The Polyhedral-Surface Cutting-Plane Method for Linear Combinatorial Optimization

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Abstract. For linear optimization on combinatorial sets inscribed into a convex surface, a Polyhedral-Surface Cutting-Plane Method (PSCM) is offered. It essentially uses representability of such sets as an intersection of a polytope with a circum surface. Three versions of PSCM are formulated depending on the choice of surface involved. A justification of applicability of PSCM for linear permutation-based and Boolean optimization problems is given, which opens perspectives to solve in a reasonable time a significantly wider class of real-world tasks modelled as combinatorial optimization problems.

Keywords: Linear Optimization, Combinatorial Optimization, vertex-located set, surface-located set, cutting-plane method, polyhedral relaxation.

1 Introduction

In Combinatorial Optimization and Convex Optimization, cutting-plane methods play a special role in illustrating how discrete optimization problems can be solved by continuous optimization methods [2, 14, 24, 26]. The effectiveness of cutting-plane methods substantially depends on the depth of the cuts and the computational complexity of their construction. In Combinatorial Optimization, when developing optimization methods, the most important component is the study and utilizing of the specifics of a feasible domain of optimization problems under consideration [5, 7, 13, 24]. The present paper is dedicated to developing a cutting-plane method for one class of combinatorial problems.

2 Problem 1 Statement

Let us consider a linear optimization problem over a finite point configuration (FPC) [11] E' in R^n:

$$cx \to min,\tag{1}$$

$$Ax \le b, \tag{2}$$

$$x \in E', \tag{3}$$

$$E' = P' \cap S, \tag{4}$$

where $A \in \mathrm{R}^{m \times n}, b \in \mathrm{R}^{m}, c, x \in \mathrm{R}^{n}$, P' is a full-dimensional polytope, S is a convex hypersurface, E' is an FPC, i.e.,

$$dim P' = n, \tag{5}$$

$$1 < |E'| < \infty, \tag{6}$$

$$\exists f : \mathrm{R}^{n} \to \mathrm{R}^{1} - convex : S = \{ x \in \mathrm{R}^{n} : f(x) = 0 \}. \tag{7}$$

A statement of this problem (further referred to as **Problem 1**) is as follows: it is necessary to find a solution x^{*} of the problem (1)-(4), which is the problem of minimizing the linear function (1) with linear constraints (2) on a finite set of points R^{n} formed as an intersection of a convex surface S and the full-dimensional polytope P expressed in the constraint (4).

This means that conditions (5)-(7) are also satisfied, in particular, there exists a convex function $f(x)$ defining this surface.

Here, the condition (4) is present that distinguishes this problem from the generic linear discrete optimization problem.

This allows offering specific methods for solving Problem 1 that use the specifics of the admissible domain.

Let $\left\langle x^{*}, z^{*} \right\rangle = \left\langle x^{*}, cx^{*} \right\rangle$ be a solution to Problem 1.

Among the features of E is that, without loss of generality, it can be assumed that P is a convex hull of E. In other words, it is a combinatorial polytope corresponding to this set. The set's representation (4) is called *polyhedral-surfaced* [19, 20]. The next important feature is that E is vertex-located set (VLS) [27], i.e., it coincides with a vertex set of its convex hull.

Respectively, a feasible domain E of Problem 1 will also be VLS. Associating a combinatorial polytope P to set E, this allows constructing a polyhedral-surface representation involving P and S. This Problem 1 specifics can be summarized as follows:

1. One can assume that P' is *a combinatorial polytope* (**CP**) associated with E', i.e.,

$$P' = conv E', \tag{8}$$

then (4) is a polyhedral-surfaced representation of E' (E'.**PSR**) as an intersection of a CP P' with a circumsurface S.

2. From (4), it follows that E' is *a vertex-located set* (**VLS**), i.e., $E' = vert P'$.

3. Let E be a feasible region of Problem 1, and P be its convex hull:

$$E = \{x \in E' : Ax \leq b\},$$

$$P = convE,$$

then E is a VLS, while P is a CP associated with E.

It is easy to see that an arbitrary VLS allows forming a polyhedral-surface representation.

Its constructing it is a separate task, which includes finding an H-representation of the polytope and equality of circumscribed surface S. Sets satisfying (4) are called *polyhedral-surface* (**PSSs**), among which are *polyhedral-spherical* (**PSSs**), *polyhedral-ellipsoidal* (**PESs**), and so on [19, 20, 29].

In this paper, special attention we will be paid to polyhedral-spherical sets (PSSs), because their specificity in solving linear combinatorial optimization problems and a variety of practical problems modelled as optimization problems on PSSs [17, 21, 29]. The class of such sets includes a set of n-multipermutations induced by a numerical multiset G containing k different elements, which is an image in R^n of the combinatorial space of permutations with repetitions [19,20]. It is a set

$$E_{nk}(G) = \{x \in \mathrm{R}^n : \{x\} = \{x_1, ..., x_n\} = G\} \tag{9}$$

induced by a multiset

$$G = \{g_i\}_{i \in J_n = \{1,...,n\}} \subset \mathrm{R}^1 : g_i \leq g_{i+1}, i \in J_{n-1},$$

with a ground set $S(G) = \{e_i\}_{i \in J_k} : e_i < e_{i+1}, i \in J_{k-1}$;

Also, this class includes a set of signed n-multipermutations [17, 23] that differs from the previous one in that signs are added to the elements of the permutation:

$$E_{nk}^{\pm}(G) = \{x \in \mathrm{R}^n : \{|x|\} = \{|x_1|, ..., |x_n|\} = G\}. \tag{10}$$

Finally, these are the well-known Boolean set B_n and binary set B_n', where

$$B_n = \{0,1\}^n; \tag{11}$$

$$B_n' = \{-1,1\}^n, \tag{12}$$

on which Boolean optimization problems are modelled [5, 13, 15, 16, 32], as well as their various subsets, such as the permutation matrices [4], even and odd permutation sets [30], a vertex set of the half-cube [6, 12], etc.

Thus, Problem 1 covers all problems of linear Boolean optimization, linear problems on permutations, allowing formulation on a set of multipermutations and signed multipermutations, and many others.

3   Cutting-Plane Approaches to VLS-Optimization

A method we propose to solve Problem 1 generalizes a method of combinatorial cuts (CCM) for solving linear programs on VLSs [8, 10]. CCM utilizes a standard idea to solve linear combinatorial optimization problems with the help of their polyhedral relaxations [5, 15]. This means that, for its effective utilization, there must be ways to solve the corresponding polyhedral relaxation problems in polynomial time on their dimension. Generally, this stage is complicated because combinatorial polytopes can contain an exponential number of constraints [1, 9, 17, 22, 31]. CCM uses an original idea of constructing deep cuts through adjacent vertices of the relaxation polytope P, since the feasible points are located at its vertices only. In this connection, one more constraint arises on the number of adjacent vertices of a a vertex of combinatorial polytopes. For example, the Birkhoff polytope is the convex hull of set Π_n, which is described by a polynomial number of constraints [4]. However, the number of adjacent vertices is exponential on n. Thus, to solve conditional linear problems on Π_n, by a method that analyzes all adjacent vertices is problematic.

CCM is applicable to solving Problem1, if :

1. its polyhedral relaxation (PR) is polynomially solvable on n (**Property 1**);

2. any point of E has a number of adjacent vertices in P' polynomial on n (**Property 2**).

Among sets with Properties 1, 2 are the listed above sets (10)-(12). Let us illustrate this by set $E_{nk}(G)$:

1. $E = B_n'$: a) an H-representation of a hypercube $PB_n' = conv\ B_n'$ is well-known and has $2n$ constraints; b) $\forall x \in B_n' \ \left| N_{PB_n'}(x) \right| = n$;

2. Despite the fact that the generalized permutohedron $P_{nk}(G)$, which is the convex hull of $E_{nk}(G)$, is generally defined by a non-polynomial number of constraints [25,31], the polyhedral relaxation problem can be solved efficiently based on the property (13), which allows using an insignificant part of the constraints of the polytope [25]. So, for $E = E_{nk}(G)$: a) $P_{nk}(G) = conv\ E_{nk}(G)$ - is the generalized permutohedron; b) $\forall x \in E_{nk}(G) \ \left| N_{P_{nk}(G)}(x) \right| = n_1 n_2 + ... + n_{k-1} n_k$; c) an H-representation is known and has up to $2^n - 1$ constraints. However, for $E_{nk}(G)$, PR of Problem 1 is polynomially solvable based on the following fact: $\forall x \in \mathrm{R}^n$ such that $x_i \le x_{i+1}, i \in J_{n-1}$,

$$x \in P_{nk}(G) \Leftrightarrow \sum_{i=1}^{j} x_i \ge \sum_{i=1}^{j} g_i, j \in J_{n-1}; \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} g_i. \tag{13}$$

Here, $N_{P^{[\cdot]}}(x^j)$ is a neighbourhood of x^j in $P^{[\cdot]}$

3.1 Modified CCM (MCCM)

In the original version of the CCM [8,10], cutting planes for a polyhedral relaxation solution, which is not a point of E', are constructed based on the analysis of the last simplex table, from which information about adjacent vertices to this point is derived. As a result, a cut of the unfeasible solution is formed through at least adjacent vertex of P'.

In this paper, we offer a modified CCM (MCCM), where geometric features of E' and P' are substantially used, resulting in a possibility of deeper cuts built on their basis than CCM-ones. So, on the initial iteration, the usual polyhedral relaxation of Problem 1 is solved. If its solution x^0 is a point of E', this problem is solved. Otherwise, we move to the construction of a cut. To do this, we construct a neighbourhood of the point x^0 in the polytope P' and draw a hyperplane Π^0 through the neighbourhood points.

MCCM outline:

Step 0. Initial iteration $j = 0$, a search domain is $P^j = P'$, the number of additional constraints is $m^j = m$.

Step 1 (Main Stage)

a) solve a problem: find $x^j = arg\min_{P^j} f(x)$;

b) if $x^j \in E'$, then terminate;

c) otherwise, form $N_{P^j}(x^j)$, choose the shortest n edges $\left[x^j, x^{ji_l} \right], l \in J_n$, of P^j intersecting at x^j. The edges' endpoints are:

$$X^j = \left\{ x^{ji_l} \right\}_{l \in J_n} \subseteq vert \ P^j; \qquad (14)$$

d) form a hyperplane through points (14):

$$\Pi^j = \left\{ x : a^j_{m^j+1} x - b^j_{m^j+1} = 0 \right\} : a^j_{m^j+1} x^j - b^j_{m^j+1} > 0; \qquad (15)$$

e) choose a subspace cutting off x^j:

$$\alpha^j : a^j_{m^j+1} x - b^j_{m^j+1} \leq 0. \qquad (16)$$

Step 2. Go to the next iteration:

$$j = j+1, \ m^j = m^{j-1}+1, \ P^j = \left\{ x \in P^{j-1} : a^j_{m^j} x - b^j_{m^j} \le 0 \right\}.$$

Repeat Steps 1-2 until a valid point of E' is obtained - $x^j \in E'$.

Output: $x^* = x^j$. (16) is a cut for x^j, if the following condition holds:

$$\forall x' \in N_{P^j}(x^j) \quad a^j_{m^j+1} x' - b^j_{m^j+1} \le 0. \tag{17}$$

For fulfilling (17), it is sufficiently that

$$n_j = \left| N_{P^j}(x^j) \right| = n. \tag{18}$$

If (17) is false, cut x^j by a hyperplane Π^j is built passing through a point of $N_{P^j}(x^j)$ in one of the ways:

According to Lemma 1, Π^0 will certainly be a cutting plane if the number of adjacent vertices to x^0 equals the dimension of the polytope. Moreover, the plane is uniquely determined. Condition (17) is a check that as a result of adding constraint (16) no other vertex of P^0 than x^0 was cut off. Using it, one can verify that the cut (16) is valid if the number of adjacent vertices exceeds n. If the condition (17) is not satisfied, we propose using the standard cut of CCM [8, 10] (further referred to as **Way 1**).

Note that, for polyhedral-spherical sets, one can use the following their feature: if there are two different hyperspheres circumscribed about E', then E' lies in an intersection plane of these hyperspheres, which equation can be easily found from the equations of these hyperspheres. So, for PSpS $E' \subset S_r(a)$, we recommend forming a hypersphere $S_{r^j}(x^j)$ centred at x^j with a radius $r^j = 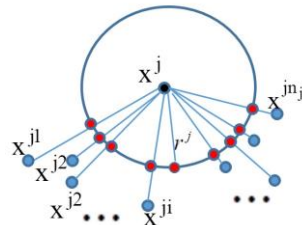\min_{i \in J_{n_j}} \left| x^{ji} - x^j \right|$ and, if $S' = S_r(a) \cap S_{r^j}(x^j)$ is $n-2$-sphere, take, as a cutting plane (15), a hyperplane where S' lies (further referred to as **Way 2**).
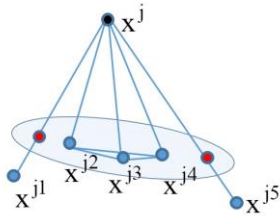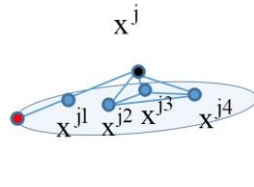
Fig. 1. (17) is true   **Fig. 2.** (17) is false   **Fig. 3.** Way 2 illustration

Figure 1 illustrates a case, where the hyperplane passing through the n nearest vertices is cutting off, while Figure 2 shows a situation, where such a plane cannot be selected because it is not valid, because the point x^j is cut off. Finally, Figure 3 illustrates one more method of constructing a cut through the adjacent vertex closest to x^i.

At the next step, the valid cut is added to the constraints, and the procedure is repeated in the same way until point of E' is found as a solution of a polyhedral relaxation problem.

Figure 4 illustrates the MCCM for the case of polyhedral-spherical E'. First, x^0 is found on iteration 0. It does not belong to E'. Thus we build a cut for it. It is formed through vertices x^{01} and x^{02} adjacent to it. Then repeat the procedure similarly until we find the optimal solution x^* to Problem 1.

3.2   Combinatorial Polytope Cutting-Plane Method (CPCM)

The second approach, a combinatorial polytope cutting-plane method (CPCM), utilizing the fact that there are no feasible points of E' in the interior of the combinatorial polytope P and all its faces of any dimension.

CPCM uses an observation that feasible points of a VLS E are absent in an interior of a CP P' and its faces of the dimension at least one. Let us represent the method as a modification of MCCM.

The modification core: at the iteration j of the main stage of MCCM, extend the edges

$$\left[x^j, x^{ji} \right], i \in J_{n_j}, \tag{19}$$

up to their intersection with a surface $\Gamma P'$ getting extended edges

$$\left[x^j, y^{ji} \right], i \in J_{n_j}, \tag{20}$$

Endpoints of first n the shortest edges are used instead of (14):

$$X^j \rightarrow Y^j = \left\{ y^{ji'_l} \right\}_{l \in J_n} \subset \Gamma P' \tag{21}$$

in constructing a cut (16).

As a result, a hyperplane (15) is built through points (21).

As a modification of the CCM, it looks like this: edges (19) of the polytope P^j are extended until they intersect a surface of polytope P' resulting on forming set (20). By a choose of the nearest n to x^j among the set's points, a set Y^j is formed, which is then used instead of X^j to form the cut.

We operate with a bound of P', therefore (18) is replaced by:

$$\forall x' \in \left\{ y^{ji} \right\}_{i \in J_{n_j}} \subset \Gamma P' \quad a^{j}_{m^j+1} x' - b^{j}_{m^j+1} \leq 0. \tag{22}$$
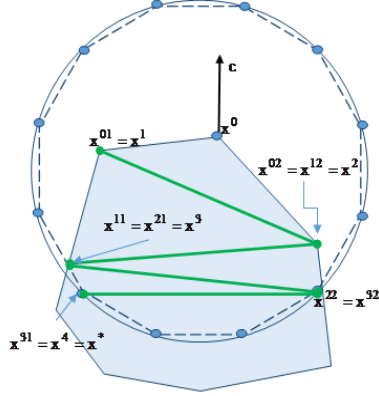
Fig. 4. The Combinatorial Polytope Cutting-Plane Method (CPCM)
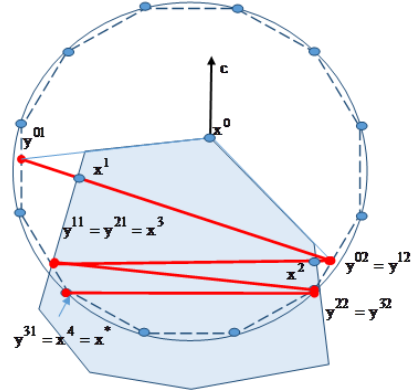
Fig. 5. The Surface-Cutting Method (SCM)

Fig. 5 shows how edges of the polytope P^0, induced by x^0, are extended to an intersection with a surface of the combinatorial polytope and the cut is conducted through the formed points. Then, the procedure is repeated.

3.3 Surface Cutting-Plane Method (SCM)

A third approach is a surface cutting-plane method (SCM), which offers a greater extension of the edges forming x^0. Namely, they go up to an intersection with surface S resulting in forming a set Z^j of intersection points. Then, the cut is formed through all or a part of these points. Here we use a VLS-feature, that feasible points can only be on a circumsurface S Not that, as a result of such a cut, not only an unfeasible part of the combinatorial polytope is cut off, but also an unfeasible piece of S.

Thus SCM exploits the fact that $x^* \subset S$ and $E \subset P \subseteq P' \subset C = conv \, S$. This inclusion suggests further prolongation of the edges $\left[x^j, x^{ji} \right], i \in J_{n_j}$, unless they intersect S and forming the plane Π^j through n of these intersection points $z^{ji}, i \in J_{n_j}$, the closest to x^j. As a result, we utilize edges $\left[x^j, z^{ji} \right], i \in J_{n_j}$, endpoints of the shortest n of which:

$$Z^j = \left\{ z^{ji''_l} \right\}_{l \in J_n} \subset S \tag{23}$$

are used for constructing a hyperplane (15), respectively, the condition (18) is replaced by:

$$\forall x' \in \left\{ z^{ji} \right\}_{i \in J_{n_j}} \subset S \ a^j_{m^j+1} x' - b^j_{m^j+1} \leq 0. \tag{24}$$

An illustration of this method is given in Figures 6, 7. It shows a comparison of cuts constructed by all listed methods. The figure illustrates that a SCM-cut is deepest.
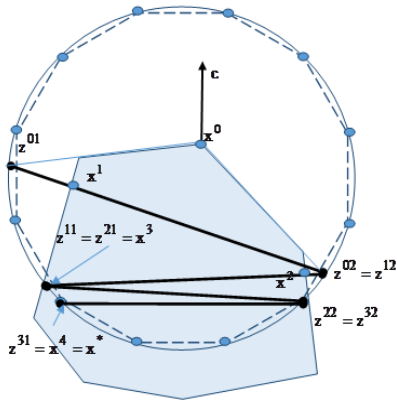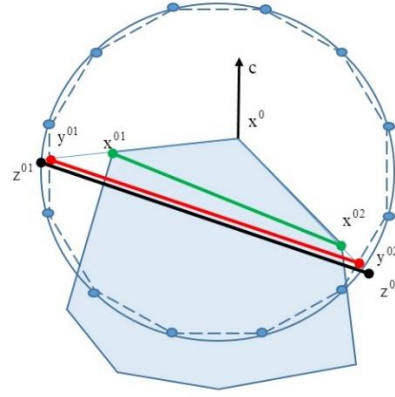

Fig. 6. SCM illustration   **Fig. 7.** An SCM comparison

3.4 Polyhedral-Surface Cutting-Plane Method (PSCM)

Let us generalize CCM, CPCM, and SCM in a method called the *Polyhedral-Surface Cutting Plane Method* (**PSCM**).

For that, we introduce a polyhedral cone with a vertex x^j:

$$Cone(x^j) = \{ x \in R^n : \tilde{A}^j x \leq \tilde{b}^j \}, \tag{25}$$

where $\tilde{A}^j x \leq \tilde{b}^j$ is the collection of constraints of an H-representation of P^j active at x^j.

Let S' be a surface:

$$S' \in \{ S, S^j, S'^j \}, \tag{26}$$

where $S^j = \Gamma D^j$, $D^j = \{x \in \mathrm{R}^n : \overline{A}^j x \le \overline{b}^j\}$,

$S^j = \Gamma D^j$, $D^j = \{x \in \mathrm{R}^n : \overline{A}^j x \le \overline{b}^j\}$, $\quad$ $S^{'j} = \Gamma D^{'j}$, $D^{'j} = \{x \in \mathrm{R}^n : \overline{A}^{'j} x \le \overline{b}^{'j}\}$,

$\overline{A}^j x \le \overline{b}^j$ and $\overline{A}^{'j} x \le \overline{b}^{'j}$ are collections of constraints of P^j and P' respectively, which are non-active at x^j. The cone (25) allows the conical combination description (**CCD**):

$$Cone(x^j) = \left\{ x^j + \sum_{l=1}^{n_j} \lambda_l v^{jl} : \lambda_l \in \mathrm{R}_+^1, v^l \in V^l, l \in J_{n_j} \right\}, \qquad (27)$$

where v^{jl} - is the direction vector of an edge $\left[x^j, x^{ji} \right]$, $x^{ji} \in N_{P^j}(x^j)$, $i \in J_{n_j}$, a set $V^j = \{v^{ji}\}_{i \in J_{n_j}}$ is found in such a way that $V^j \subset \mathrm{S}^j$.

PSCM outline: On each iteration, choose a surface (26).

Find a cone (25) and its CCD (27). Form a cutting plane (15) through points $V^{'j} = \{v^{ji_l}\}_{l \in J_n} \subseteq V^j$, which are the closest to x^j.

Verify a condition: if $\forall x' \in V^j$ $a^j_{m^j+1} x' - b^j_{m^j+1} \le 0$ holds, for instance, if $V^{'j} = V^j$, then the corresponding new constraint (16) is added to current constraints. Otherwise, form a hyperplane through a point $v^j \in V^j$, which is the closest to x^j and cut it off. Then the corresponding cut is added to the current collection of constraints.

For instance, if $E \subset S_r(a)$ form a hypersphere $S_{r^j}(x^j)$ centred at x^j with a radius $r^j = \left| v^j - x^j \right|$, as (15) it can be taking a hyperplane, where $n-2$-sphere $S_r(a) \cap S_{r^j}(x^j)$ is situated.

In (26), if at step j $S = S^j$ is chosen, PSCM becomes MCCM; if $S = S^{'j}$ - CPCM; if $S = S$ - SCM. Thus, PSCM covers all the above cutting-plane approaches. Moreover, it allows choosing different surfaces from a family (26) at various stages of implementing PSCM, thus combining MCCM, CPSM, SCM together.

4 Discussion

Problem 1 can be solved to optimality by another well-known approach – Branch and Bound [5, 13, 24]. For sets (9)-(12), the corresponding methods can utilize partitions of these sets into lower-dimensional ones of the same combinatoral type [19, 20]. Other original approaches to linear constrained permutation optimization are

described in [25, 29, 30]. Among cutting-plane methods attacking Problem 1 is the one presented in [18], where another generalization of CCM is presented called a Spherical Cutting-plane Method (SCPM). Along with polyhedral sphericity of (9)-(12), it utilizes another important feature of these sets - the simplicity of solving unconstrained linear problems. This results in a possibility to form cuts of unfeasible solutions of polyhedral relaxations by cutting planes, which do not require a search of their neighbourhood. In turn, an advantage of PSCM over SCPM is a depth of cuts. The listed approaches can be combined with approximate and heuristics ones [25,28,30], which expectedly will result in an expansion of a class of Boolean, permutation-based, and other combinatorial problems having wide real-world applications solvable in a reasonable time.

5   Conclusion

In this paper, we propose a new cutting plane method for solving linear optimization problems on VLSs, using the ability to fit these sets into a convex hypersurface. The scope of this method is not limited to such sets, since the optimization problem on an arbitrary discrete set can be reduced to optimization on one or more VLSs. In the future, we plan to develop PSCM into new classes of combinatorial sets, in particular, we are working on the search for new cutting plane methods.

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