A Decidable Theory of Differentiable Functions with Convexities and Concavities on Real Intervals*

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Abstract. We enrich a pre-existing decision algorithm, which in its turn augmented a fragment of Tarski's elementary algebra with one-argument functions enjoying significant properties such as continuity and differentiability. We also pave the way to further enrichments, embodying symbols to designate certain operations on functions such as pointwise addition.

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1 Introduction

Around 1930, Alfred Tarski put forward an axiomatic system of elementary geometry based on first-order predicate calculus and proved, through a quantifier-elimination method, the completeness of his axiomatic theory. Completeness entailed the algorithmic solvability of the truth problem as referred to sentences in the elementary algebra of real numbers [14,15]. Subsequent improvements of the decision algorithm aimed, on the one hand, at making its complexity affordable in special subcases; on the other hand, at broadening its applicability beyond algebra, so as to handle entities and constructs relevant to the realm of real analysis. One of the decidable extensions was the *RMFC* theory (Theory of Reals with Monotone and Convex Functions), first investigated by D. Cantone, A. Ferro, E.G. Omodeo and J.T. Schwartz [3], whose language extends the existential theory of real numbers with various predicates about real functions. Other decidability results were achieved through ideas close to the ones which had led to the decidability of *RMFC*; those results regard: $RMFC^{+}$ (Augmented theory of Reals with Monotone and Convex Functions) by D. Cantone, G. Cincotti and G. Gallo [1], *RDF* (Theory of Reals with Differentiable Functions) by D. Cantone and G. Cincotti [5,2], and finally the RDF^+ theory (Augmented Theory of Reals with Differentiable Functions) [5], the subject of this paper.

Let us briefly explain the basic idea through which one gets the decidability of *RMFC*, $RMFC^+$, *RDF*, and RDF^+. As a preliminary, notice that if $\psi = \exists x_1 \dots \exists x_n \theta$ is an existential sentence of EAR, the elementary algebra of real numbers, and hence θ is a quantifier-free formula all of whose free variables are among $x_1, \dots x_n$, then ψ is *true* iff there exist real numbers $a_1, \dots, a_n$ which satisfy θ, to wit, iff θ is *satisfiable*; and, since by Tarski's result we have a decision algorithm for existential sentences of EAR, we also have a satisfiability algorithm for quantifier-free formulas of EAR. All four theories mentioned above extend the existential fragment of EAR with a new sort of variables, for functions, and with new predicate symbols, for special relations regarding functions; in order to establish their decidability, we translate their formulas into logically equivalent formulas of the existential theory of real numbers and thus can rely upon Tarski's original decision result.

2 The RDF^+ Theory

The augmented version RDF^+ of the theory *RDF* of reals with differentiable functions, is an unquantified first-order theory involving various predicates on real functions of class C^1 of one real variable, namely functions with continuous first derivative. Predicates of RDF concern comparison of functions, strict and non-strict monotonicity, strict and non-strict convexity (and concavity), and comparison of first derivatives with real constants. In this section we introduce the language of RDF^+, explain its intended meaning, and illustrate its use through quick examples.

Syntax and semantics

The RDF^+ language involves variables of two different sorts: *numerical variables*, denoted by $x, y, \dots$, range over numbers; *functional variables*, denoted by $f, g, \dots$, range over functions. Six constants are also available:

- the symbols 0 and 1, designating the numbers 0 and 1;
- the symbols $+\infty$ and $-\infty$, occurring only as endpoints of interval domains (whose definition will be given later);

We next specify the syntax of *term*, *atom*, and *formula* for RDF^+:

Definition 21 Numerical terms *are recursively defined as follows:*

1. *numerical variables and the constants* $0, 1$ *are numerical terms;*
2. *if* t_1 *and* t_2 *are numerical terms, then*

$$t_1 + t_2 \, , \, t_1 - t_2 \, , \, and \, t_1 \cdot t_2$$

also are numerical terms;

3. *if t is a numerical term and f is a functional variable, then*

$$f(t) \text{ and } D[f](t)$$

also are numerical terms;

4. *numerical terms are all and only the expressions built up according to the above criteria 1., 2., and 3.*

In the ongoing, we will refer by the locution "numerical variable" to either a numerical variable proper or to one of the constants 0 and 1. Moreover, we will refer by the locution "extended numerical variable" to either a numerical variable or one of the symbols $-\infty$, $+\infty$; and by the locution "extended numerical term" to either a numerical term or one of the symbols $-\infty$, $+\infty$.

Definition 22 *An* ATOM *of RDF^+ is an expression of one of the forms*

$$\begin{array}{rr}
t_1 = t_2 \ , & t_1 > t_2 \ , \\
(f = g)_A \ , & (f > g)_B \ , \\
\mathrm{Up}(f)_A \ , & \mathrm{Strict\_Up}(f)_A \ , \\
\mathrm{Down}(f)_A \ , & \mathrm{Strict\_Down}(f)_A \ , \\
\mathrm{Convex}(f)_A \ , & \mathrm{Strict\_Convex}(f)_A \ , \\
\mathrm{Concave}(f)_A \ , & \mathrm{Strict\_Concave}(f)_A \ , \\
(D[f] \bowtie t)_A \ , &
\end{array}$$

where: A designates an interval, either bounded or unbounded, open, half-open or closed; B designates a bounded interval, open, half-open, or closed; t_1, t_2, t are numerical terms, and $\bowtie\, \in \{=, <, \ >, \ \leqslant, \ \geqslant\}$. Specifically, A can be of any of the forms $[e_1, e_2]$, $[e_1, e_2[$, $]e_1, e_2]$, $]e_1, e_2[$, where e_1 is either a numerical term or $-\infty$, and e_2 is either a numerical term or $+\infty$; as for B, it can be of any of the forms $[s_1, s_2]$, $[s_1, s_2[$, $]s_1, s_2]$, $]s_1, s_2[$, where s_1 and s_2 are numerical terms. We dub the extended numerical terms e_1, e_2 ENDS *of A; likewise, we dub the numerical terms s_1, s_2* ENDS *of B.* (When we will come to the semantics—we will call the quantities designated by the ends of an interval specification the *endpoints* of the designated interval).

Definition 23 *A* FORMULA *of RDF^+ is any truth-functional combination of atoms of RDF^+.*

For definiteness, we will build up the RDF^+ formulas from RDF^+ atoms by means of the usual propositional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Observation 24 (Derived relators) *For convenience, we enrich our language with derived symbols by means of the following shortening definitions:*

$$\begin{array}{rcl}
t_1 \neq t_2 & \leftrightarrow_{\mathit{Def}} & \neg(t_1 = t_2)\,, \\
t_1 < t_2 & \leftrightarrow_{\mathit{Def}} & t_2 > t_1 \ , \\
t_1 \leqslant t_2 & \leftrightarrow_{\mathit{Def}} & (t_1 < t_2) \vee (t_1 = t_2)\,, \\
t_1 \geqslant t_2 & \leftrightarrow_{\mathit{Def}} & (t_1 > t_2) \vee (t_1 = t_2)\,, \\
t_1 = t_2/t_3 & \leftrightarrow_{\mathit{Def}} & (t_2 = t_1 \cdot t_3) \wedge (t_3 \neq 0)\,, \\
t_1 > t_2/t_3 & \leftrightarrow_{\mathit{Def}} & \big((t_2 < t_1 \cdot t_3) \wedge (t_3 > 0)\big) \vee \big((t_2 > t_1 \cdot t_3) \wedge (t_3 < 0)\big)\,, \\
t_1 < t_2/t_3 & \leftrightarrow_{\mathit{Def}} & \big((t_2 > t_1 \cdot t_3) \wedge (t_3 > 0)\big) \vee \big((t_2 < t_1 \cdot t_3) \wedge (t_3 < 0)\big)\,.
\end{array}$$

When A is as in Def. 22, we also introduce the abbreviation:[4]

$$(D[f] \neq t)_A \ \leftrightarrow_{\mathit{Def}} (D[f] < t)_A \vee (D[f] > t)_A \,.$$

Remark 21 Notice that although we have chosen to regard them as primitive constructs, the relators Up ad Down are expressible in terms of differentiation, via the equivalences:

$$\begin{aligned}\mathrm{Up}(f)_A &\leftrightarrow (D[f] \geqslant 0)_A \vee (e_1 = e_2)\,,\\ \mathrm{Down}(f)_A &\leftrightarrow (D[f] \leqslant 0)_A \vee (e_1 = e_2)\,,\end{aligned}$$

where e_1, e_2 are the ends of A.

Constructs which could easily be brought into play as derived relators designate the notions of *upward monotonicity* in a point and of *inflection point*:

$$\begin{aligned}
&\mathrm{up}(f,c) && \leftrightarrow_{\mathrm{Def}} \ (D[f] \geqslant 0)_{[c,c]}\,,\\
&\mathrm{strict\_up}(f,c)_{[x,y]} && \leftrightarrow_{\mathrm{Def}} \ x < c \wedge c < y \wedge \mathrm{Strict\_up}(f)_{[x,y]}\,,\\
&\mathrm{flex}(f,c)_{[x,y]} && \leftrightarrow_{\mathrm{Def}} \ (\ x < c \ \wedge \ c < y\ ) \ \wedge\\
& && \Big(\big(\mathrm{Strict\_Convex}(f)_{]x,c[} \wedge \mathrm{Strict\_Concave}(f)_{]c,y[}\big) \vee\\
& && \quad \big(\mathrm{Strict\_Concave}(f)_{]x,c[} \wedge \mathrm{Strict\_Convex}(f)_{]c,y[}\big) \Big)\,.
\end{aligned}$$

The semantics of RDF^+ (in light of which the above abbreviations should become clear) revolves around the designation rules listed in our next definition, with which any truth-value *assignment* for the formulas of RDF^+ must comply.

Definition 25 *An* ASSIGNMENT *for RDF^+ is a mapping M whose domain consists of all terms and formulas of RDF^+, satisfying the following conditions:*

0. *$M0$ and $M1$ are the real additive zero and multiplicative unit;*
1. *For each numerical variable x, Mx is a real number.*
2. *For each functional variable f, (Mf) is an everywhere defined differentiable real function of one real variable, endowed with continuous derivative.*
3. *For each numerical term of the form $t_1 \otimes t_2$ with $\otimes \in \{+, -, \cdot\}$, $M(t_1 \otimes t_2)$ is the real number $Mt_1 \otimes Mt_2$.*
4. *For each numerical term of the form $f(t)$, $M(f(t))$ is the real number $(Mf)(Mt)$; for each numerical term $D[f](t)$, $M(D[f](t))$ is the real number $D[(Mf)](Mt)$.*
5. *For each interval specification A, MA is an interval of $\mathbb{R}$ of the appropriate kind, whose endpoints are the evaluations via M of the ends of A.*[5] *For example, when $A =]t_1, t_2]$, then $MA =]Mt_1, Mt_2]$.*

[4] The rationale of the abbreviation $(D[f] \neq t)_A$ will become clearer in light of the semantics: in fact, it relies on the continuity of the derivative of f.

[5] It goes without saying what is meant when M is undefined at either end of A (actually, $M(-\infty)$ and $M(+\infty)$ are undefined).

6. *Truth values are assigned to formulas of RDF^+ according to the following rules (where t, t_1, t_2 stand for numerical terms, and f, g for functional variables):*
   - ***a)*** *$t_1 = t_2$ (resp. $t_1 > t_2$) is true iff $Mt_1 = Mt_2$, (resp. $Mt_1 > Mt_2$);*
   - ***b)*** *$(f = g)_A$ is true iff $(Mf)(x) = (Mg)(x)$ holds for all x in MA;*
   - ***c)*** *$(f > g)_B$ is true iff $(Mf)(x) > (Mg)(x)$ holds for all x in MB;*
   - ***d)*** *$(D[f] \bowtie t)_A$, with $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, is true iff $D[(Mf)](x) \bowtie Mt$ holds for all x in MA;*
   - ***e)*** *$Up(f)_A$ (resp. $Strict\_Up(f)_A$) is true iff (Mf) is a monotone nondecreasing (resp. strictly increasing) function in MA;*
   - ***f)*** *$Convex(f)_A$ (resp. $Strict\_Convex$) is true iff (Mf) is a convex (resp. strictly convex) function in MA;*
   - ***g)*** *the truth values of $Down(f)_A$ and $Concave(f)_A$ (resp., of $Strict\_Down(f)_A$ and $Strict\_Concave(f)_A$), are defined in close analogy with points **e)** and **f)**;*
   - ***h)*** *The truth value which M assigns to a formula whose lead symbol is any of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ complies with the usual semantics of the propositional connectives.*

An assignment M is said to MODEL *a set Φ of formulas when $M\varphi$ is true for every φ in Φ.*

Remark 22 Notice that when an interval turns out to have a left endpoint greater than or equal to its right endpoint, then it is either empty or a singleton. Accordingly, most atoms involving that interval turn out to be vacuously true: this is the case, e.g., with all atoms of the forms $\mathrm{Up}(f)_{[t,t]}$, $\mathrm{Up}(f)_{]t,t[}$, $\mathrm{Strict\_Up}(f)_{[t,t]}$, $\mathrm{Strict\_Up}(f)_{]t,t[}$, and variants thereof (on the other hand, formulae $\mathrm{up}(f,t)$ and $\mathrm{strict\_up}(f,t)$ as introduced in Remark 21 are at times false and at times true).

However, an atom of any of the forms

$$(f = g)_{[t_1,t_2]}\,, (f > g)_{[t_1,t_2]}\,, (D[f] = t)_{[t_1,t_2]}\,, (D[f] \bowtie t)_{[t_1,t_2]}\,,$$

with $t_1 = t_2$, can be false.

Example 21 *Notice that any constant function $x \mapsto q$, where q is a rational constant, can be easily characterized by means of an RDF^+ formula. E.g., when q is a positive integer, such a function can be described by the derived relator:*

$$f = q := (D[f] = 0)_{]-\infty,+\infty[} \ \wedge \ f(0) = \underbrace{1 + \cdots + 1}_{q\ times}\,.$$

In analogous terms one can express the fact f is a first-degree polynomial with fixed rational coefficients. E.g. the function $x \mapsto 2x - \frac{1}{3}$ can be defined as:

$$f = 2x - \tfrac{1}{3} := (D[f] = 1 + 1)_{]-\infty,+\infty[} \ \wedge \ f(0) = (0 - 1) \cdot \tfrac{1}{1+1+1}\,.$$

Some examples

We show next how formulas of RDF^+ can express basic theorems of analysis.

Example 22 *Let f be a real function defined in the closed and bounded interval $[a,b]$, differentiable with a continuous derivative f'. If $f'(x) > 0$ for all $x \in \,]a,b[$ then f is monotone increasing in $[a,b]$.*

This theorem can be translated into the following formula of RDF^+:

$$(D[f] > 0)_{]a,b[} \quad \rightarrow \quad \mathrm{Strict\_Up}(f)_{[a,b]} \ .$$

Notice here that the truth of the claim as stated in words amounts to the *validity* of the RFD^+ formula into which we have translated it, i.e., it amounts to the fact that the RFD^+ formula evaluates to *true* in any assignment M. Checking automatically that this is indeed the case will be possible by means of the decision algorithm that we are about to describe, as will be illustrated in Sec. 5.

Example 23 *Let f be a real function defined in a closed bounded interval $[a,b]$, differentiable with a continuous derivative f'. Then there is $c \in \,]a,b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Here is how we can state this weak version of the *mean value theorem* in RDF^+:

$$a < b \quad \rightarrow \quad \neg\left(D(f) \neq \frac{f(b)-f(a)}{b-a}\right)_{]a,b[} \ .$$

As with the preceding example, the validity of this formula can be established automatically, by means of the decision algorithm to be presented next.

3 The decision algorithm

When one deals with an unquantified language such as RDF^+, which is closed with respect to propositional connectives, being able to determine algorithmically whether or not a formula is valid amounts to establishing whether the negation thereof is *satisfiable* or unsatisfiable. We prefer to address the satisfiability problem for RDF^+ in what follows, and our algorithm will produce a yes/no answer, where 'yes' means that φ admits a model. Hence, indirectly, if we were to test a formula for validity, a negative response would mean that a counterexample exists.

The idea is to transform, through a finite number of steps, the given RDF^+ formula φ to be tested for satisfiability into a finite collection of formulas ψ_i, still devoid of quantifiers, each belonging to the elementary algebra of real numbers; this will be done so that φ is satisfiable if and only if one of the resulting ψ_i is satisfiable. Each resulting ψ_i can be tested via Tarski's decision algorithm.

First we reduce our formula φ to a particular format, called *ordered form*; second we explain the algorithm.

Normalization

Let T be an unquantified, possibly multi-sorted, first-order theory, endowed with: equality $=$, a denumerable infinity of individual variables $x_1,\ x_2,\ldots,$ function symbols $F_1,\ F_2,\ldots,$ and predicate symbols $P_1,\ P_2,\ldots.$

Definition 31 *A formula φ of T is said to be in* NORMAL FORM *if:*

(1) every term occurring in φ either is an individual variable or has the form $F(x_1,\ x_2,\ldots,\ x_n)$, where $x_1,\ x_2,\ldots,\ x_n$ are individual variables and F is a function symbol.

(2) every atom in φ either has the form $x = t$, where x and t are an individual variable and a term, or has the form $P(x_1,\ x_2,\ldots,\ x_n)$, where $x_1,\ x_2,\ldots,\ x_n$ are individual variables and P is a predicate symbol.

Via routinary flattening techniques, one proves the following:

Lemma 1. *There is an effective procedure to transform any formula φ of T into a formula ψ in normal form so that φ and ψ are equisatisfiable.*

A more restrained format than just "normal form" is defined here:

Definition 32 *A formula φ of T is said to be in* STANDARD NORMAL FORM *if it is a conjunction of literals of the forms:*

$$x = y\,,\, x = F(x_1,\ldots,x_n)\,,\, x \neq y\,,$$
$$P(x_1,\ldots,x_n)\,,\ \neg P(x_1,\ldots,x_n)\,,$$

where $x, y, x_1, \ldots, x_n$, stand for individual variables, F for a function symbol, and P for a predicate symbol.

Let S be the class of all formulas of T in standard normal form; the following holds:

Lemma 2. *T is decidable if and only if S is decidable.*

Proof: Clearly any algorithm for formulas in T is also an algorithm for formulas in S. For the converse, suppose that an algorithmic satisfiability test for S is available, and let φ be any formula of T. By applying the normalization process to φ, we get a formula ψ, in normal form such that φ and φ are equisatisfiable. We can now bring ψ to disjunctive normal form, thus obtaining a formula $\psi_1 \vee \cdots \vee \psi_\kappa$ where all ψ_i's are conjunctions, and we may assume that each ψ_i is in standard normal form, because any literal of type $\neg\, x = F(y_1,\ldots,y_n)$ within it can be replaced by the conjunction $\neg\, x = z \wedge z = F(y_1,\ldots,y_n)$, where z is a brand new variable. Since

$$\varphi \ \text{ is satisfiable} \leftrightarrow\ \psi \text{ is satisfiable } \ \leftrightarrow\ \psi_i \text{ is satisfiable for some } i$$

and since all transformations used to build conjunctions $\psi_1,\ldots,\psi_n$ are effective, our lemma follows. □

We are now almost ready to define an *ordered form* for RDF^+ formulas.

Definition 33 *A* DOMAIN VARIABLE *in a formula φ of RDF^+ is a numerical variable x which occurs in φ either as the argument of a term of one of the forms $f(x)$, $D[f](x)$, with f a functional variable, or as an end of some interval mentioned in φ (as exemplified by $Convex(f)_{[x,+\infty]}$).*

Definition 34 *An RDF^+ formula is said to be in* ORDERED FORM *if it is in standard normal form and has the form:*

$$\varphi \quad \wedge \quad \bigwedge_{i=1}^{n-1}(x_i < x_{i+1}),$$

where $\{x_1, \dots, x_n\}$ is the set of all distinct domain variables in φ.

The family RDF^+_{ord} of all ordered formulas of RDF^+ is a proper subset of RDF^+, nevertheless the following holds:

Lemma 3. *RDF^+ is decidable if and only if RDF^+_{ord} is decidable.*

Proof: omitted. □

The algorithm

We describe next the decision algorithm for satisfiability of formulas of RDF^+. In view of Lemma 3, we assume w.l.o.g. that φ is given in ordered form. The algorithm takes a formula φ of RDF^+ and reduces it, via a series $\varphi \rightsquigarrow \varphi_1 \rightsquigarrow \varphi_2 \rightsquigarrow \varphi_3 \rightsquigarrow \varphi_4 = \psi$ of transformations, to a formula ψ such that:

1. φ and ψ are equisatisfiable,
2. ψ is an existentially quantified Tarski formula, i.e. it contains only real variables, arithmetic operators $+, *$ and predicates $=, <$.

As recalled in the introduction, there exists a decision algorithm for Tarski formulas (cf. [15] by Tarski, and [6] by Collins). A decision algorithm for RDF^+ results from integrating Tarski's decision algorithm with the reduction $\varphi \rightsquigarrow \psi$ we are going to present.

In the following, w_i denotes a numerical variable, while z_i denotes an extended numerical variable.

The series of transformations we need goes as follows:

1. $\varphi \rightsquigarrow \varphi_1$: BEHAVIOR AT THE ENDPOINTS.
   (a) Let a, b be real numbers such that $a < b$, and f, g be real continuous functions in the closed interval $[a, b]$; then $f > g$ holds in the open interval $]a, b[$ if and only if one of the followings holds:
      i. $f > g$ all over $[a, b]$;
      ii. $f > g$ all over $[a, b[$, and $f(b) = g(b)$;
      iii. $f > g$ all over $]a, b]$, and $f(a) = g(a)$;
      iv. $f > g$ all over $]a, b[$, and $f(a) = g(a) \wedge f(b) = g(b)$.

(a$_1$) We can hence rewrite a conjunct of this or of an alike form, namely an atom of one of the forms

$$(f > g)_{]w_1,w_2[}\,,\ (f > g)_{[w_1,w_2[}\,,\ (f > g)_{]w_1,w_2]}\,,$$

as an equivalent disjunction comprising 4 or just 2 alternatives; in particular:

$$\begin{aligned}(f > g)_{]w_1,w_2[} \rightsquigarrow\ & (f > g)_{[w_1,w_2]} \vee \\ & ((f > g)_{[w_1,w_2[} \wedge f(w_2) = g(w_2)) \vee \\ & ((f > g)_{]w_1,w_2]} \wedge f(w_1) = g(w_1)) \vee \\ & ((f > g)_{]w_1,w_2[} \wedge f(w_1) = g(w_1) \wedge f(w_2) = g(w_2))\,, \\ (f > g)_{[w_1,w_2[} \rightsquigarrow\ & (f > g)_{[w_1,w_2]} \vee ((f > g)_{[w_1,w_2[} \wedge f(w_2) = g(w_2))\,.\end{aligned}$$

(a$_2$) Each such rewriting disrupts the structure of the overall formula, which we can readily restore by bringing it again to disjunctive normal form $\delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$ (where $n \in \{2,4\}$) by means of the distributive law

$$(\alpha \vee \beta) \wedge \gamma \ \leftrightarrow\ (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)\,,$$

and then working on each δ_i separately in the sequel of this algorithm.

(a$_3$) Let w_1, w_2 be numerical variables and f, g be functional variables. In each δ_i where the literals $(f > g)_{]w_1,w_2[}$, $f(w_1) = g(w_1)$, and $f(w_2) = g(w_2)$ occur together, when $w_1 < w_2$ as ordered domain variables and there are no domain variables between w_1 and w_2, we add the literals $w_1 < w$, $w < w_2$ and $f(w) = z$, where w and z are new numerical variables.[6] Plainly, the resulting formula and the original one are equisatisfiable.

(b) We then rewrite each atom of the form

$$(D[f] > t)_{]-\infty,w_2[}\,,$$

where f is a functional variable and t, w_2 are numerical variables, as the formula

$$(D[f] > t)_{]-\infty,w_1]} \ \wedge\ (D[f] > t)_{[w_1,w_2[} \ \wedge\ w_1 < w_2\,,$$

where w_1 is the first variable in the ordering of domain variables if w_2 is preceded by at least one such variable; otherwise, w_1 is a brand new domain variable.

We also perform the specular rewriting:

$$(D[f] > t)_{]w_1,+\infty[} \rightsquigarrow (D[f] > t)_{]w_1,w_2]} \ \wedge\ (D[f] > t)_{[w_2,+\infty[} \ \wedge\ w_1 < w_2\,.$$

[6] Notice that when, here as well as in steps (**b**) and (**c**) below, a new domain variable happens to enter into play, its position w.r.t. the ordering of the domain variables (see Def. 34) is mandatory.

We proceed similarly also in the two cases:

$$(D[f] < t)_{]-\infty, w_2[},\ (D[f] < t)_{]w_1, +\infty[}\,.$$

By these transformations we obtain an equisatisfiable formula.

(c) Let a, b and t be real numbers and f a function, $f \in C^1([a,b])$. Then $f' > t$ in $]a,b[$ if and only if one of the following holds:

i. $f' > t$ in $[a,b]$,
ii. $f' > t$ in $[a,b[$ and $f'(b) = t$,
iii. $f' > t$ in $]a,b]$ and $f'(a) = t$,
iv. $f' > t$ in $]a,b[$, $f'(a) = t$ and $f'(b) = t$.

The action to be made is similar to the one made under (**a**), i.e.:

(c$_1$) we rewrite conjuncts of the forms

$$(D[f] > t)_{]w_1,w_2[}\,,\quad (D[f] > t)_{[w_1,w_2[}\,,\quad (D[f] > t)_{]w_1,w_2]}\,,$$

as equivalent disjunctions, for example:

$$\begin{aligned}(D[f] > t)_{]w_1,w_2]} \rightsquigarrow\ & (D[f] > t)_{[w_1,w_2]} && \vee \\ & ((D[f] > t)_{]w_1,w_2]}\ \wedge\ D[f](w_1) = t)\,; \end{aligned}$$

(c$_2$) we bring again the overall formula into disjunctive normal form taking the distributive law

$$(\alpha \vee \beta) \wedge \gamma\ \leftrightarrow\ (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$$

into account.

(c$_3$) If in a formula three literals $(D[f] > t)_{]w_1,w_2[}$, $D[f](w_1) = t$, and $D[t](w_2) = t$ occur together, and are such that $w_1 < w_2$ as ordered domain variables, and there are no domain variables between w_1 and w_2, we add the following ones: $w_1 < w$, $w < w_2$ and $f(w) = z$.
The same for $(D[f] < t)_{]w_1,w_2[}$.

By applying rules (**a**), (**b**), and (**c**) to a formula ψ in ordered form, we obtain a finite disjunction of ψ_i formula; moreover ψ is satisfiable if and only if at least one of the ψ_i is satisfiable.
To each ψ_i we apply the rest of the algorithm.

2. $\varphi_1 \rightsquigarrow \varphi_2$: NEGATIVE-CLAUSE REMOVAL.
From φ_1 we construct an equisatisfiable formula φ_2 with only positive predicates. The general idea applied in this step is to substitute every negative clause involving a functional symbol with an implicit existential assertion.
For the sake of simplicity, in the following:

- $x, x_1, x_2, x_3, y_1, y_2, y_3$ will be numerical variables, new with respect the formula we will be considering,
- we will use the notation $x \preccurlyeq y$ as a shorthand for $x \leqslant y$ when $x,\ y$ are both numerical variables; otherwise (when either x is $-\infty$ or y is $+\infty$), $x \preccurlyeq y$ stands for $0 = 0$.

(a) Replace any literal $\neg(f = g)_{[z_1,z_2]}$ occurring in φ_1 by the formula:

$$(z_1 \preccurlyeq x \preccurlyeq z_2) \wedge y_1 = f(x) \wedge y_2 = g(x) \wedge \neg(y_1 = y_2).$$

(b) Replace any literal $\neg(f > g)_{[w_1,w_2]}$ occurring in φ_1 by the formula:

$$(w_1 \leqslant x \leqslant w_2) \wedge y_1 = f(x) \wedge y_2 = g(x) \wedge (y_1 \leqslant y_2),$$

In the same way we can remove literal on open and semi-open intervals.

(c) Replace any literal $\neg(D[f] = t)_{[z_1,z_2]}$ (resp. $<, \leqslant, >, \geqslant$) occurring in φ_1 by the formula:

$$(z_1 \leqslant x \leqslant z_2) \wedge y_1 = D[f](x) \wedge \neg(y_1 = t) \text{ (resp. } <, \leqslant, >, \geqslant \text{).}$$

(d) Replace any literal $\neg$Strict_Up $(f)_{[z_1,z_2]}$ (resp. $\neg$ Strict_Down $(f)_{[z_1,z_2]}$) occurring in φ_1 by the formula:

$$\Gamma \wedge y_1 \geqslant y_2 \quad \text{(resp. } \Gamma \wedge y_1 \leqslant y_2) \,,$$

where

$$\Gamma \coloneqq (z_1 \leqslant x_1 < x_2 \leqslant z_2) \wedge y_1 = f(x_1) \wedge y_2 = f(x_2).$$

(e) Replace any literal $\neg$ Convex$(f)_{[z_1,z_2]}$ (resp. $\neg$ Strict_Convex $(f)_{[z_1,z_2]}$) occurring in φ_1 by the formula:

$$\Gamma \wedge (y_2 - y_1)(x_3 - x_1) > (x_2 - x_1)(y_3 - y_1)$$

$$\text{(resp. } \Gamma \wedge (y_2 - y_1)(x_3 - x_1) \geqslant (x_2 - x_1)(y_3 - y_1)) \,,$$

where

$$\Delta \coloneqq (z_1 \leqslant x_1 < x_2 < x_3 \leqslant z_2),$$

$$\Gamma \coloneqq \Delta \wedge y_1 = f(x_1) \wedge y_2 = f(x_2) \wedge y_3 = f(x_3).$$

Literals of the forms $\neg$ Concave$(f)_{[z_1,z_2]}$, $\neg$ Strict_Concave $(f)_{[z_1,z_2]}$ are handled similarly.

Equisatisfiability of the formulas φ_1 and φ_2 is straightforward to prove. Using Lemma 2 and Lemma 3 we can normalize φ_2, to obtain an equivalent formula in ordered form with domain variables $v_1, v_2, \ldots, v_r$.

3. $\varphi_2 \rightsquigarrow \varphi_3$: EXPLICIT EVALUATION OF FUNCTIONAL VARIABLES.
This step is preparatory to the elimination of the functional clauses, by explicit evaluation of functional variables over domain variables. For each such variable v_j and for every functional variable f occurring in φ_2, introduce two new numerical variables y_j^f, t_j^f and add the literals $y_j^f = f(v_j)$ and $t_j^f = D[f](v_j)$ to φ_2. Moreover, for each literal $x = f(v_j)$ already occurring in φ_2 add the literal $x = y_j^f$; and for each literal $x = D[f](v_j)$ already occurring in φ_2 insert the literal $x = t_j^f$ into φ_3.
The formula φ_3 resulting at the end of these insertions and the original φ_2 clearly are equisatisfiable.
4. $\varphi_3 \rightsquigarrow \varphi_4$: ELIMINATION OF FUNCTIONAL VARIABLES.
As a final step, we get rid of all literals containing functional variables.

Define the index function $ind\colon V \cup \{-\infty, +\infty\} \to \{1, 2, \ldots, r\}$ over the set $\{v_1, v_2, \ldots, v_r\}$ of distinct domain variables of φ_3 as follows:

$$ind(x) := \begin{cases} 1 \text{ if } x = -\infty, \\ l \text{ if } x = v_l \text{ for some } l \in \{1, 2, \ldots, r\}, \\ r \text{ if } x = +\infty. \end{cases}$$

For each functional symbol f occurring in φ_3, let introduce the new numerical variables γ_0^f, γ_r^f and proceed as follows:

a) For each literal $(f = g)_{[z_1,z_2]}$ occurring in φ_3, add all literals $y_i^f = y_i^g$, $t_i^f = y_i^g$ whose subscript i satisfies $ind(z_1) \leqslant i \leqslant ind(z_2)$; moreover, if $z_1 = -\infty$ introduce the literal $\gamma_0^f = \gamma_0^g$, and if $z_2 = +\infty$ introduce the literal $\gamma_r^f = \gamma_r^g$.

b) For each literal $(f > g)_{[w_1,w_2]}$ (resp. $(f > g)_{]w_1,w_2[}$, $(f > g)_{[w_1,w_2[}$, $(f > g)_{]w_1,w_2]}$) occurring in φ_3, add the following literal:

$$y_i^f > y_i^g$$

for each $ind(w_1) \leqslant i \leqslant ind(w_2)$ (resp. $ind(w_1) < i < ind(w_2)$, $ind(w_1) \leqslant i < ind(w_2)$, $ind(w_1) < i \leqslant ind(w_2)$). Moreover if $w_1 < w_2$ as domain variables, in the case $(f > g)_{]w_1,w_2[}$ (resp. $(f > g)_{[w_1,w_2[}$, $(f > g)_{]w_1,w_2]}$) add the following literals:

$$t_i^f \geqslant t_i^g, \quad t_j^f \leqslant t_j^g \quad (\text{resp. } t_j^f \leqslant t_j^g \quad \text{or} \quad t_i^f \geqslant t_i^g)$$

with $i = ind(w_1)$ and $j = ind(w_2)$.

c) For each literal $(D[f] \bowtie y)_{[z_1,z_2]}$ occurring in φ_3, where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, add the following formulas:

$$t_i^f \bowtie y, \quad \frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j} \bowtie y,$$

for each $ind(z_1) \leqslant i, j \leqslant ind(z_2)$, $j \neq ind(z_2)$, and if $\bowtie \in \{\leqslant, \geqslant\}$ add the formulas:

$$\left(\frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j} = y\right) \to (t_j^f = y \wedge t_{j+1}^f = y);$$

moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^f \bowtie y$, and if $z_2 = +\infty$, introduce the literal $\gamma_r^f \bowtie y$.

d) For each literal $(D[f] \bowtie y)_{]w_1,w_2[}$ (resp. $(D[f] \bowtie y)_{]w_1,w_2]}$, $(D[f] \bowtie y)_{[w_1,w_2[}$) occurring in φ_3, where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, add the formulas:

$$t_i^f \bowtie y, \quad \frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j} \bowtie y$$

for each $ind(w_1) \leqslant j < ind(w_2)$ and for each $ind(w_1) < i < ind(w_2)$ (resp. $ind(w_1) < i \leqslant ind(w_2)$ and $ind(w_1) \leqslant i < ind(w_2)$).

e) For each literal Strict_Up$(f)_{[z_1,z_2]}$ (resp. Strict_Down$(f)_{[z_1,z_2]}$) occurring in φ_3, add the following literals:

$$t_i^f \geqslant 0 \ \ (\text{resp. } \leqslant), \quad y_{j+1}^f > y_j^f \ \ (\text{resp. } <),$$

for each $ind(z_1) \leqslant i, j \leqslant ind(z_2),\ j \neq ind(z_2)$; moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^f > 0$ (resp. $<$) and, if $z_2 = +\infty$, introduce the formula $\gamma_r^f > 0$ (resp. $<$).

f) For each literal Convex$(f)_{[z_1,z_2]}$ (resp. Concave$(f)_{[z_1,z_2]}$) occurring in φ_3, add the following formulas:

$$t_i^f \leqslant \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} = y \leqslant t_{i+1}^f \ \ (\text{resp. } \geqslant),$$

$$\left(\frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} = t_i^f \ \vee \ \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} = t_{i+1}^f \right) \to (t_i^f = t_{i+1}^f),$$

for each $ind(z_1) \leqslant i < ind(z_2)$; moreover, if $z_1 = -\infty$ introduce the literal $\gamma_0^f \leqslant t_1^f$ (resp. $\geqslant$), and, if $z_2 = +\infty$, introduce the literal $\gamma_r^f \geqslant t_r^f$ (resp. $\leqslant$).

g) For each literal Strict_Convex$(f)_{[z_1,z_2]}$ (resp. Strict_Concave$(f)_{[z_1,z_2]}$) occurring in φ_3, add the following formulas:

$$t_i^f < \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} < t_{i+1}^f \ \ (\text{resp. } >),$$

for each $ind(z_1) \leqslant i < ind(z_2)$; moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^f < t_0^f$ (resp. $>$), and, if $z_2 = +\infty$, introduce the literal $\gamma_r^f > t_r^f$ (resp. $<$).

h) Drop all literals involving any functional variable.

The formula φ_4 resulting at the end involves only real variables, hence it can be decided by means of Tarski's method.

4 Correctness of the algorithm

In order to prove the correctness of the algorithm, it is enough to show that each one of the (*terminating*) transformations $\varphi \rightsquigarrow \varphi_1$, $\varphi_1 \rightsquigarrow \varphi_2$, $\varphi_2 \rightsquigarrow \varphi_3$, $\varphi_3 \rightsquigarrow \varphi_4$ is *satisfiability preserving*. Regarding the first three of them (behavior at the endpoints, negative-clause removal, explicit evaluation of functional variables), this should emerge from the comments in the algorithm description section.

We must focus on the equisatisfiability of the formulas φ_3 and φ_4, because the transformation $\varphi_3 \rightsquigarrow \varphi_4$ is less transparent than the previous ones: we are, in fact, comparing a formula whose predicates regard the behavior of functions in real intervals with another one which only involves relations between numerical variables. Let us sketch the idea behind the proof (the full proof is unaffordably long for a conference paper), which consists as usual of two parts: *soundness* and *completeness*. Recall that φ_4 is obtained from φ_3 by adding some formulas that involve only numerical variables, and by removing all predicates which refer to functional variables. **Soundness:** If a model exists for φ_3, it can be extended to a model that also verifies the numerical formulas added in φ_4, since these formulas reflect properties of the functions in φ_3 at specific points of real intervals. **Completeness:** Conversely, if there exists a model for φ_4, it is possible to extend it to φ_3 by interpreting the functional variables with suitable interpolation functions. In particular, in the proof we use a family of interpolation functions constructed as the sum of a linear component with a perturbation (that we call "elastic") piecewise-defined starting from quadratic and exponential expressions. These functions, and their derivatives, consistently express the properties of φ_3 at points and real intervals.

5 An example

To see the decision algorithm for RDF^+ at work on a paradigmatic instance, we will carry out one by one the transformations leading from φ to φ_4, where φ results from negation of the formula

$$(D[f] > 0)_{]a,b[} \quad \rightarrow \quad \text{Strict\_Up}(f)_{[a,b]}\,, \text{ with } a, b \text{ numerical variables}\,,$$

produced in Example 22 at the end of Sec. 2.

To automatically prove that this formula is true under any value-assignment to a, b, and f, we use our algorithm and Tarski's decision method to check the unsatisfiability of its opposite statement

$$(D[f] > 0)_{]a,b[} \quad \wedge \quad \neg\text{Strict\_Up}(f)_{[a,b]}\,.$$

Obviously, in fact, if no assignment makes the latter formula true, then every assignment makes our claim true. We hence apply the decision algorithm to the conjunction φ just seen.

$\varphi \rightsquigarrow \varphi_1$: BEHAVIOR AT ENDS.
According to action (**c**), we split the atom $(D[f] > 0)_{]a,b[}$ into a disjunction:

$$\begin{aligned}(D[f] > 0)_{]a,b[} \rightsquigarrow\ & (D[f] > 0)_{[a,b]} \vee \big[(D[f] > 0)_{[a,b[} \wedge D[f](b) = 0\big] \\ & \vee \big[(D[f] > 0)_{]a,b]} \wedge D[f](a) = 0\big] \\ & \vee \big[(D[f] > 0)_{]a,b[} \wedge D[f](a) = 0 \wedge D[f](b) = 0\big]\,;\end{aligned}$$

then we rearrange the overall formula into disjunctive normal form:

$$\begin{aligned}&[(D[f] > 0)_{[a,b]} \wedge \neg\text{Strict\_Up}(f)_{[a,b]}] \\ \vee\ & [(D[f] > 0)_{[a,b[} \wedge D[f](b) = 0 \wedge \neg\text{Strict\_Up}(f)_{[a,b]}] \\ \vee\ & [(D[f] > 0)_{]a,b]} \wedge D[f](a) = 0 \wedge \neg\text{Strict\_Up}(f)_{[a,b]}] \\ \vee\ & [(D[f] > 0)_{]a,b[} \wedge D[f](a) = 0 \wedge D[f](b) = 0 \wedge \neg\text{Strict\_Up}(f)_{[a,b]}];\end{aligned} \qquad (\dagger)$$

finally, to each disjunct φ_1 of this formula we apply the rest of the algorithm. For the sake of brevity, let us consider only the first disjunct.

$\varphi_1 \rightsquigarrow \varphi_2$: NEGATIVE-CLAUSE REMOVAL.
In order to remove the literal $\neg\mathrm{Strict\_Up}(f)_{[a,b]}$, we can perform the rewriting

$$\begin{aligned}(D[f] > 0)_{[a,b]} \wedge \neg\mathrm{Strict\_Up}(f)_{[a,b]} \rightsquigarrow{} & (D[f] > 0)_{[a,b]} \wedge a \leqslant x_1 < x_2 \leqslant b \\ & \wedge f(x_1) = y_1 \wedge f(x_2) = y_2 \wedge y_2 \leqslant y_1,\end{aligned}$$

thus getting our φ_2.

$\varphi_2 \rightsquigarrow \varphi_3$: EXPLICIT EVALUATION OF FUNCTIONAL VARIABLES.
To describe the evaluation more transparently, let us rename our variables as follows:

$$a \rightsquigarrow v_1, \quad x_1 \rightsquigarrow v_2, \quad x_2 \rightsquigarrow v_3, \quad b \rightsquigarrow v_4, \quad y_1 \rightsquigarrow y_2, \quad y_2 \rightsquigarrow y_3\,,$$

so that our formula φ_2 becomes:

$$(D[f] > 0)_{[v_1,v_4]} \;\wedge\; v_1 \leqslant v_2 < v_3 \;\leqslant v_4 \;\wedge\; f(v_2) = y_2 \wedge\; f(v_3) = y_3 \;\wedge\; y_3 \leqslant y_2.$$

We can now evaluate the function f over the domain variables $V = \{v_1, v_2, v_3, v_4\}$, by adding the necessary literals:

$$\begin{aligned}\varphi_3 := {} & (D[f] > 0)_{[v_1,v_4]} \wedge v_1 \leqslant v_2 < v_3 \leqslant v_4 \wedge f(v_2) = y_2 \wedge f(v_3) = y_3 \wedge y_3 \leqslant y_2 \\ & \wedge f(v_1) = y_1^f \wedge f(v_2) = y_2^f \wedge f(v_3) = y_3^f \wedge f(v_4) = y_4^f \wedge y_2 = y_2^f \wedge y_3 = y_3^f \\ & \wedge D[f](v_1) = t_1^f \wedge D[f](v_2) = t_2^f \wedge D[f](v_3) = t_3^f \wedge D[f](v_4) = t_4^f.\end{aligned}$$

$\varphi_3 \rightsquigarrow \varphi_4$: ELIMINATION OF FUNCTIONAL VARIABLES.
Finally, we can remove all literals containing functional variables, by substituting suitable clauses in their place. We thus get:

$$\begin{aligned}\varphi_4 := {} & v_1 \leqslant v_2 < v_3 \leqslant v_4 \wedge y_3 \leqslant y_2 \wedge y_2 = y_2^f \wedge y_3 = y_3^f \\ & \wedge t_1^f > 0 \wedge t_2^f > 0 \wedge t_3^f > 0 \wedge t_4^f > 0 \\ & \wedge \tfrac{y_2^f - y_1^f}{v_2 - v_1} > 0 \wedge \tfrac{y_3^f - y_2^f}{v_3 - v_2} > 0 \wedge \tfrac{y_4^f - y_3^f}{v_4 - v_3} > 0,\end{aligned}$$

a formula containing only numerical variables which can be submitted to Tarski's decision method. To see that this φ_4 is false, just focus on its sub-conjunction

$$v_2 < v_3 \wedge y_2 = y_2^f \wedge y_3 = y_3^f \wedge y_3 \leqslant y_2 \wedge \frac{y_3^f - y_2^f}{v_3 - v_2} > 0\,,$$

whose constraints $v_2 < v_3$ and $y_3^f = y_3 \leqslant y_2 = y_2^f$ yield that $\frac{y_3^f - y_2^f}{v_3 - v_2} \leqslant 0$.
By performing a similar analysis of the remaining three disjuncts of (†), one proves the validity of our example.

Related works and complexity issues

The decidability of *RDF*$^+$ theory treated above is a follow-up of a series of previous results, regarding the *RMFC*, *RMFC*$^+$, and *RDF* theories [3,1,5,2].

A general survey on those results can be found in [4], where, other decidability results on real analysis are also treated, in particular the FS theory [9,10].

Since the decidability of RDF^+ is obtained via an explicit algorithm, some complexity issues are worth being discussed here. Since Tarski's decision method enters into ours, our algorithm inherits its complexity as a lower bound. The first complexity amelioration w.r.t. Tarski's historical result is due to Collins [6], whose procedure has doubly exponential complexity relative to the number of variables occurring in the sentence (or just exponential, if the endowment of variables is finite and fixed). A refinement of this result was achieved with Grigoriev's algorithm [11], applicable to sentences in prenex normal form, whose complexity is doubly exponential relative to the number of quantifier alternations. If we merely focus on the *existential* theory of reals, the known decision algorithms have a complexity at best exponential relative to the number n of variables [8]; however, if one fixes beforehand how many variables can be used, then the algorithmic complexity becomes polynomial [12].

Finally, while Tarski himself proved the undecidability of EAR extended with some real functions [15] (for example $sin\ x$), in [13] Richardson proved the undecidability of the existential theory of reals extended with the numbers $\log 2$, π and with the functions e^x, $sin\ x$.

Conclusions and future work

This article has presented a decision algorithm for a fragment, RDF^+, of real analysis, which extends the elementary algebra EAR of real numbers with variables designating functions of a real variable endowed with a continuous derivative. After introducing the syntax and semantics of RDF^+, we have shown how to reduce a generic formula φ of RDF^+ to a logically equivalent formula ψ of EAR, thereby showing the decidability of RDF^+. The proof of correctness of the decision algorithm relies upon a particular class of functions, the $[\alpha, \theta_1, \theta_2]$-defined functions.

While writing this article, we began to spot out further promising extensions of the decidable theory. The main next extension we are envisaging will allow one to designate pointwise function addition by means of functional terms of the form $f+g$ and to constrain the new functions through atoms such as $(f+g > t)_{[w_1,w_2]}$.

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