting method [29,30], combinatorial polytope cutting method [31], surface cutting method [31]. They are based on the absence of admissible solutions in an interior of faces on any dimension, as well as on most of circumsphere. All the exact methods are intended to solve combinatorial programs only. Thus, they require developing relevant generalization to the partially combinatorial case. The only exception is the combinatorial cutting method, which has been reformulated for partially combinatorial programs in [29]. An issue of applying the listed cutting-plane techniques is that they require finding a set of adjacent vertices to solutions of auxiliary polyhedral relaxation problems (the solutions' neighborhood). It is caused by, generally, the exponential on N number of constraints in an H-representation of $P_{Nk}(G)$ making impossible processing the whole collection of $P_{Nk}(G)$-constraints. It turns out that it is sufficient to involve inconsiderable part of the H-representation [21]. However, in this case, to extract the above-mentioned neighborhood becomes problematic. Therefore, in this paper, we aim to develop a new cutting-plane method SCPM for solution linear constraint permutation-based and partially permutation-based problems, which utilizes solutions of polyhedral relaxation problems, spherical locality of $E_{Nk}(G)$, and properties of linear functions over the set. Our final goal is to adapt this method for solving Problem 2.

4 Cutting-plane method for linear optimization on WD-SpLSs

Consider an optimization problem of finding x such that

$$f(x) = cx \to min \tag{13}$$

subject to constraints

$$Ax \le b,\ A \in R^{M \times n},\ b \in R^M, \tag{14}$$

$$x \in E \subseteq S_r(a) \subset R^n, \tag{15}$$

$$E \text{ is a well-described set (WDS),} \tag{16}$$

where $S_r(a)$ – is a hypersphere centered at $a \in R^n$ with a radius $r > 0$. The condition (16) means that the problem (13) is effectively solvable on , i.e., it is polynomially solvable [32]. The condition (15) implies that is a spherically-located set (SpLS) [16,17]. Thus, the problem (13)-(16) is a general linear constraint optimization problem (further **Problem 3**) on SpLS and WDS E (further **WD-SpLS**). The conditions (15), (16) allow using specifics of WD-SpLS in optimization, in particular, when cutting-plane optimization schemes are developed.

Theorem 1. If E is SpLS, then for any $x^0 \in R^n$, there exists $c \in R^n$ such that problems (13) and

$$h(x) = \left\| x - x^0 \right\|^2 \to \min_{x \in E} \tag{17}$$

are equivalent.

Proof. Let us assume that SpLS E satisfies (15), wherefrom

$$r^2 \underset{E}{=} \left\| x - a \right\|^2 = x^2 - 2ax + a^2. \tag{18}$$

Single outing x^2 from (18) and substituting it in (17) yield

$$h(x) = \left(x - x^0\right)^2 = x^2 - 2xx^0 + \left(x^0\right)^2 = r^2 + 2ax - a^2 - 2xx^0 + \left(x^0\right)^2 =$$
$$= 2\left(a - x^0\right)x + \left(r^2 - a^2 + \left(x^0\right)^2\right).$$

The expression can be rewritten as follows:

$$h(x) = cx + d, \text{ where } c = 2\left(a - x^0\right),\ d = r^2 - a^2 + \left(x^0\right)^2. \tag{19}$$

In (19), d is a constant; hence the projection problem (17) has been reduced to a minimization of linear function $cx + d$, which is equivalent to the problem (13), where c is found from (19).

Corollary 1. If E is WD-SpLS, then, for any $x^0 \in R^n$, the projection problem (17) is polynomially solvable.

Indeed, in this case, (17) is reducible to a linear program over E, which is effectively solvable by definition of WDS.

4.1 SCPM outline

The spherical cutting-plane method (SCPM) is an iterative approach, and it will be stated in terms of a single iteration.

Let $l \in J_L^0 = J_L \cup \{0\}$ be an iteration number, where an iteration numbered 0 is initial, while the one numbered L is last. On iteration l, a linear program (13) under constraints (15), (16),

$$A^l x \le b^l,\ A^l \in R^{m^l \times n},\ b^l \in R^{m^l} \tag{20}$$

is solved (further **Problem 3.l**), which is equivalent to Problem 3, through its continuous relaxation. For that, the constraint (15) is replaced by

$$x \in P = conv\ E \subseteq S_r(a). \tag{21}$$

For instance, if E is finite, P will be a polytope, respectively, the problem (13), (16), (20), (21) (further **Problem 4.l**) is a polyhedral relaxation of Problem 3.l.

Let a solution of Problem 3 be denoted $\langle x^*, z^* \rangle = \langle x^*, cx^* \rangle$, of Problem 4.l – $\langle x^l, z^l \rangle = \langle x^l, cx^l \rangle$.

Step 0. On initial iteration $l = 0,\ A^0 = A,\ b^0 = b,\ m^0 = M$.

Step l. On iteration l, if $x^l \in E$, then $\langle x^*, z^* \rangle = \langle x^l, z^l \rangle$, end. If $x^l \notin E$, we form a cut for x^l. For that, find a projection y^l of the point x^l onto E:

$$y^l = \Pr_E x^l. \tag{22}$$

By construction, $y^l \ne x^l$, thus there exists a sphere S^l of a positive radius centered at x^l having no points in common with E, which can be cut off from a feasible domain of the Problem 4.l. Choose $S^l = S_{r^l}(x^l)$, where

$$\left(r^l\right)^2 = \left(y^l - x^l\right)^2, \tag{23}$$

because this sphere contains no points of E in an interior. So, a deep nonlinear cut of $x^l \notin E$ is

$$\left(x-x^l\right)^2 \ge \left(r^l\right)^2, \tag{24}$$

which can be added to the current constraints. An issue is that the constraint (24) is nonlinear. Moreover, it is concave. Therefore, the utilization of (24) makes a new problem harder than it was. Meanwhile, it is easy to see that the cut is not unique, and it is possible to find a linear constraint cutting off x^l and the relevant cutting plane.

Let us construct the cutting plane based on Theorem 1. Preliminarily, the theorem will be generalized as follows, $0 \to l,\ h(x) \to h^l(x),\ c \to c^l,\ d \to d^l,\ l \in J_L^0$.

Corollary 2. If E is SpLS, then for any $x^l \in R^n$, there exists $c^l \in R^n$ such that problems $c^l x \to \min\limits_{x \in E}$ and $h^l(x) = \left\|x - x^l\right\|^2 \to \min\limits_{x \in E}$ are equivalent, namely,

$$h^l(x) = c^l x + d^l, \text{ where } c^l = 2\left(a - x^l\right),\ d^l = r^2 - a^2 + \left(x^l\right)^2. \tag{25}$$

Now, inequality (24) can be rewritten equivalently on E $\left(r^l\right)^2 \le \left(x - x^l\right)^2 = = h^l(x) \underset{E}{=} c^l x + d^l$ or

$$c^l x \ge d^l - \left(r^l\right)^2, \tag{26}$$

where r^l is given by (23), c^l, d^l – by (25).

By construction, $c^l x^l < d^l - \left(r^l\right)^2$, hence $c^l x = d^l - \left(r^l\right)^2$ is a cutting plane for x^l. Instead of (24), let us add inequality (26) to the current constraints (20) obtaining input data $A^{l+1},\ b^{l+1}$ for Problem 3.(l+1).

Set $l = l+1,\ m_l = m_{l-1} + 1$. Go to solving Problem 3.l through Problem 4.l. Repeat until the method terminates, which can occur, if the maximal number of iteration has been reached, x^* was found, the current lower and upper bound coincide (then $\left\langle x^*, z^* \right\rangle = \left\langle x^{**}, z^{**} \right\rangle$), or incompatibility of Problem 4.l was proven.

Remark 1. Throughout the iterative process, a lower bound z^{lb} on z^* are constantly improved. Namely, by construction, $z^0 < z^1 < \ldots < z^L$ that is why: a) initially $z^{lb} = z^0$; b) on iteration 1, $z^{lb} = \max\left\{z^1, z^{lb}\right\} = z^1$; …; c) on iteration L $z^{Lb} = \max\left\{z^L, z^{Lb}\right\} = z^L$. In order to reduce a search domain, it makes sense to solve a feasibility problem of finding admissible point x^{**} of Problem 3 and then to monitor improving the initial upper bound $z^{ub} = z^{**} = cx^{**}$. The current upper bound is improved on iteration l, if the point (22) satisfies (14), and $z^{ub} > z\left(y^l\right) = cy^l$, wherefrom $z^{ub} = min\left\{z\left(y^l\right), z^{lb}\right\} = z\left(y^l\right)$.

For the increasing probability of improving the upper bound in such a way, it is worthful to explore a whole projection $Pr_E\ x^l$ of x^l onto E, which implies replacing formula (22) by $Y^l = Pr_E\ x^l$. Now, if $Y^l \cap E \neq \{\varnothing\}$, there is a chance to improve the current lower bound if the whole neighborhood is examined.

5 Adaptation SCPM to Problem 2

Let us adjust the SCPM to solving the general linear partially combinatorial problem (10) (12). The following substitution will be made in the SCPM and formulations of Problems 3.l, 4.1: $n \to N$, $x^{[\cdot]} \to \left(x^{[\cdot]}, x'^{[\cdot]}\right)$, $c \to (c, c')$; (20) should be replaced by

$$A^l\ (x, y) \le b^l,\ A^l \in R^{m^l \times (N+n')},\ b^l \in R^{m^l}, \tag{27}$$

$z^{[\cdot]} = cx^{[\cdot]} + c'\ x'^{[\cdot]}$, where $A^0 = (A, A')$, $[\cdot] \in \{l, *, **\}$.

Step l is reformulated as follows: on iteration l, if $x^l \in E$, then $\left\langle\left(x^*, y^*\right), z^*\right\rangle = \left\langle\left(x^l, y^l\right), z^l\right\rangle$, end. If $x^l \notin E$, then find y^l by (22) and form a cut for x^l in accordance to (26).

To the generalization of the SCPM (further referred to as a **generalized SCPM** (**GSCPM**)). It is directly applicable to solving Problem 2. For that, constraints (8), (9) are presented in the form of (11), where $n' = 1, x^l = y$. Matrix A^0 is of the dimension $m^0 = m + m'$ by $N+1$, the objective function vector in (10) is $(c, c') = (\mathbf{0}, 1)$, where $\mathbf{0} \in R^N$ is a zero-vector.

Remark 2. When solving Problem 2 by the GSCPM, a search domain can be reduced depending on the type of values of elements in (4). Without loss of generality assume that a greatest common divisor $GCD\left(\{t_i\}_{i \in J_n}\right) = 1$, then an initial lower bownd will be $z^{lb} = \left\lceil \frac{1}{m} \sum_{i=1}^{n} t_i \right\rceil$. At the same time, an initial upper bound can be found by a well-known heuristic, where the least filled bin associated with a machine is filled first, while jobs are considered in random order. Then the initial bin packing may be improved by adjacent transposition of a vector x^{**} associated with this packing.

Conclusion

An actual problem of organizing effective parallelization of a job batch is considered. This problem is modeled as linear constrained partially combinatorial. For linear constrained problems over well-described spherically-located sets, such as permutation set, Boolean set, or permutation matrices' set, a special exact solution method is offered, called a spherical cutting-plane method (SCPM).

The SCPM is generalized to solving partially combinatorial problems resulting in the generalized SCPM (GSCPM) and is adapted for the scheduling problem under consideration. SCPM and GSCPM can be applied to a wide class of real-world problems in which combinatorial structures are singled out, such as permutations and Boolean vectors [2-13,23-26,33-35]. It can also be generalized to nonlinear combinatorial and partially combinatorial problems [36-40], where, in order to solve optimization problems globally, our method should be combined with the convex extension theory [16,17,20,23] and continuous functional representation theory [17,20,22].

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