Analytical and numerical solutions of a fractional conformable derivative of the modified Burgers equation using the Cole-Hopf transformation[1]

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Abstract

In this paper, we deal with a study of modified time-fractional Burgers equations. The idea is based on the use of a Cole-Hopf transformation which transforms the time-fractional modified Burgers equations into linear parabolic time fractional equations. To solve the latter, we use the Fourier transformation. Therefore, the solution of the modified time-fractional Burgers equations can be found by using the solution of parabolic equation and the inverse Cole-Hopf transformation.

Keywords

Cole-Hopf transformation, Conformable derivative, Modified Burgers' equation.

1. Introduction

This work considers the following modified Burgers equations [?]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (\nu + u)u_x = r u_{xx}, \tag{1.1}$$

where ν and r are nonnegative parameters, α is the fractional derivative, $0 < \alpha \leq 1$, $x \in [0, b]$, $t > 0$, $\partial^\alpha u/\partial t^\alpha$ mean conformable derivative of the function $u(x, t)$.
When $\nu = 0$, we get the conformable derivative Burgers equation. So the term νu translates the modifying equation and can provide an interesting improvement concerning the numerical solution. If the viscosity parameter r approaches zero the equation models a inviscid fluid.

Subject to the initial and the boundary conditions

$$\begin{cases} u(x, 0) = u_0(x), \ \ x \in [0, b], \\ \\ u(x, t) = f(x, t), \ \ x \in \partial([0, b]), t > 0. \end{cases} \tag{1.2}$$

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2. Preliminaries

We briefly recall a definition and some properties of fractional derivatives which can be used in the sequel.

([? ?]) Given a function $f : [0 : \infty) \longrightarrow \mathbb{R}$, then the conformable fractional derivative of f of order α is defined by:

$$T_\alpha(f)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in (0.1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \longrightarrow 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \longrightarrow 0^+} f^{(\alpha)}(t). \tag{2.2}$$

Let's give some properties which are summarized in the above theorem.

([? ?]) Let $0 < \alpha \leq 1$ and f, g be α-differentiable at a point $t > 0$. Then,

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
4. $T_\alpha(fg) = f\,T_\alpha(g) + g\,T_\alpha(f)$.
5. $T_\alpha(\frac{f}{g}) = \dfrac{g\,T_\alpha(f) - f\,T_\alpha(g)}{g^2}$.
6. If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \dfrac{df}{dt}(t)$.

Note that the Property 6 of Theorem 1 is very important and it will often use in the sequel of this study.

3. A linearized Cole-Hopf transformation

In this section, we introduce the Cole-Hopf transformation in order to linearize the modified Burgers equations (??).

Using the Property 6 of Theorem 1, we can rewrite Eq.(??) as follows

$$t^{(1-\alpha)} \frac{\partial u}{\partial t} + (\nu + u)u_x = ru_{xx}. \tag{3.1}$$

The Cole-Hopf transformation is performed in two steps:

First step: Suppose that $u = \psi_x$ thus Eq.(??) becomes:

$$t^{(1-\alpha)} \psi_{xt} + (\nu + \psi_x)\psi_{xx} = r\left(\psi_{xxx}\right), \tag{3.2}$$

which can be rewritten as:

$$t^{(1-\alpha)} \psi_{xt} + \frac{\partial}{\partial x}\left(\frac{1}{2}(\nu + \psi_x)^2\right) = r\left(\psi_{xxx}\right), \tag{3.3}$$

we integrate Eq.(??) with respect to x, we obtain:

$$t^{(1-\alpha)}\psi_t + \left(\frac{1}{2}(\nu + \psi_x)^2\right) = r\,(\psi_{xx}) + \eta(t), \tag{3.4}$$

where $\eta(t)$ is arbitrary function depending of t.

Second step: Introducing now, the transformation $\psi = -2r\ln\phi$, we obtain

$$u = -2r\frac{\phi_x}{\phi}. \tag{3.5}$$

The derivatives of the function ψ are

$$\psi_t = -2r\frac{\phi_t}{\phi}, \quad \psi_x = -2r\frac{\phi_x}{\phi}, \quad \psi_{xx} = -2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}. \tag{3.6}$$

Substituting the derivatives ψ_t,ψ_x and ψ_{xx} in Eq.(??) , we obtain

$$t^{(1-\alpha)}\left(-2r\frac{\phi_t}{\phi}\right) + \left(\frac{1}{2}(\nu - 2r\frac{\phi_x}{\phi})^2\right) = r\left(-2r\frac{\phi_{xx}}{\phi} + 2r\frac{\phi_x^2}{\phi^2}\right) + \eta(t). \tag{3.7}$$

Eq.(??) can be reduced to :

$$\frac{\partial^\alpha\phi}{\partial t^\alpha} = r\phi_{xx} - \nu\phi_x + \frac{\nu^2}{4r}\phi + \zeta(t)\phi, \tag{3.8}$$

where $\zeta(t) = \dfrac{-\eta(t)}{2r}$.

Let's give the following theorem which shows that the cancel of function $\zeta(t)$ in Eq.(??) has no effect on the solution of Eq. (??). Let $\phi(x,t)$ be the solution of Eq.(??), $u(x,t)$ is defined in (??), then the solutions u is independent of $\zeta(t)$.

Let

$$\beta(t) = \int \frac{1}{t^{1-\alpha}}\zeta(t)dt,$$

then,

$$\beta'(t) = \frac{1}{t^{1-\alpha}}\zeta(t).$$

Multiply by $e^{-\beta(t)}$ to both sides of Eq.(??), yields

$$\frac{\partial^\alpha\phi}{\partial t^\alpha}e^{-\beta(t)} = r\phi_{xx}e^{-\beta(t)} - \nu\phi_x e^{-\beta(t)} + \frac{\nu^2}{4r}\phi e^{-\beta(t)} + \zeta(t)\phi e^{-\beta(t)}. \tag{3.9}$$

By using the Property 6 of Theorem 1, Eq.(??) becomes

$$t^{1-\alpha}\frac{\partial\phi}{\partial t}e^{-\beta(t)} - \zeta(t)\phi e^{-\beta(t)} = r\phi_{xx}e^{-\beta(t)} - \nu\phi_x e^{-\beta(t)} + \frac{\nu^2}{4r}\phi e^{-\beta(t)}. \tag{3.10}$$

Then,

$$t^{1-\alpha}\frac{\partial}{\partial t}\left(e^{-\beta(t)}\phi\right) = r\phi_{xx}e^{-\beta(t)} - \nu\phi_x e^{-\beta(t)} + \frac{\nu^2}{4r}\phi e^{-\beta(t)}. \tag{3.11}$$

Let $\psi(x,t) = e^{-\beta(t)}\phi(x,t)$, then $\psi(x,t)$ satisfies the following linear parabolic equation

$$t^{1-\alpha}\frac{\partial\psi}{\partial t} = r\psi - \nu\psi + \frac{\nu^2}{4r}\psi. \tag{3.12}$$

According to Property 6, Eq. (??) becomes

$$\frac{\partial^\alpha\psi}{\partial t^\alpha} = r\psi - \nu\psi + \frac{\nu^2}{4r}\psi. \tag{3.13}$$

We can see that the difference between the solution of Eq.(??) and Eq.(??) is the factor $e^{-\beta(t)}$. Therefore, we have

$$u(x,t) = \frac{\phi_x}{\phi} = \frac{e^{-\beta(t)}\phi_x}{e^{-\beta(t)}\phi} = \frac{\psi_x}{\psi}. \tag{3.14}$$

It is clear that the solutions $u(x,t)$ and is independent of the function $\zeta(t)$. In order to simplify the study, we can take $\zeta(t) = 0$ in Eq.(??). Then it is written as

$$\frac{\partial^\alpha\phi}{\partial t^\alpha} = r\phi_{xx} - \nu\phi_x + \frac{\nu^2}{4r}\phi. \tag{3.15}$$

Initial and boundary conditions for Eq. (??)

In the order to determinat the initial condition (IC) and boundary condition (BC), of the Eq.(??), we use

$$\frac{\phi_x}{\phi} = \frac{u(x,t)}{-2r}. \tag{3.16}$$

Integrating both sides of Eq.(??) with respect to x, we obtain

$$\phi(x,t) = \phi(t)\exp\left(\frac{-1}{2r}\int_0^x u(s,t)ds\right), \tag{3.17}$$

where $\phi(t)$ is constant of integration, and at $t = 0$ in Eq.(??), we obtain then the initial condition

$$\phi(x,0) = \phi(0)\exp\left(\frac{-1}{2r}\int_0^x u(s,0)ds\right). \tag{3.18}$$

From Eq.(??), it is clear that $\phi(0)$ has not effect on the final solution of System (??). So, we can consider $\phi(0) = 1$, it yields

$$\phi_0(x) = \exp\left(\frac{-1}{2r}\int_0^x u_0(s)ds\right). \tag{3.19}$$

Now, the transformed boundary condition (BC), can reduced to

$$\phi_x = -\frac{1}{2r}u(x,t)\phi(x,t),\ (x,t) \in (\partial\Omega \times (0,T)). \tag{3.20}$$

Therefore, the time-fractional parabolic equation with the initial and Neumann boundary conditions is given by.

$$\begin{cases} Eq. : & \dfrac{\partial^\alpha \phi}{\partial t^\alpha} = r\phi_{xx} - v\phi_x + \dfrac{v^2}{4r}\phi. \\ IC : & \phi_0(x) = \exp\left(\dfrac{-1}{2r}\displaystyle\int_0^x u_0(s)ds\right). \\ BC : & \phi_x = -\dfrac{1}{2r}u(x,t)\phi(x,t),\ (x,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{3.21}$$

Reformulating the Problem (??) by using the Property 6 of Theorem ??, it yields

$$\begin{cases} Eq. : & t^{(1-\alpha)}\dfrac{\partial \phi}{\partial t} = r\phi_{xx} - v\phi_x + \dfrac{v^2}{4r}\phi. \\ IC : & \phi_0(x) = \exp\left(\dfrac{-1}{2r}\displaystyle\int_0^x u_0(s)ds\right). \\ BC : & \phi_x = -\dfrac{1}{2r}u(x,t)\phi(x,t),\ (x,t) \in (\partial\Omega \times (0,T)). \end{cases} \tag{3.22}$$

4. Analytical solution of the (??) and (??)

We introduce the Fourier transform (F.T)

$$\widehat{\phi}(k_x,t) \overset{F.T}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(x,t)e^{-ik_x x}dx, \tag{4.1}$$

and the inverse Fourier transformation (F.T^{-1})

$$\phi(x,t) \overset{F.T^{-1}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \widehat{\phi}(k_x,t)e^{ik_x.x}\ dk_x. \tag{4.2}$$

In first, we apply the F.T to the term ϕ_x,

$$F.T(\phi_x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial x}e^{-ik_x x}dx, \tag{4.3}$$

Integrating by part with respect to x, then we obtain

$$F.T(\phi_x) = \frac{1}{\sqrt{2\pi}}\phi(x,t)\Big|_{-\infty}^{+\infty} - \frac{1}{\sqrt{2\pi}}(ik_x)\int_{-\infty}^{+\infty} \phi(x,t)e^{-ik_x x}dx,$$

According to [?], the boundary conditions for the heat equation on the infinite interval: $\phi = 0$ as $|x| = \infty$, so we get,

$$F.T(\phi_x) = -(ik_x)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(x,t)e^{-ik_x x}dx = (ik_x)\widehat{\phi}. \tag{4.4}$$

In same way and by integration twice by part with respect to x, we have

$$F.T(\phi_{xx}) = -k_x^2\widehat{\phi}. \tag{4.5}$$

Substituting the above results into Eq.(??), we obtain

$$\frac{\partial^\alpha \widehat{\phi}}{\partial t^\alpha} + v(ik_x\widehat{\phi}) - \frac{v^2}{4r}\widehat{\phi} = -rk_x^2\widehat{\phi}. \tag{4.6}$$

Using the Property 6 of Theorem 1, the Eq.(??) becomes

$$t^{(1-\alpha)}\frac{\partial \widehat{\phi}}{\partial t} + v(ik_x\widehat{\phi}) - \frac{v^2}{4r}\widehat{\phi} = -rk_x^2\widehat{\phi}. \tag{4.7}$$

Thus, the solution of Eq.(??) is given by

$$\widehat{\phi} = A(k_x)\; e^{(-ik_x v + v^2/4r - rk_x^2)t^\alpha/\alpha}, \tag{4.8}$$

where $A(k_x) = \widehat{\phi}_0(k_x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi_0(y)e^{-ik_x x}dy$ is the integration constant.

Applying F.T^{-1} to Eq.(??), then we obtain

$$\begin{aligned}\phi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ik_x.x}\;\widehat{\phi}_0(k_x)e^{-(ik_x v - v^2/4r + rk_x^2)t^\alpha/\alpha}dk_x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi_0(y)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ik_x.(y-x)\; -(ik_x v - v^2/4r + rk_x^2)t^\alpha/\alpha}dk_x\right]dy.\end{aligned} \tag{4.9}$$

By using the program of Maple, the solution of Eq.(??) is given by

$$\phi(x,t) = \frac{1}{2\sqrt{\pi r t^\alpha/\alpha}}\int_{-\infty}^{+\infty} \exp\left[\frac{-\alpha^2(x-y)^2 + 2\alpha t^\alpha v(x-y)}{4r\alpha t^\alpha}\right]\phi_0(y)dy. \tag{4.10}$$

After obtaining the linear time fractional parabolic equations ,the combining of the obtained solution of parabolic equation and the inverse Cole-Hopf transformation will allow us a solution of the modified time-fractional Burgers equations.

To calculate the analytical solution of Eq.(??), we calculate first

$$\phi_x(x,t) = \frac{1}{\sqrt{2\pi r t^\alpha/\alpha}} \int_{-\infty}^{+\infty} c \ \exp\left[\frac{-\alpha^2(x-y)^2 + 2\alpha t^\alpha \nu(x-y)}{4r\alpha t^\alpha}\right] \phi_0(y)dy, \tag{4.11}$$

where $c = \dfrac{-2\alpha^2(x-y) + 2\alpha\nu t^\alpha}{4r\alpha t^\alpha}$.

Once the functions $\phi(x,t)$ and $\phi_x(x,t)$ are known and by using (??), therefore the solutions is

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} c' \ \exp\left[\frac{-\alpha^2(x-y)^2 + 2\alpha t^\alpha \nu(x-y)}{4r\alpha t^\alpha} - \frac{1}{2r}\int_0^y u_0(s)ds\right] dy}{\int_{-\infty}^{+\infty} \exp\left[\frac{-\alpha^2(x-y)^2 + 2\alpha t^\alpha \nu(x-y)}{4r\alpha t^\alpha} - \frac{1}{2r}\int_0^y u_0(s)ds\right] dy}, \tag{4.12}$$

where $c' = [\alpha(x-y)t^{-\alpha} - \nu]$.

5. Numerical schemes for (??)

We discretize the domain Ω by the finite difference method (FDM) into nx, each of length $\Delta x = (b-a)/nx$ along the x-axis, and define the discrete mesh points (x_i, t_n) by $(a + i\Delta x, n\Delta t)$, where $i = 0, ..., nx$ and $n = 0, ..., T$.

5.1. An explicit scheme

By using a simple forward in time and centered in space discretization at point (x_i, t_n), the explicit scheme for the Eq.(??) is

$$t_n^{(1-\alpha)} \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\nu \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \frac{\nu^2}{4r}\phi_i^n + r\left(\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right).$$

So that, for every interior point (x_i, t_n), with $i = 1, ..., nx-1$, we obtain

$$\phi_i^{n+1} = (\alpha + \beta)\phi_{i-1}^n + (1 + \gamma - 2\beta)\phi_i^n - (\alpha - \beta)\phi_{i+1}^n, \tag{5.1}$$

where

$$\alpha = \frac{\nu\Delta t}{\Delta x\, t_n^{(1-\alpha)}}, \ \beta = \frac{r\Delta t}{\Delta x^2\, t_n^{(1-\alpha)}} \ \text{ and } \ \gamma = \frac{\nu^2\Delta t}{4r\, t_n^{(1-\alpha)}}.$$

Now, let us consider the so-called BC described as

$$\phi_x(x_i, t_n) \simeq \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} = -\frac{1}{2r}u_i^n\phi_i^n, \tag{5.2}$$

which can be rewritten as:

$$\phi_{i+1}^n = \phi_{i-1}^n - \frac{\Delta x}{r} u_i^n \phi_i^n, \tag{5.3}$$

For $i = 0$ and $i = nx$, Eq.(??), respectively becomes

$$\phi_1^n = \phi_{-1}^n - \frac{\Delta x}{r} u_0^n \phi_0^n \quad \text{and} \quad \phi_{nx+1}^n = \phi_{nx-1}^n - \frac{\Delta x}{r} u_{nx}^n \phi_{nx}^n \tag{5.4}$$

Substituting this constraint into Eq.(??) at the boundary points, we obtain respectively

$$\begin{aligned} \phi_0^{n+1} &= 2\beta\phi_1^n + \left(1 + \gamma - 2\beta + (\alpha + \beta)\frac{\Delta x}{r} u_0^n\right)\phi_0^n, \\ \phi_{nx}^{n+1} &= 2\beta\phi_{nx-1}^n + \left(1 + \gamma - 2\beta + (\alpha - \beta)\frac{\Delta x}{r} u_{nx}^n\right)\phi_{nx}^n. \end{aligned} \tag{5.5}$$

5.2. An implicit scheme

By using a simple forward in time and centered in space discretization at point (x_i, t_n), the implicit scheme for Eq.(??) is

$$t_n^{(1-\alpha)} \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\nu \frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \frac{\nu^2}{4r}\phi_i^{n+1} + r\left(\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right).$$

which can rewrite as

$$-(\alpha + \beta)\phi_{i-1}^{n+1} + \gamma\phi_i^{n+1} + (\alpha - \beta)\phi_{i+1}^{n+1} = \phi_i^n \tag{5.6}$$

where

$$\alpha = \frac{\nu\Delta t}{2\Delta x t_n^{1-\alpha}}, \quad \beta = \frac{r\Delta t}{\Delta x^2 t_n^{1-\alpha}}, \quad \gamma = \left(1 - \frac{\nu^2\Delta t}{4rt_n^{1-\alpha}} + 2\beta\right).$$

5.3. Calculating the required solution

The calculation of solution to the Eq. (??) can be obtained by the inverse Cole-Hopf transformation. Let $D_x\phi_i^n$ denote the derivative of ϕ, at point (x_i, t_n) with respect to x. Then, $D_x\phi_i^n$ can be calculated from the first order centered difference formula, for $i = 1, .., nx - 1$

$$D_x\phi_i^n = \frac{\partial\phi}{\partial x} \simeq \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x}, \tag{5.7}$$

Note that the derivatives: $D_x\phi_0^n$ and $D_x\phi_{nx}^n$ at the end points are known.

Once the approximated values of ϕ and ϕ_x are known at any discrete point (x_i, t_n), then the approximated values of u at discrete points can be calculated from the following discrete version of Eq.(??), for $i = 1, .., nx$,

$$u_i^n = -2r\frac{D_x\phi_i^n}{\phi_i^n}. \tag{5.8}$$

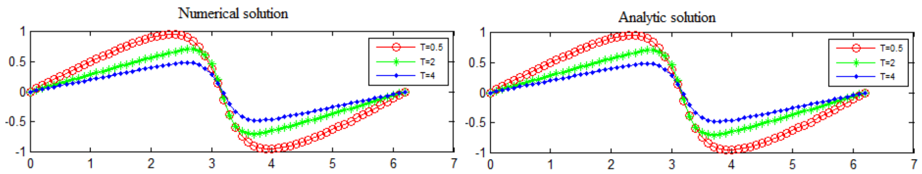

Figure 1: Numerical and analytical solution of modified Burger's equation at $T = 0.5, 2, 4$ for $\alpha = 0.75$.

6. Numerical experiment and discussion

In this section, we discuss a example to test the performance and accuracy of the method. The numerical results arrived by this method are compared with analytic solution for various values of α, and T. To show the accuracy of the method, both the relative error L_1-norm and L_∞-norm respectively are given by

$$||Erreuru||_{L_1} = \frac{||u_a - u_n||_{L_1}}{||u_a||_{L_1}}, \tag{6.1}$$

$$||Erreuru||_{L_\infty} = \frac{||u_a - u_n||_{L_\infty}}{||u_a||_{L_\infty}}, \tag{6.2}$$

where u_a represents the analytical solution (**??**) and u_n represents the computed solution (**??**) for Eq.(**??**). We use the Matleb program to calculate the u_n.

Considering modified Burgers equation Eq.(**??**), with the initial conditions:

$$u(x, t) = \sin(x), \ x \in [0, 2\pi], \ t > 0, \tag{6.3}$$

and boundary conditions:

$$u(0, t) = u(2\pi, t) = 0. \tag{6.4}$$

After computing, let's give in Figure **??** and Figure **??** respectively the graphs of the numerical solution and the exact solution. For simulation, we take the following data, $r = 0.1$, $\nu = 0.1$, $\alpha = 0.25$, 0.5, 0.75 and 0.9 respectively,

It can be see from Figure **??**, at different time, there is no difference between the numerical and the exact solution curve. In addition, as the time increases, the solution curve approach the x -axis and the viscosity value becomes smaller. More, from each of the graphs in Figure **??**, we can be observe that as α increases, the numerical solution curve are in good agreement with the exact solution curve.

7. Conclusion

In this paper, we deal with a study of the modified Burgers equation with fractional conformal derivatives with respect to time. The presence of both the fractional time derivative and the

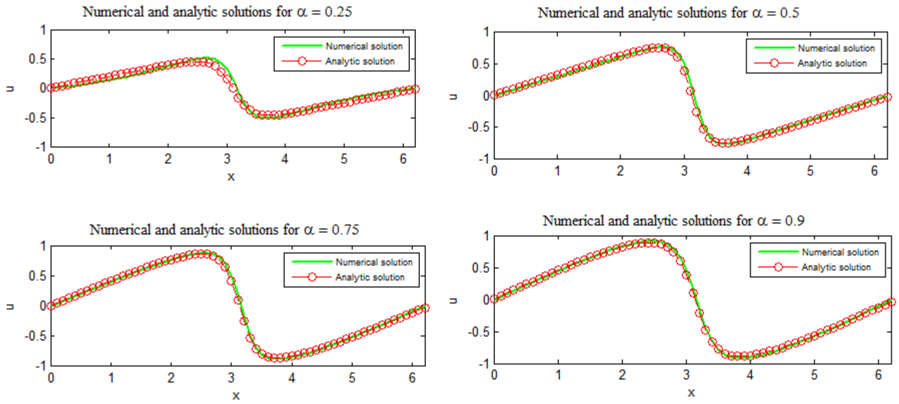

Figure 2: Numerical and analytical solution of modified Burger's equation at $T = 1$ for different value of α.

nonlinear term in this equation makes solving the problem more difficult. The idea is to use the Cole-Hopf transform to reduce the modified Burgers equation of temporal fractional conformable derivative to a modified linear equation of temporal fractional conformable derivative. Then we can solve the latter using the Fourier transformation. Therefore, the solution of the time-fractional conformable modified Burgers equation can be found using both the solution of the parabolic equation and the inverse Cole-Ho transformation. For illustration, the experimental simulations are given to show the interest of this approach.

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