On the Modeling Process of Ultrasonic Wave Propagation in a Relaxation Medium by the Three-Point in Time Problem

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Abstract

A mathematical model of the process of ultrasonic oscillations in a relaxation medium with known acoustic wave profiles at three points in time is proposed. The model is reduced to the study of a three-point problem for a hyperbolic equation of third order, which is widely used in ultrasound diagnostics. A differential-symbol method for constructing a solution of the three-point problem is proposed and a class of quasipolynomials as the class of uniqueness solvability of the problem is found. The technique which specified in the work allows to investigate in detail the main parameters of acoustic oscillations in problems of ultrasonic diagnostics. The method is demonstrated on specific examples of three-point problems.

Keywords [1]

Mathematical model, acoustic oscillations, three-point problem, differential-symbol method, ultrasound diagnostics

1. Introduction

In the theory of mathematical modeling there are many models of processes of various nature. Increasingly, these models from some areas of knowledge are used in other areas. In particular, modeling of hydromechanics and gas dynamics problems [1, 2] is successfully used in modeling biomechanical and medical processes [3-5].

Modern mathematical models increasingly contain partial differential equations, both linear and nonlinear. Therefore, the research of such models is quite complex and their study involves powerful numerical, qualitative and asymptotic methods (in particular, see [6, 7]). In addition to traditional partial differential equations of the second order, which are actively studied in the equations of mathematical physics, there are often partial differential equations of the third order in time in mechanical, biomedical and geophysical models [8-11]. Among the problems of ultrasonic diagnostics in [12-14] the Cauchy problem for the hyperbolic equation of the third order of the form

$$\left[\tau\partial_t^3 - \tau c_1^2\,\partial_t\Delta + \partial_t^2 - c_2^2\Delta\right]u(t,\bar{x}) = 0,\quad \bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{1}$$

is investigated. In equation (1), τ is relaxation time, $u(t,\bar{x})$ is dynamic pressure, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is three-dimensional Laplace operator, $\partial_t = \partial / \partial t$, constants c_1 and c_2 are limiting phase speeds of sound.

In addition to the Cauchy problem for partial differential equations, multipoint in time problems with the given values of the unknown solution not at only one time point, but at several moments of time are intensively studied. In particular, papers [15–18] and [19, 20] are devoted to problems with multipoint in time conditions in bounded and unbounded domains respectively. Problems with n-point time conditions have a simple physical interpretation, namely in these problems the state of the

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research process at n different time points are given. However, despite of the simplicity of physical interpretation, multipoint problems are not easy to study. In contrast to the Cauchy problem, the kernel of multipoint in time problems is nontrivial [21, 22]. Therefore, the study of the corresponding incorrect multipoint problems for partial differential equations requires new research methods, among of them, the differential-symbol method is especially effective [23–26].

The aim of this work is:

- study of a mathematical model describing the motion of an ultrasonic wave in a relaxation medium with given profiles of the wave at three time points;
- establish the class of existence and uniqueness of the solution of the corresponding three-point in time problem;
- recommend the method for constructing the solution of the problem;
- development of a method for determining the influence of the wave process parameters under the condition of specific initial data of the three-point problem.

2. Posing of the problem and main results

By replacing $x = \dfrac{\overline{x}}{c_1\tau}$, $\alpha = \dfrac{c_1}{c_2}$, ultrasonic wave equation (1) is transformed in the one-parameter hyperbolic equation

$$\left[\partial_t^3 - \Delta\partial_t + \partial_t^2 - \alpha\Delta\right]u(t,x) = 0, \ \ (t,x) \in (0,+\infty)\times\mathbb{R}^3. \tag{2}$$

Let's note that the spatial variable x and the parameter α in equation (2) are dimensionless, moreover α belongs to the interval $(0,1)$.

We consider the mathematical model of the process of an ultrasonic wave propagation which is described by equation (2), if the profiles of wave are given at three equidistant moments of time $t = jh,\ j \in J = \{0,1,2\},\ h > 0$:

$$u(jh,x) = f_j(x), \quad j \in J, \quad x \in \mathbb{R}^3. \tag{3}$$

To study the three-point problem (2), (3), we use the differential-symbol method which was previously [23] used to solve the two-point in time problem. Based on the differential equation (2), for the unknown function $\Gamma = \Gamma(t,\beta)$ we write the corresponding ordinary differential equation with the parameter μ

$$\left[d_t^3 + d_t^2 - \beta d_t - \alpha\beta\right]\Gamma(t,\beta) = 0, \tag{4}$$

in which $\beta = |\mu|^2 \equiv \mu_1^2 + \mu_2^2 + \mu_3^2$, $\mu = (\mu_1,\mu_2,\mu_3) \in \mathbb{C}^3$, $d_t = d/dt$.

Taking into account the dependence $\beta = |\mu|^2$, let us denote the roots of the characteristic equation for (4)

$$\lambda^3 + \lambda^2 - \beta\lambda - \alpha\beta = 0, \tag{5}$$

by $\lambda_1 = \lambda_1(\mu)$, $\lambda_2 = \lambda_2(\mu)$, $\lambda_3 = \lambda_3(\mu)$. These roots have the following form

$$\begin{aligned}
\lambda_1 &= \lambda_1(\mu) = -\frac{1}{3} + \frac{\sqrt[3]{2}(1+3\beta)}{3A} + \frac{A}{3\sqrt[3]{2}}, \\
\lambda_2 &= \lambda_2(\mu) = -\frac{1}{3} - \frac{(1+i\sqrt{3})(1+3\beta)}{3A\sqrt[3]{4}} - \frac{(1-i\sqrt{3})A}{6\sqrt[3]{2}}, \\
\lambda_3 &= \lambda_3(\mu) = -\frac{1}{3} - \frac{(1-i\sqrt{3})(1+3\beta)}{3A\sqrt[3]{4}} - \frac{(1+i\sqrt{3})A}{6\sqrt[3]{2}},
\end{aligned} \tag{6}$$

where $i^2 = -1$, $A = \sqrt[3]{B + \sqrt{-4(3\beta+1)^3 + B^2}}$, $B = 27\alpha\beta - 9\beta - 2$.

Remark 1. The roots (6) coincide only in the case $\alpha=\frac{1}{9}$ and $\beta=-\frac{1}{3}$. Then $\lambda_1=\lambda_2=\lambda_3=-\frac{1}{3}$. If $(\alpha,\beta)\neq\left(\frac{1}{9},-\frac{1}{3}\right)$, there are at least two different roots among the roots of (6). If $\beta(4\beta^2-27\alpha^2\beta+18\alpha\beta-4\alpha+\beta)\neq 0$ there are simple roots.

Remark 2. For $\beta=0$ equation (5) has the form

$$\lambda^3+\lambda^2=0$$

and the roots $\lambda_1=-1$, $\lambda_2=\lambda_3=0$ do not depend on α.

For the ordinary differential equation of third order (4) we construct a fundamental system of solutions $\{\Gamma_0(t,\mu),\Gamma_1(t,\mu),\Gamma_2(t,\mu)\}$, which satisfies local three-point conditions

$$\Gamma_k\left(jh,\mu\right)=\begin{cases}1, & k=j,\\ 0, & k\neq j,\end{cases}\quad k,j\in J\ . \tag{7}$$

This system can be formed only for vectors $\mu\in\mathbb{C}^3$ which fulfill the condition

$$\Delta(\mu)\equiv\det\left(g_k(jh,\mu)\right)_{k,j\in J}\neq 0\ . \tag{8}$$

We get:

$g_k(t,\mu)=e^{\lambda_{k+1}(\mu)t}$ for simple roots $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$,

$g_k(t,\mu)=t^k e^{\lambda_1(\mu)t}$ for triple root $\lambda_1(\mu)$,

$g_0(t,\mu)=e^{\lambda_1(\mu)t}$, $g_1(t,\mu)=e^{\lambda_2(\mu)t}$, $g_2(t,\mu)=t\,e^{\lambda_2(\mu)t}$ for $\lambda_1\neq\lambda_2=\lambda_3$.

Let's denote $E_1=e^{\lambda_1(\mu)h}$, $E_2=e^{\lambda_2(\mu)h}$, $E_3=e^{\lambda_3(\mu)h}$ for $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$. For condition (8) the elements of system $\{\Gamma_0(t,\mu),\Gamma_1(t,\mu),\Gamma_2(t,\mu)\}$ have the following form

$$\begin{aligned}\Gamma_0(t,\mu)&=\frac{E_3E_2(E_3-E_2)e^{\lambda_1(\mu)t}-E_3E_1(E_3-E_1)e^{\lambda_2(\mu)t}+E_2E_1(E_2-E_1)e^{\lambda_3(\mu)t}}{\Delta(\mu)},\\ \Gamma_1(t,\mu)&=\frac{(E_3^2-E_2^2)e^{\lambda_1(\mu)t}-(E_3^2-E_1^2)e^{\lambda_2(\mu)t}+(E_2^2-E_1^2)e^{\lambda_3(\mu)t}}{\Delta(\mu)},\\ \Gamma_2(t,\mu)&=\frac{(E_3-E_2)e^{\lambda_1(\mu)t}-(E_3-E_1)e^{\lambda_2(\mu)t}+(E_2-E_1)e^{\lambda_3(\mu)t}}{\Delta(\mu)},\end{aligned} \tag{9}$$

where $\Delta(\mu)\equiv\left(E_3-E_1\right)\left(E_3-E_2\right)\left(E_2-E_1\right)$.

If $\lambda_1\neq\lambda_2=\lambda_3$, then $\Delta(\mu)=hE_2\left(E_2-E_1\right)^2$. The functions $\Gamma_0(t,\mu)$, $\Gamma_1(t,\mu)$, $\Gamma_2(t,\mu)$ take the form

$$\begin{aligned}\Gamma_0(t,\mu)&=\frac{hE_2^3e^{\lambda_1(\mu)t}-hE_1E_2(2E_2-E_1)e^{\lambda_2(\mu)t}+E_1E_2(E_2-E_1)\,t\,e^{\lambda_2(\mu)t}}{\Delta(\mu)},\\ \Gamma_1(t,\mu)&=\frac{-2hE_2^2e^{\lambda_1(\mu)t}+2hE_2^2e^{\lambda_2(\mu)t}-(E_2^2-E_1^2)\,t\,e^{\lambda_2(\mu)t}}{\Delta(\mu)},\\ \Gamma_2(t,\mu)&=\frac{hE_2e^{\lambda_1(\mu)t}-hE_2e^{\lambda_2(\mu)t}+(E_2-E_1)\,t\,e^{\lambda_2(\mu)t}}{\Delta(\mu)}.\end{aligned} \tag{10}$$

In the case of triple root of equation (5), that is $\lambda_1=\lambda_2=\lambda_3$, we have $\Delta(\mu)=2h^3E_1^3\neq 0$ for each $\mu\in\mathbb{C}^3$, for which $|\,\mu\,|^2=-\frac{1}{3}$. The functions $\Gamma_0(t,\mu),\Gamma_1(t,\mu),\Gamma_2(t,\mu)$ get the following form

$$\Gamma_0(t,\mu)=\left(1-\frac{3}{2h}t+\frac{1}{2h^2}t^2\right)e^{\lambda_1(\mu)t},$$
$$\Gamma_1(t,\mu)=\left(2-\frac{t}{h}\right)\frac{t}{h}e^{\lambda_1(\mu)(t-h)}, \tag{11}$$
$$\Gamma_2(t,\mu)=(-h+t)\frac{t}{2h^2}e^{\lambda_1(\mu)(t-2h)}.$$

According to Remark 1, all roots of equation (5) are equal to $-\frac{1}{3}$ for $\beta=-\frac{1}{3}$, $\alpha=\frac{1}{9}$. Then $E_1=e^{-\frac{h}{3}}$, $\Delta(\mu)=2h^3e^{-h}\neq 0$ and functions (11) can be written as follows

$$\Gamma_0(t,\mu)\big|_{|\mu|^2=-\frac{1}{3}}=\left(1-\frac{3}{2h}t+\frac{1}{2h^2}t^2\right)e^{-\frac{1}{3}t},$$
$$\Gamma_1(t,\mu)\big|_{|\mu|^2=-\frac{1}{3}}=\left(2-\frac{t}{h}\right)\frac{t}{h}e^{-\frac{1}{3}(t-h)}, \tag{12}$$
$$\Gamma_2(t,\mu)\big|_{|\mu|^2=-\frac{1}{3}}=(-h+t)\frac{t}{2h^2}e^{-\frac{1}{3}(t-2h)}.$$

According to Remark 2, for $\beta=0$ we get $\lambda_1=-1$, $\lambda_2=\lambda_3=0$, $E_1=e^{-h}$, $E_2=1$. Then $\Delta(\mu)=h\left(1-e^{-h}\right)^2\neq 0$. The functions (8) take the form

$$g_0(t)\equiv\Gamma_0(t,\mu)\big|_{|\mu|^2=0}=\frac{he^{-t}-he^{-h}(2-e^{-h})+e^{-h}(1-e^{-h})t}{h(1-e^{-h})^2},$$
$$g_1(t)\equiv\Gamma_1(t,\mu)\big|_{|\mu|^2=0}=\frac{-2he^{-t}+2h-(1-e^{-2h})t}{h(1-e^{-h})^2}, \tag{13}$$
$$g_2(t)\equiv\Gamma_2(t,\mu)\big|_{|\mu|^2=0}=\frac{he^{-t}-h+(1-e^{-h})t}{h(1-e^{-h})^2}$$

and do not depend on parameter α. The graphical representations of these functions for $h=1$ are given in Figure 1. Dashed lines on Figure 1 indicate asymptotes.

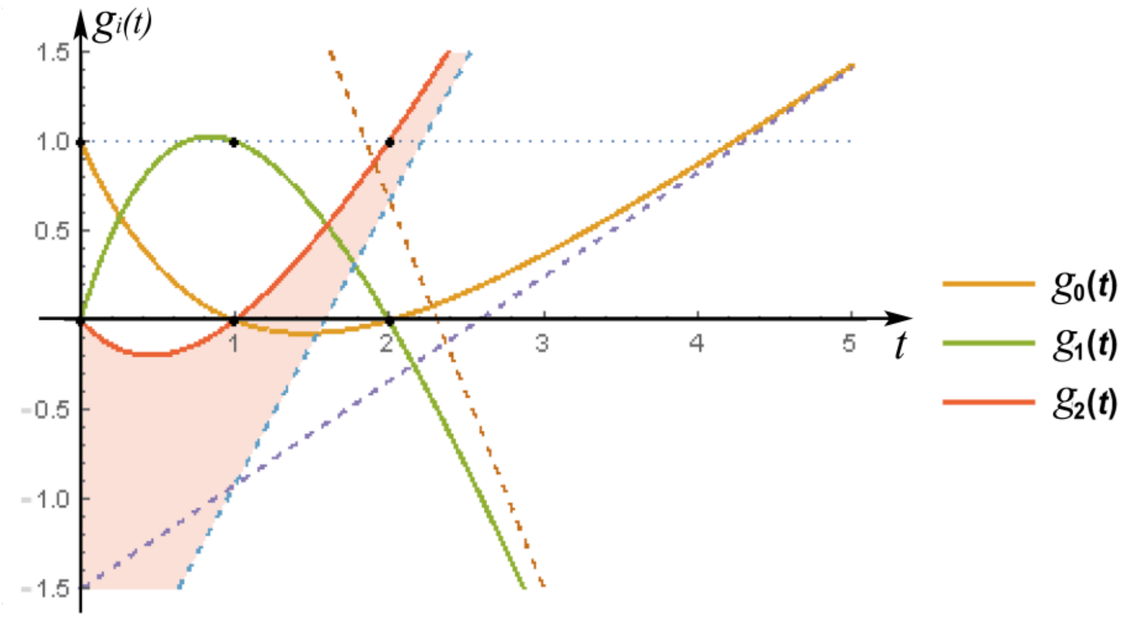

Figure 1: The graphs of the functions $g_0(t), g_1(t), g_2(t)$

Let us consider nontrivial quasipolynomials of the form

$$f(x) = Q(x)e^{\nu \cdot x}, \quad \nu \cdot x = \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3, \tag{14}$$

in which $Q(x)$ is nonzero polynomial of variables x_1, x_2, x_3 with complex coefficients, ν_1, ν_2, ν_3 are complex parameters, the vector $\nu = (\nu_1, \nu_2, \nu_3)$ satisfies condition (8), that is $\nu \in M$, where the set M is determined by the formula

$$M = \left\{ \mu \in \mathbb{C}^3 : \Delta(\mu) \neq 0 \right\}. \tag{15}$$

Note that the set (15) is nonempty, since the vectors $\mu \in \mathbb{C}^3$ in the case $|\mu|^2 = 0$ and for $\alpha = \frac{1}{9}$, $|\mu|^2 = -\frac{1}{3}$ belong to this set.

For the set (15), let K_M is the class of functions which can be represented as a finite sum of quasipolynomials of the form (14) that differ from each other by different vectors ν. So, K_M is the class of quasipolynomials of the variables x_1, x_2, x_3 and the zero quasipolynomial belongs to K_M.

Let the right-hand sides of conditions (3), namely the functions $f_0(x), f_1(x), f_2(x)$ belong to the class K_M. Then there is an unique solution of problem (2), (3) in the class of quasipolynomials of variables t, x_1, x_2, x_3 which belong to K_M for each fixed t. This solution can be represented by the formula

$$u(t,x) = \sum_{k=0}^{2} f_k\left(\partial_\mu\right)\left\{ \Gamma_k(t,\mu) e^{\mu \cdot x} \right\}\Big|_{\mu = O}, \tag{16}$$

where $\mu \cdot x = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$, $O = (0,0,0)$, $\partial_\mu = (\partial_{\mu_1}, \partial_{\mu_2}, \partial_{\mu_3})$.

The differential expressions $f_0\left(\partial_\mu\right)$, $f_1\left(\partial_\mu\right)$, $f_2\left(\partial_\mu\right)$ are obtained from the functions $f_0(x)$, $f_1(x)$, $f_2(x)$ by replacing the vector x by vector-derivative ∂_μ. For each summand of the form (14) in the quasipolynomials $f_0(x)$, $f_1(x)$, $f_2(x)$ we put in correspondence the differential expression $Q(\partial_\mu) e^{\nu_1 \partial_{\mu_1} + \nu_2 \partial_{\mu_2} + \nu_3 \partial_{\mu_3}}$, which acts on the function $\Gamma_k(t,\mu) e^{\mu \cdot x}$ by formula

$$\begin{aligned} Q(\partial_\mu) e^{\nu_1 \partial_{\mu_1} + \nu_2 \partial_{\mu_2} + \nu_3 \partial_{\mu_3}} \left\{ \Gamma_k(t,\mu) e^{\mu \cdot x} \right\}\Big|_{\mu = O} &\equiv Q(\partial_\mu)\left\{ \Gamma_k(t,\mu) e^{\mu \cdot x} \right\}\Big|_{\mu = \nu} \\ &= e^{\nu \cdot x} Q(x + \partial_\mu)\left\{ \Gamma_k(t,\mu) \right\}\Big|_{\mu = \nu}, \end{aligned}$$

that is, the differential polynomial $Q(\partial_\mu)$ acts onto the function $\Gamma_k(t,\mu) e^{\mu \cdot x}$, then we set the vector-parameter μ equals to $\nu = (\nu_1, \nu_2, \nu_3)$. If among the functions $f_0(x)$, $f_1(x)$, $f_2(x)$ is zero, then the corresponding summand in formula (16) is zero. Therefore, formula (16) for finding the solution of problem (2), (3) assumes the implementation of a finite number of differentiation of functions $\Gamma_0(t,\mu)$, $\Gamma_1(t,\mu)$, $\Gamma_2(t,\mu)$ by parameters μ_1, μ_2, μ_3.

The fact that function (16) is the solution of problem (2), (3) follows from the commutativity of the differentiation operators ∂_t, ∂_x, ∂_μ, taking into account that the functions $\Gamma_0(t,\mu)$, $\Gamma_1(t,\mu)$, $\Gamma_2(t,\mu)$ satisfy equations (4) and conditions (7).

The fact that the found solution of three-point problem (2), (3) is unique in the specified class of quasipolynomials can be proved by contradiction method (see, for example, [23]). The choice of quasipolynomials $f_0(x)$, $f_1(x)$, $f_2(x)$ exactly from the class K_M is significant.

Thus, if the functions $f_0(x), f_1(x), f_2(x)$ in conditions (3) belong to the set K_M, then quasipolynomial solutions of problem (2), (3) are constructed by formula (16).

Note that numerous studies have been devoted to the construction of quasipolynomial solutions of partial differential equations and boundary value problems for these equations [27–31].

Main result. The process of propagation of an ultrasonic wave in a relax medium with given wave profiles at three time points is modeled by problem (2), (3) for the hyperbolic partial differential equation of the third order in time (2) with three-point time conditions (3). The method of constructing the solution of problem (2), (3) is proposed. The class of quasipolynomials as a class of uniqueness solvability of problem (2), (3) is indicated. Equation (2) belongs to the important partial differential equations which are used in the problems of ultrasound diagnostics.

3. The examples of application of the method to constructing solution of the problem with given profiles of the ultrasonic wave at three moments of time

Let us investigate the process of acoustic oscillations for specifically given right-hand sides of three-point conditions and parameters of the differential equation. We use the method which is proposed in the previous section to construct the solution of problem (2), (3).

Example 1. Let us consider problem (2), (3) for $h=1$, $f_0(x)=x_1^2$, $f_1(x)=x_2$, $f_2(x)=2$. The functions $f_0(x)$, $f_1(x)$, $f_2(x)$ are polynomials, therefore they have the form (14) and $\nu=O$. Since $\Delta(O)=\left(1-e^{-1}\right)^2\neq 0$, then these functions belong to K_M. So, the unique solution of the problem exists in indicated above class of quasipolynomials (in particular, in a subclass of polynomials). The solution we can find by formula (16):

$$\begin{aligned}u(t,x)&=\partial_{\mu_1}^2\left\{\Gamma_0(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=O}+\partial_{\mu_2}\left\{\Gamma_1(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=O}+2\left\{\Gamma_2(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=O}\\&=\partial_{\mu_1}^2\Gamma_0(t,\mu)\Big|_{\mu=O}+2x_1\partial_{\mu_1}\Gamma_0(t,\mu)\Big|_{\mu=O}+x_1^2\Gamma_0(t,O)\\&\quad+x_2\Gamma_1(t,O)+\partial_{\mu_2}\Gamma_1(t,\mu)\Big|_{\mu=O}+2\Gamma_2(t,O)\\&=\partial_{\mu_1}^2\Gamma_0(t,\mu)\Big|_{\mu=O}+0+x_1^2\frac{e^{-t}-e^{-1}(2-e^{-1})+e^{-1}(1-e^{-1})t}{(1-e^{-1})^2}\\&\quad+x_2\frac{-2e^{-t}+2-(1-e^{-2})t}{(1-e^{-1})^2}+0+2\frac{e^{-t}-1+(1-e^{-1})t}{(1-e^{-1})^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}u(t,x)&=\partial_{\mu_1}^2\Gamma_0(t,\mu)\Big|_{\mu=O}\\&\quad+x_1^2\frac{e^{-t}-e^{-1}(2-e^{-1})+e^{-1}(1-e^{-1})t}{(1-e^{-1})^2}\\&\quad+x_2\frac{-2e^{-t}+2-(1-e^{-2})t}{(1-e^{-1})^2}+2\frac{e^{-t}-1+(1-e^{-1})t}{(1-e^{-1})^2}.\end{aligned}\tag{17}$$

In formula (17), the function $\xi(t)\equiv\partial_{\mu_1}^2\Gamma_0(t,\mu)\Big|_{\mu=O}$ is the solution of three-point problem for nonhomogeneous ordinary differential equation

$$\left(d_t^3+d_t^2\right)\xi(t)=\left(2d_t+\alpha\right)\Gamma_0(t,O),$$
$$\xi(0)=\xi(h)=\xi(2h)=0.$$

This function is obtained by differentiating the problem (4), (5) for $\Gamma_0(t,\mu)$ by the parameter μ_1 at the point $\mu=O$. Calculations show that the function $\xi(t)$ is a quasipolynomial of the form

$$\xi(t)=c_1\left(e^{-t}-1\right)+c_2t+Ate^{-t}+Bt^3+Ct^2,$$

where $A=\dfrac{2(1-\alpha)e^2}{(e-1)^2}$, $B=\dfrac{\alpha e}{3(e-1)}$, $C=\dfrac{e-1-3e\alpha+2\alpha}{(e-1)^2}$, constants c_1 and c_2 satisfy a nondegenerate system of algebraic equations

$$\begin{cases} c_1\left(e^{-1}-1\right)+c_2=-Ae^{-1}-B-C, \\ c_1\left(e^{-2}-1\right)+2c_2=-2Ae^{-2}-8B-4C. \end{cases}$$

Note that found solution (17) of the problem is a linear function of the parameters A, B, C and α.

Example 2. We consider the process of propagation of an ultrasonic wave in the relaxation medium which is described by the problem (2), (3), where

$$\alpha=\frac{1}{9},\ f_0(x)=\cos\frac{x_1+x_2+x_3}{3},\ f_1(x)=f_2(x)=0,\ h=1.$$

The function $f_0(x)$ is a sum of two quasipolynomials of form (14), namely

$$\cos\frac{x_1+x_2+x_3}{3}=\frac{1}{2}e^{\frac{x_1+x_2+x_3}{3}i}+\frac{1}{2}e^{-\frac{x_1+x_2+x_3}{3}i},$$

in which the vectors of parameters equal to $\nu=\frac{i}{3}(1,1,1)$ and to $-\nu$. Since $|\pm\nu|^2=-\frac{1}{3}$ and $\alpha=\frac{1}{9}$, then $\Delta(\pm\nu)=2e^{-1}\neq 0$. Therefore, $\pm\nu\in M$.

The solution of problem (2), (3) for these data is found by formulas (16), (12):

$$u(t,x)=\cos\left[\frac{1}{3}\left(\partial_{\mu_1}+\partial_{\mu_2}+\partial_{\mu_3}\right)\right]\left\{\Gamma_0(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=O}$$

$$=\frac{1}{2}\left\{\Gamma_0(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=\frac{i}{3}(1,1,1)}+\frac{1}{2}\left\{\Gamma_0(t,\mu)e^{\mu\cdot x}\right\}\Big|_{\mu=-\frac{i}{3}(1,1,1)},$$

where

$$\left\{\Gamma_0(t,\mu)\right\}\Big|_{\mu=\frac{i}{3}(1,1,1)}=\left\{\Gamma_0(t,\mu)\right\}\Big|_{\mu=-\frac{i}{3}(1,1,1)}$$

$$=\left(1-\frac{3}{2}t+\frac{1}{2}t^2\right)e^{-\frac{1}{3}t}.$$

Finally, we obtain the following solution of the problem

$$u(t,x)=\left(\frac{1}{2}-\frac{3}{4}t+\frac{1}{4}t^2\right)\left(e^{\frac{x_1+x_2+x_3-t}{3}}+e^{\frac{-x_1-x_2-x_3-t}{3}}\right)$$

$$=\left(1-\frac{3}{2}t+\frac{1}{2}t^2\right)e^{-\frac{1}{3}t}\cos\frac{x_1+x_2+x_3}{3}.$$

For graphically illustrating the process of acoustic oscillations, we consider the solution on parallel planes $x_1+x_2+x_3-3\eta=0$ of the space $\mathbb{R}^3$, where $\eta\in\mathbb{R}$, $\sqrt{3}|\eta|$ is the distance from the origin of coordinates to the plane.

The solution of problem in variables t and η is 2π-periodical function by η and has the such analytical factorized form

$$u(t,\eta)=\frac{(t-1)(t-2)}{2}e^{-\frac{1}{3}t}\cos\eta. \tag{18}$$

The graph of function (18) of two variables t and η is depicted in Figure 2.

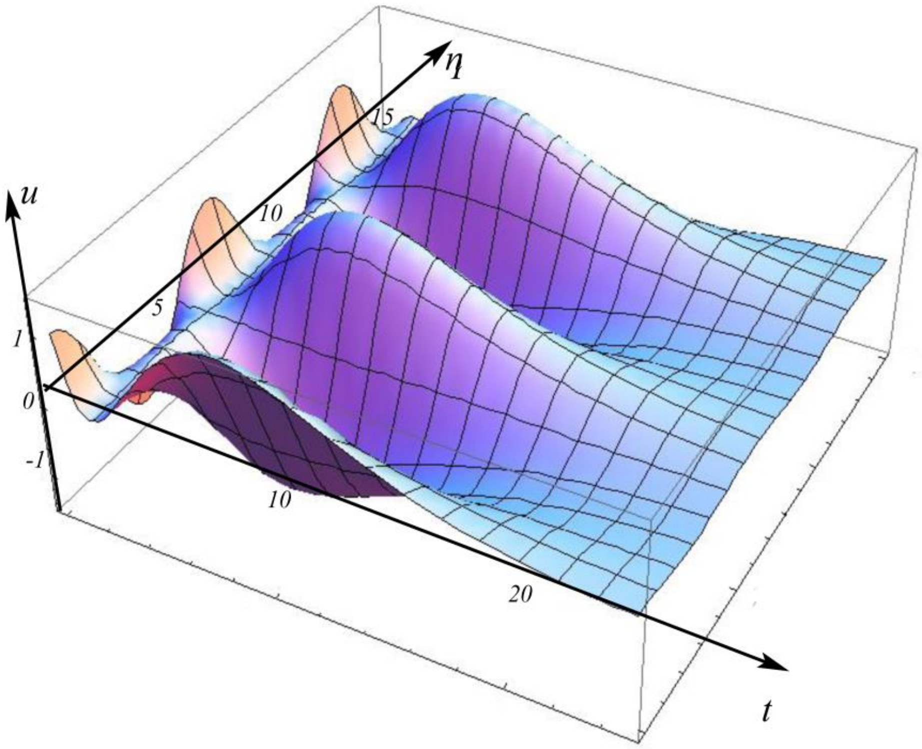

Figure: 2. Graphical dependence of the solution $u(t,\eta)$ on time t and distance η .

Therefore, function (18) describes the periodic oscillations of the ultrasonic wave with the period $T = 2\pi$ by the variable η . The amplitudes of these oscillations for $\eta = 0$ and $\eta = \frac{\pi}{3}$ are determined by the corresponding formulas

$$A_1(t) = \left|1 - \frac{3}{2}t + \frac{1}{2}t^2\right| e^{-\frac{1}{3}t}, \quad A_2(t) = \frac{1}{2}\left|1 - \frac{3}{2}t + \frac{1}{2}t^2\right| e^{-\frac{1}{3}t}.$$

These amplitudes are depicted in Figure 3 by a top and bottom lines.

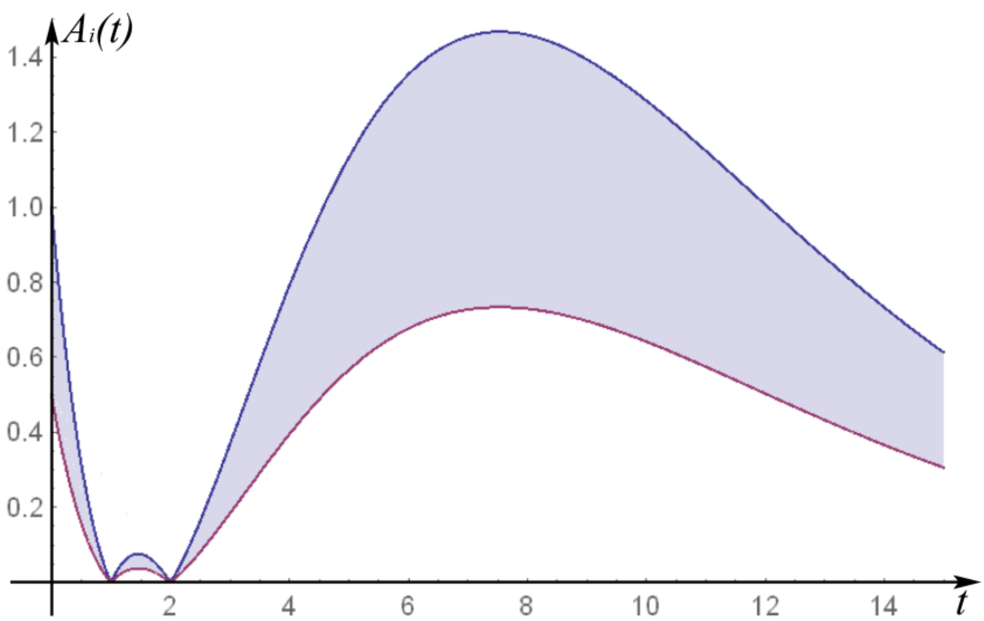

Figure 3: Graphs of amplitudes of the oscillating process

As noted above, the found solution is unique in the class of quasipolynomials which for the fixed t belong to K_M.

Note that the ultrasonic wave oscillates in a limited range at arbitrary time moment and goes exponentially to zero for $t \to +\infty$ on planes which are parallel to the plane $x_1 + x_2 + x_3 = 0$.

4. Conclusions

The mathematical model of the process of ultrasonic wave propagation in a relaxation medium under the condition of setting the wave profile at three time points is investigated. The model is reduced to a problem with three-point time conditions for a hyperbolic partial differential equation of the third order.

The class of quasipolynomials as a class of uniqueness solvability of the problem is established and a practically effective method of constructing the solution in this class is proposed. The examples of application of the specified technique are given.

The proposed method is important in mathematical modeling of acoustic oscillatory processes in relaxation environments. The results of the research can be used in medicine, in particular, in the theory of ultrasound diagnostics.

5. References

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