Interval Exchange Transformations: from Symbolic Dynamics to Combinatorics

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Abstract: An Interval Exchange Transformations, or IET, is a dynamical system obtained by iterating simple geometric transformations on an interval. Such kind of system can be seen as a generalization of the rotation of the circle. Using an operation called natural coding, one can obtain from a IET a formal language and a shift space, both satisfying interesting and peculiar properties. In this contribution we use IETs to describe the connections between symbolic dynamics and combinatorics on words, using tools ranging from formal language theory to ergodic theory. We also introduce Rauzy induction on a IET, which can be viewed as a generalization of the continued fraction expansion.

1 Interval exchange transformations

A *symbolic dynamical system* is a pair (X,σ) formed of a topological space X and a continuous transformation σ. A classical example of symbolic dynamical systems is given by Interval Exchange Transformations.

Let us consider the semi-interval $[0,1[$ and a partition $(I_a)_{a\in\mathscr{A}}$ of $[0,1[$, with $\mathscr{A}$ an ordered alphabet. The *interval exchange transformation* (or *IET*) relative to $(I_a)_{a\in\mathscr{A}}$ is the map $T : [0,1[\to [0,1[$ defined by

$$T(z) = z+\alpha_a \qquad \text{if } z\in I_a$$

where α_a is the length of the semi-interval I_a. Observe that the restriction of T to I_a is just a translation to $T(I_a)$.

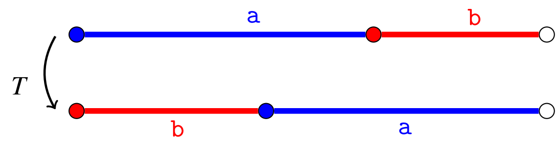
A IET can be represented by two copies of the semi-interval $[0,1[$, one showing the partition $(I_a)_{a\in\mathscr{A}}$ and the other the partition given by $(T(I_a))_{a\in\mathscr{A}}$.

Example 1. *Let R be the interval exchange transformation corresponding to $\mathscr{A} = \{a,b\}$, with $a<b$, and let $I_a = [0,1-\alpha[$ and $I_b = [1-\alpha,1[$. The transformation R is the rotation of angle α defined by*

$$R(z) = z+\alpha \bmod 1$$

and it is represented in Figure 1 (here $\alpha < \frac{1}{2}$).

The *orbit* of a point z is the set $\mathscr{O}(z) = \{T^n(z) \mid n\in\mathbb{Z}\}$. A transformation T is called *minimal* if the orbit of every point is dense in $[0,1[$. It is *regular* if the orbits of



the starting points of the semi-intervals I_a, for $a\in\mathscr{A}$, are infinite and disjoint. Keane proved in [3] that a regular IET is minimal (the opposite is, in general, not true).

Example 2. *Every rotation of irrational angle α is a regular IET (see Example 1).*

2 Natural coding

The *natural coding* of a IET T with respect to a point $z\in[0,1[$ is the infinite word $\Sigma_T(z) = a_0a_1\cdots$ on the alphabet $\mathscr{A}$ defined by

$$a_n = a \qquad \text{if } T^n(z)\in I_a.$$

Note that a bi-infinite version of such a coding can be easily defined by considering the transformation T^{-1}.

Given a (finite or infinite) word w, we say that u is a *factor* of w if it is possible to write $w = xuy$ for some words x,y. The set

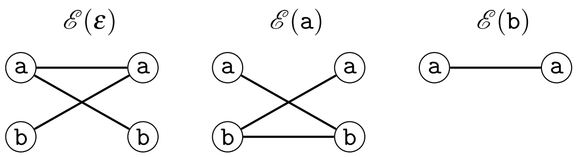
$$\mathscr{L}(T) = \bigcup_{z\in[0,1[} \mathrm{Fac}\,(\Sigma_T(z))$$

of all factors of all possible natural codings of a IET T is called the *language* associated to T. When T is minimal (resp. regular), this language does not depend on the choice of the point, but only of T; in this case we call $\mathscr{L}(T)$ a *minimal* (resp. *regular*) *interval exchange set*.

3 Extension graphs

Given a language (factorial set) $\mathscr{L}$ and a word $w\in\mathscr{L}$, it is possible to define a bipartite graph, called the *extension graph* of w (with respect to $\mathscr{L}$), denoted $\mathscr{E}(w)$, describing the extensions of the word. Such a graph has on the left (resp. on the right) the left-extensions (resp. the right-extensions) of w, and the edges are given by the bi-extensions.

Example 3. *Let us consider a language $\mathscr{L}$ containing as elements of length* 3 *the words* `aab`, `aba`, `baa`, `bab` *(but not* `aaa` *nor any word containing* `bb`*). The extension graphs of the empty word* ε *is represented in Figure 3 on the left, while the empty graph of the letters* `a` *and* `b` *are represented respectively on the same figure on the center and on the right.*



A language such that the extension graph of every word in it is a tree (acyclic and connected graph) is called *dendric*.

Theorem 1 ([2, 1])**.** *Regular interval exchange sets are dendric.*

4 Subshifts

Let T be a regular IET and $z \in [0,1[$. The closure $\overline{\mathscr{O}(z)}$ of the orbit of z, together with the *shift transformation* $\sigma : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$, defined by $\sigma(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$ is a symbolic dynamical system, called a *IET subshift*. It is known that its entropy is zero. Moreover it is easy to find an invariant probability measure μ on (X,σ) by associating to each word $w = a_0a_1\cdots a_{m-1}$ a semi-interval I_w as

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \cdots \cap T^{-m+1}(I_{a_{m-1}})$$

and defining $\mu([w]) = |I_w|$, where $|\cdot|$ is the classical Lebesgue measure. Keane conjectured in 1975 that every symbolic dynamical system associated to a regular IET is uniquely ergodic, that is that μ is the only possible invariant probability measure one can define on it. However, Keynes and Newton proved in [4] that this is not the case, even though almost all (in probability) such systems are uniquely ergodic ([5, 7]).

5 Rauzy induction

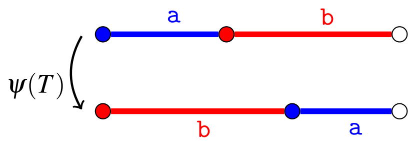
Let $J \subset [0,1[$ be semi-interval. If a IET T is minimal, for each $z \in [0,1[$ there is a $n \geq 0$ such that $T^n(z) \in J$. The *transformation induced* by T on J is the IET $S : J \to J$ defined for $z \in J$ by $S(z) = T^n(z)$ with $n = \min\{k > 0 \mid T^k(z) \in J\}$.

The *Rauzy induction* ([6]) is the map ψ that send a IET T to the IET $\psi(T)$ induced by T on the domain $J = [0,r[$, where $r < 1$ is the right-most above all the starting points of the semi-interval in the partition $(I_a)_{a\in\mathscr{A}}$ and the images of such starting points. The new partition is given by $(K_a)_{a\in\mathscr{A}}$ where $K_a = S^{-1}(T(I_a) \cap J)$.

Example 4. *The transformation* $\psi(T)$*, where* T *is the IET defined in Example 1, is given by*

$$\psi(T)(z) = \begin{cases} T^2(z) & \text{if } z \in I_{\mathtt{b}} \\ T(z) & \text{otherwise}. \end{cases}$$

Note that J *is the semi-interval starting with* 0 *and ending at the starting point of* $I_{\mathtt{b}}$*. The transformation is also represented in Figure 5.*



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