Analytical Model with Interruption of Service of Short-term Objects with Temporary Reservation

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Abstract
The article presents an analytical model for the maintenance of short-term facilities with temporary redundancy, which are used not continuously, but occasionally. These systems perform tasks that arrive at random times and take some time to complete. Examples of such systems are communication systems as well as various automated control systems. In the developed model, various factors of the real functioning of this class of systems are taken into account: the possibility of interrupting the maintenance of an object upon receipt of a task and two components of the replenished reserve of time, one of which is provided in the system itself, and the second is due to the random nature of the receipt of tasks.
A method is proposed for determining the optimal maintenance frequency and extreme values of the indicators of the systems under consideration: a complex reliability indicator − the coefficient of technical utilization and a cost indicator − the average costs per unit time of the object's stay in a working condition (average unit costs).

Keywords
Information and communication networks, refusal, failure, readiness index, reliability index.

1. Introduction

The need for maintenance arises at the stage of operation of any technical system. The main purpose of maintenance is to ensure the maximum efficiency from the usage of the system during operation. This goal can be achieved by purposeful intervention in the functioning of the system through a rational choice of the type of maintenance, justification of the optimal timing and content of maintenance. Therefore, the solution to the problem of optimal maintenance, in which the extreme values of the selected indicators are provided, is relevant.

2. Formulation of the problem

Consider a system with a replenishment reserve of time (system “object-time”), in which an object is represented by one generalized structural element [1, 2]. Let the operating time of an object to failure t_0 be distributed according to an arbitrary law $F(t) = P\{t_0 < t\}$ with a finite mathematical expectation $\bar{t}_0$, and a failure manifests itself at the moment of its occurrence (a system with instant indication of failures). The system performs tasks arriving at random moment of time, and the time intervals between the times of arrival are distributed exponentially with a parameter γ_f. We will

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assume that the duration of the task is so short in comparison with the average operating time of the object to failure that it can be practically neglected (short-term system) [3]. This means that if an incoming task finds the system in a working state, then it is executed with a probability of one.

Suppose that at the initial moment of time $t = 0$ the object is operational and a scheduled preventive maintenance is assigned at a deterministic time T, which determines the frequency of maintenance. Before the start of service execution, one of two independent events can occur: object failure or task arrival. Let the object fail is first. In this case, the restoration of its performance immediately begins, the duration of which is a random variable t_R, distributed according to an exponential law with a parameter μ and a finite mathematical expectation $\bar{t}_R = 1/\mu$. If a task is received during the recovery process, then a delay (delay) in its execution for a time τ_a is allowed, which can be a random variable, exponentially distributed with a parameter γ, or a deterministic value $t_a = \text{const}$. If the restoration of the object's operability is completed before the use of the time reserve τ_a (or t_a) that determines the permissible lag time, then the task is executed with probability one, otherwise, at the moment the condition $t_R > \tau_a$ (or $t_R > t_a$) is fulfilled, the system fails (task execution failure) [4]. Let the object fail until the moment T (an event $t_0 > T$ has occurred). Then, at the appointed time T, maintenance starts, the duration of which is exponentially distributed with a parameter θ. If a task arrives in the process of servicing, then servicing is terminated and the object is transferred to the main (operating) mode in a random time t_z, exponentially distributed with a parameter γ_z. At the same time, the system provides for a permissible time for bringing an object to readiness to perform a task τ_{a1}, which can be a random variable with an exponential distribution law with a parameter γ_1 or a deterministic value $t_{a1} = \text{const}$. If $t_z < \tau_{a1}$ (or $t_z < t_{a1}$), then the received task is executed with probability one, after which the object is transferred to the maintenance completion mode. Otherwise (at $t_z > \tau_{a1}$ or $t_z > t_{a1}$) at the moment when the time reserve is used up, the system malfunctioning (refusal to complete the task) occurs. Thus, the system uses two components of the replenished reserve of time: one component is provided in the system itself (τ_a and τ_{a1} or t_a and t_{a1}), and the other is due to the random nature of the arrival of tasks (the time from the moment of object failure or the start of maintenance to the moment the task is received) [5,6]. It is necessary to obtain analytical expressions for the service quality indicators [7]: technical usage coefficient $K_{tu}(T)$ and average unit costs $\overline{C}(T)$ (objective functions) and determine the optimal values of the service frequency T^* and T_1^*, at which the quality indicators take extreme values: the maximum value $K_{tu}(T^*)$ for the technical utilization factor and the minimum value $\overline{C}(T_1^*)$ for the average unit costs.

2.1. Obtaining target functions

Let us introduce into consideration a random process $x(t)$ describing the functioning of the system, the graph of states and transitions of which is shown in Fig. 1, where:

e_0 – the state in which the object is operational;

e_1 – the state in which the object is undergoing maintenance and there are no tasks;

e_2 – the state in which the object is being restored and there are no tasks;

e_3 – the state in which the object is inoperable and the received task is waiting for the end of the restoration of operability within the admissible time τ_a (or t_a);

e_4 – the state in which the received task is waiting for the end of the transfer of the object from the service mode to the operating mode within a permissible time τ_{a1} (or t_{a1});

e_5, e_6 – the states in which the system is, respectively, under recovery or under maintenance after spending the replenished reserve of time τ_a or τ_{a1} (t_a or t_{a1});

E_+, E_- – subsets of healthy and faulty states of the system, respectively.

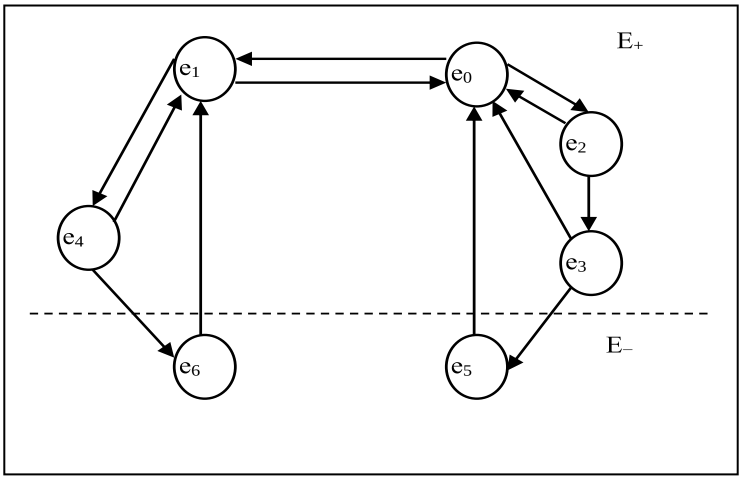

Figure 1: The graph of states and transitions of a random process $x(t)$ describing the operation of a system with service interruption

The technical usage coefficient $K_{tu}(T)$ is defined as the stationary probability of the $x(t)$ process staying in the subset of operable states E_+

$$K_{tu}(T) = \sum_{i=0}^{4} \pi_i a_i \left(\sum_{i=0}^{6} \pi_i a_i \right)^{-1}, \tag{1}$$

where π_i the stationary probabilities of the embedded Markov chain $\pi_i\left(i=\overline{0,6}\right)$; a_i − average residence time of process $x(t)$ in state e_i, $i=\overline{0,6}$ [8].

The probability $\pi_i\left(i=\overline{0,6}\right)$can be found using the well-known system of equations [9]:

$$\pi_i = \sum_{j \in E} p_{ij} \pi_j ,$$

where p_{ij} stationary transition probabilities of the embedded Markov chain.

This system of equations taking into account the graph of states and transitions of process $x(t)$ (Fig. 1) takes the form:

$$\left.\begin{aligned}
\pi_0 &= \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30} + \pi_5 p_{50}, \\
\pi_1 &= \pi_0 p_{01} + \pi_4 p_{41} + \pi_6 p_{61}, \\
\pi_2 &= \pi_0 p_{02}, \\
\pi_3 &= \pi_2 p_{23}, \\
\pi_4 &= \pi_1 p_{14}, \\
\pi_5 &= \pi_3 p_{35}, \\
\pi_6 &= \pi_4 p_{46}
\end{aligned}\right\} \tag{2}$$

taking into account the normalization condition $\sum_{i=0}^{6} \pi_i = 1$.

Solving this system, we get

$$\begin{gathered}
\pi_0 = B; \quad \pi_1 = \left(\frac{p_{01}}{p_{10}}\right) B; \quad \pi_2 = p_{02} B; \quad \pi_3 = p_{02} p_{23} B; \\
\pi_4 = p_{14} \left(\frac{p_{01}}{p_{10}}\right) B; \quad \pi_5 = p_{02} p_{23} p_{35} B; \quad \pi_6 = p_{14} p_{46} \left(\frac{p_{01}}{p_{10}}\right) B,
\end{gathered} \tag{3}$$

where $B=\left[1-\left(\dfrac{p_{01}}{p_{10}}\right)p_{14}(1-p_{46})+p_{02}(1-p_{23})+p_{02}p_{23}p_{35}\right]^{-1}$.

Let us now substitute the found values of $\pi_i\left(i=\overline{0,6}\right)$ into the general formula (1) and obtain an expression for the coefficient of technical use in general form:

$$K_{tu}(T)=\frac{a_0+p_{02}(a_2+p_{23}a_3)+\left(\dfrac{p_{01}}{p_{10}}\right)(a_1+p_{14}a_4)}{a_0+\left(\dfrac{p_{01}}{p_{10}}\right)[a_1+p_{14}(a_4+p_{46}a_6)]+p_{02}[a_2+p_{23}(a_3+p_{35}a_5)]}. \tag{4}$$

Using formulas $a_i=\int_0^\infty x dF_i(x)$ and $p_{ij}=\lim_{t\to\infty}P_i(t)$ [10] and taking into account the initial conditions of the problem and the accepted assumptions and constraints, we obtain the following expression for the stationary transition probabilities ρ_{ij} and average sojourn times a_i in states: $e_i,\left(i=\overline{0,6};\ i\neq j\right)$

$$\left.\begin{array}{l}
p_{01}=\overline{F}(T);\ p_{02}=F(T);\ p_{10}=\dfrac{\theta}{\theta+\gamma_f};\ p_{14}=\dfrac{\gamma_f}{\theta+\gamma_f};\ p_{20}=\dfrac{\mu}{\mu+\gamma_f};\\[2ex]
p_{23}=\dfrac{\gamma_f}{\mu+\gamma_f};\ p_{30}=\dfrac{\mu}{\mu+\gamma};\ p_{35}=\dfrac{\gamma}{\mu+\gamma};\ p_{41}=\dfrac{\gamma_z}{\gamma_z+\gamma_1};\ p_{46}=\dfrac{\gamma_1}{\gamma_z+\gamma_1};\\[2ex]
p_{50}=p_{61}=1;\ a_0=\displaystyle\int_0^T\overline{F}(t)dt;\ a_1=\dfrac{1}{\theta+\gamma_f};\ a_2=\dfrac{1}{\mu+\gamma_f};\ a_3=\dfrac{1}{\mu+\gamma};\\[2ex]
a_4=\dfrac{1}{\gamma_z+\gamma_1};\ a_5=\dfrac{1}{\mu};\ a_6=\dfrac{1}{\gamma_z}.
\end{array}\right\} \tag{5}$$

After substituting the formulas for p_{ij} and $a_i,\left(i=\overline{0,6};i\neq j\right)$ in (4), obtain the final calculated ratio for the coefficient of technical utilization of the system under consideration:

$$K_{tu}(T)=\frac{\displaystyle\int_0^T\overline{F}(t)dt+\frac{F(T)}{\mu+\gamma_f}\left(1+\frac{\gamma_f}{\mu+\gamma}\right)+\frac{\overline{F}(T)}{\theta}\left(1+\frac{\gamma_f}{\gamma_z+\gamma_1}\right)}{\displaystyle\int_0^T\overline{F}(t)dt+\frac{F(T)}{\mu}+\frac{\overline{F}(T)}{\theta}\left(1+\frac{\gamma_f}{\gamma_z}\right)}, \tag{6}$$

where $\overline{F}(t)=1-F(T)$.

In formula (6), $F(t)$ denotes in general form the distribution function of the operating time of the object to failure. For concrete forms of this function, various particular solutions of practical interest can be obtained from formula (6). So, for the case when $F(t)$ has an exponential distribution with a parameter $\lambda=\dfrac{1}{\bar{t}_0}$, we have

$$F(t)=1-e^{-\lambda t},\ \int_0^T\overline{F}(t)dt=\frac{1}{\lambda}\left(1-e^{-\lambda T}\right), \tag{7}$$

and for the case when $F(t)$ has a second-order Erlang distribution with a parameter $\lambda=\dfrac{2}{\bar{t}_v}$, can be obtained

$$F(t)=1-e^{-\lambda t}(1+\lambda t),\ \int_0^T\overline{F}(t)dt=\frac{2}{\lambda}\left(1-e^{-\lambda T}\right)-Te^{-\lambda T}. \tag{8}$$

Substituting expressions (7) or (8) into formula (6), we obtain the calculated ratios for the coefficient of technical use:

a) for the case of exponential distribution of mean time to failure $F(t)$:

$$K_{tu}(T)=\frac{\left(1-e^{-\lambda T}\right)\left[\frac{1}{\lambda}+\frac{1}{\mu+\gamma_f}\left(1+\frac{\gamma_f}{\mu+\gamma_f}\right)\right]+\frac{1}{\theta}\left(1+\frac{\gamma_f}{\gamma_z+\gamma_1}\right)e^{-\lambda T}}{\left(1-e^{-\lambda T}\right)\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)+\frac{1}{\theta}\left(1+\frac{\gamma_f}{\gamma_z}\right)e^{-\lambda T}};\tag{9}$$

b) for the case when $F(t)$ has a 2nd order Erlang distribution:

$$K_{tu}(T)=\frac{x+\frac{\mu+\gamma+\gamma_f}{\left(\mu+\gamma_f\right)\left(\mu+\gamma\right)}y+\left[\frac{(1+\lambda T)\left(\gamma_z+\gamma_f+\gamma_1\right)}{\theta\left(\gamma_z+\gamma_1\right)}-T\right]e^{-\lambda T}}{x+\frac{1}{\mu}y+\left[\frac{(1+\lambda T)\left(\gamma_f+\gamma_z\right)}{\theta\gamma_z}-T\right]e^{-\lambda T}},\tag{10}$$

where $x=\frac{2}{\lambda}\left(1-e^{-\lambda T}\right);\ \ y=\left[1-e^{-\lambda T}(1+\lambda T)\right]$.

The obtained design ratios (6), (9) and (10) are target functions for optimizing the frequency of maintenance in terms of the complex reliability indicator − the coefficient of system technical use.

We now obtain an objective function for optimizing the frequency of servicing according to the criterion of the minimum average unit costs $\overline{C}(T)$ [11].

In the problem under consideration, the average unit costs $\overline{C}(T)$ can be defined as the ratio of the sum of the average costs of restoring the facility's operability, performing maintenance, and making the facility ready from the maintenance mode to the operating mode to the average residence time of the system in a subset of operable states [12, 13].

Proceeding from this and using the graph of states and transitions of the system (Fig.1), you can write the following formula for $\overline{C}(T)$:

$$\overline{C}(T)=\frac{c_R\left(\pi_2a_2+\pi_3a_3+\pi_5a_5\right)+c_m\pi_1a_1+c_z\left(\pi_4a_4+\pi_6a_6\right)}{\pi_0a_0+\pi_1a_1+\pi_2a_2+\pi_3a_3+\pi_4a_4}$$

or, taking into account expressions (3) and (5), after simple transformations, we obtain:

$$\overline{C}(T)=\frac{c_R\frac{F(T)}{\mu}+\frac{\overline{F}(T)}{\theta}\left(c_m+c_z\frac{\gamma_f}{\gamma_z}\right)}{\int_0^T\overline{F}(t)dt+F(T)\frac{\mu+\gamma+\gamma_f}{(\mu+\gamma)\left(\mu+\gamma_f\right)}+\overline{F}(T)\frac{\gamma_z+\gamma_1+\gamma_f}{\theta\left(\gamma_z+\gamma_1\right)}},\tag{11}$$

where c_R, c_m, c_z are the average costs per unit of time, respectively, when restoring operability, when performing maintenance and when transferring an object from maintenance mode to operating mode.

The resulting relationship (11) is an objective function for optimizing the frequency of maintenance for the selected cost indicator.

2.2. Optimization problem solution

Let us bring formula (6) to the following form:

$$K_{tu}(T)=\frac{\int_0^T\overline{F}(t)dt+F(T)(A-B)+B}{\int_0^T\overline{F}(t)dt+F(T)D+E},\tag{12}$$

where

$$A=\frac{\mu+\gamma+\gamma_f}{(\mu+\gamma)\left(\mu+\gamma_f\right)};\ B=\frac{\gamma_z+\gamma_1+\gamma_f}{\theta\left(\gamma_z+\gamma_1\right)};\ D=\frac{\theta\gamma_z-\mu\left(\gamma_z+\gamma_f\right)}{\theta\mu\gamma_z};\ E=\frac{\gamma_z+\gamma_f}{\theta\gamma_z}.\tag{13}$$

Formulas (13) for A and B are written for the case of exponential distributions of quantities τ_a and τ_{a1} with parameters γ and γ_1, respectively. In the case of degenerate distribution functions (the quantities t_a and t_{a1} are deterministic), we obtain

$$A=\frac{\mu+\gamma_f(1-q)}{\mu(\mu+\gamma_f)};\quad B=\frac{\gamma_z+\gamma_f(1-q_1)}{\theta\gamma_z}, \tag{14}$$

where

$$q=P\{t_R>t_a\}=e^{-\mu t_a};\quad q_1=P\{t_z>t_{a1}\}=e^{-\gamma_z t_{a1}}. \tag{15}$$

Differentiating expressions (12) with respect to T and equating the derivative to zero, we obtain the following equation for finding the optimal value of the service frequency:

$$\frac{E-B}{D-(A-B)}=-F(T)+\lambda(T)\left[\int_0^T \overline{F}(t)dt+\frac{DB-(A-B)E}{D-(A-B)}\right], \tag{16}$$

where the coefficients A, B, D, E are expressed by formulas (13) − (15), and is the object failure rate, i.e.

$$\lambda(T)=\frac{F'(T)}{\overline{F}(T)}=\frac{f(T)}{\overline{F}(T)},\ \overline{F}(T)=1-F(T).$$

Equation (16) is a necessary condition for the existence of an optimal value of the service periodicity.

Let us investigate the question of the existence of roots of equation (16). Therefore, it can be argued that if for $T=0$

$$\frac{E-B}{D-(A-B)}=\frac{\dfrac{\gamma_z+\gamma_f}{\theta\gamma_z}-\dfrac{\gamma_z+\gamma_1+\gamma_f}{\theta(\gamma_z+\gamma_1)}}{\dfrac{\theta\gamma_z-\mu(\gamma_z+\gamma_f)}{\theta\mu\gamma_z}-\left[\dfrac{\mu+\gamma+\gamma_f}{(\mu+\lambda)(\mu+\gamma_f)}-\dfrac{\gamma_z+\gamma_1+\gamma_f}{\theta(\gamma_z+\gamma_1)}\right]}>0 \tag{17}$$

and if $\lim\limits_{T\to\infty}\lambda(t)=\infty$, then equation (16) has at least one root. This can be verified by comparing the left and right sides of the equation when $T=0$ and $T=\infty$.

If, in addition, the monotonicity condition $\lambda'(T)>0$ is added, then the absolute maximum of the function $K_{tu}(T)$ is attained at the smallest root of equation (16).

For the case when $\lim\limits_{T\to\infty}\lambda(t)=\infty$ it is necessary to check the fulfillment of the inequality, which is a sufficient condition for the existence of an extremum of the objective function (12) [14, 15]:

$$\frac{E-B}{D-(A-B)}<\lim_{T\to\infty}V(T), \tag{18}$$

where $V(T)$ is the right side of equation (16).

If condition (18) is satisfied, then it can be concluded that it is expedient to carry out maintenance after a finite time. Let T^* be the point at which the absolute maximum of the objective function (12) is achieved. Then, to determine this maximum, following formulas can be used:

$$\max_T K_{tu}(T)=K_{tu}(T^*)=\begin{cases}\dfrac{\bar{t}_0+A}{\bar{t}_0+\dfrac{1}{\mu}}, & T^*=\infty;\\[2ex] \dfrac{1+\lambda(T^*)(A-B)}{1+\lambda(T^*)D}, & T^*<\infty,\end{cases} \tag{19}$$

where the coefficients A, B, D are determined using formulas (13) − (15).

Let us now turn to the study of the cost indicator $\overline{C}(T)$ − the average costs per unit time of the system's stay in the subset of operable states (formula (11)).

Transform the formula (11) to the following form:

$$\overline{C}(T)=\frac{F(T)K+L}{\int_0^T \overline{F}(t)dt+F(T)(A-B)+B}, \tag{20}$$

where

$$K=\frac{c_R}{\mu}-\left(\frac{c_m}{\theta}+\frac{c_z\gamma_f}{\theta\gamma_z}\right), \tag{21}$$

$$L=\frac{c_{\text{то}}}{\theta}+\frac{c_{\text{пг}}\gamma_{\text{з}}}{\theta\gamma_{\text{пг}}}, \tag{22}$$

and A, B are determined using formulas (13) − (15).

Differentiating expression (20) and equating the derivative to zero, we obtain an equation for determining the optimal value of the maintenance frequency:

$$\frac{L}{K}=-F(T)+\lambda(T)\left[\int_0^T \overline{F}(t)dt+B-\frac{(A-B)L}{K}\right]. \tag{23}$$

By its structure, equation (23) coincides with equation (16), therefore, repeating the previous reasoning, we arrive at the following inequality:

$$\frac{L}{K}<-1+\lambda(\infty)\left[\bar{t}_v+B-\frac{(A-B)L}{K}\right], \tag{24}$$

which is a sufficient condition for the existence of a finite value of the optimal frequency of servicing in terms of average unit costs.

If T_1^* is the point at which the absolute minimum of the function (20) is reached, then, taking into account that for $T_1^*\neq\infty$, the quantity T_1^* satisfies equation (23), we obtain:

$$\min_T \overline{C}(T)=\overline{C}(T_1^*)=\begin{cases}\dfrac{\frac{c_R}{\mu}}{\bar{t}_0+A}, & T^*=\infty;\\[2ex] \dfrac{\lambda(T^*)K}{1+\lambda(T^*)(A-B)}, & T^*<\infty;\end{cases} \tag{25}$$

where the coefficients *A, B, K* are determined by formulas (13) − (15) and (21).

3. Methodology for determining the optimal frequency of maintenance and extreme values of quality indicators

Required initial data for the calculation:

- distribution function of the operating time t_0 of the object to failure $F(t)$ (or $\overline{F}(t)=1-\overline{F}(t)$) with mathematical expectation $\bar{t}_0$;
- density of distribution $f(t)=F'(t)$ of a random variable t_0;
- the rate of object failures $\lambda(t)=\dfrac{F'(t)}{\overline{F}(t)}=\dfrac{f(t)}{\overline{F}(t)}$;
- intensity of maintenance $\theta=1/\bar{t}_m$;
- the intensity of transferring the object to the main (working) mode $\gamma_z=1/\bar{t}_z$;
- intensity of receipt of the task γ_f;
- parameter of the exponential distribution law of the admissible recovery time of the object's operability $\gamma=1/\bar{\tau}_a$ or the value of the reserve time $t_a=\text{const}$, if the reserve time is a deterministic value;
- parameter of the exponential law of distribution of the allowable time for transferring the

object to the main (operating) mode $\gamma_1 = 1/\bar{\tau}_{a1}$ or the value of the reserve time $t_{a1} = \text{const}$, if the reserve time is a deterministic value;

- average costs per unit of time, respectively, when restoring the facility's operability c_R, when performing maintenance c_m and when transferring the facility to operating mode c_z.

1. Limitations and assumptions:

- after the end of maintenance and restoration of operability, the system (object + standby time) is completely updated;
- the average recovery time is greater than the average service duration ($\bar{t}_R > \bar{t}_m$, where $\bar{t}_R = 1/\mu$, $\bar{t}_m = 1/\theta$).
- the time to complete the task is so short compared to the average operating time of the object to failure that it can be practically neglected;
- the following conditions must be met:

$$D > (A - B);\ \frac{c_R}{\mu} > \frac{c_m}{\theta} + \frac{c_z \gamma_f}{\theta \gamma_z}.$$

2. Formulas for calculating the optimal values of the frequency of maintenance and extreme values of the coefficient of technical use. Verification of the fulfillment of the accepted restrictions and assumptions. Checking the fulfillment of a sufficient condition for the existence of an optimal periodicity of servicing in a finite time (inequality (18)). Determination of the optimal value of the service frequency T^* as a root of the equation:

$$\frac{E - B}{D - (A - B)} = -F(T) + \lambda(T)\left[\int_0^T \overline{F}(t)dt + \frac{DB - (A - B)E}{D - (A - B)}\right], \tag{26}$$

where the coefficients A, B, D, E are determined by formulas (13) − (15).

Note that if the facility failure rate $\lambda(t)$ is an unboundedly increasing monotonic function (т.е. $\lim_{t \to \infty} \lambda(t) = \infty$), then equation (26) has a single root. Calculation of the maximum value of the coefficient of technical use. If equation (26) has one root T^*, then the maximum value of the coefficient of technical use is determined by the formula:

$$\max_T K_{tu}(T) = K_{tu}(T^*) = \frac{1 + \lambda(T^*)(A - B)}{1 + \lambda(T^*)D}. \tag{27}$$

If the equation has no roots, then $T^* = \infty$ and:

$$\max_T K_{tu}(T) = K_{tu}(\infty) = K_h = \frac{\bar{t}_0 + A}{\bar{t}_0 + \frac{1}{\mu}}. \tag{28}$$

This means that the absolute maximum of the comprehensive reliability index is achieved when the facility is not undergoing maintenance. This indicator is the system availability factor K_h.

If equation (26) has several roots T_1^*, $T_2^*, \ldots, T_k^*$, then the value of the optimal periodicity will be numerically equal to one of these roots or $T^* = \infty$ depending on the point at which the absolute maximum of the function $K_{tu}(T^*)$ is reached. It is easy to find out by comparing the values $K_{tu}(T_i^*)$, $i = 1, 2, \ldots, k$, and $K_{tu}(\infty)$.

3. Formulas for calculating the optimal values of the frequency of maintenance T^* and the minimum values of the average unit costs. Verification of the fulfillment of the accepted restrictions and assumptions. Inequality check:

$$\frac{\frac{c_m}{\theta} + \frac{c_z \lambda_f}{\theta \gamma_z}}{\frac{c_R}{\mu} - \left(\frac{c_m}{\theta} + \frac{c_z \gamma_3}{\theta \gamma_z}\right)} < -1 + \lambda(\infty)\left[\bar{t}_0 + B - \frac{L(A - B)}{K}\right], \tag{29}$$

which is a sufficient condition for the existence of an extremum (minimum) of the objective function (20). The fulfillment of this inequality testifies to the fact that the optimal frequency of servicing T_1^* has a finite value.

Determination of the optimal value of the periodicity T_1^* as a root of the equation:

$$\frac{\frac{c_m}{\theta}+\frac{c_z\gamma_3}{\theta\gamma_z}}{\frac{c_R}{\mu}-\left(\frac{c_m}{\theta}+\frac{c_z\gamma_3}{\theta\gamma_z}\right)}=-F(T)+\lambda(T)\left[\int_0^T \bar{F}(t)dt+B-\frac{(A-B)L}{K}\right], \tag{30}$$

where the coefficients A, B, K, L are determined by formulas (13) − (15), (21), (22).

If the equation has one root T_1^*, then the minimum value of the average unit cost is determined using the expression:

$$\min_T \bar{C}(T)=\bar{C}(T_1^*)=\frac{\lambda(T_1^*)\left[\frac{c_R}{\mu}-\left(\frac{c_m}{\theta}+\frac{c_z\gamma_f}{\theta\gamma_z}\right)\right]}{1+\lambda(T_1^*)(A+B)}, \tag{31}$$

if equation (30) has no roots, then $T_1^*=\infty$ and

$$\min_T \bar{C}=(T)\bar{C}(\infty)=\frac{(c_R/\mu)}{\bar{t}_0+A}. \tag{32}$$

This corresponds to the case where maintenance of the facility is impractical. If equation (30) has several roots, then the value of the optimal servicing frequency will be numerically equal to one of them, depending on the point at which the absolute minimum of the function $\bar{C}(T^*)$ is achieved.

Note that formulas (27) and (31) for calculating the extreme values of the quality indices $K_{tu}(T^*)$ and $\bar{C}(T_1^*)$ are obtained under the assumption that the optimal values of the servicing frequency T^* and T_1^* satisfy equations (26) and (30), respectively.

4. Numerical example.

Consider a system characterized by the following data:

- the operating time of the object to failure is distributed according to the second order Erlang law $F(T)=1-[\exp(-\lambda t)](1+\lambda t)$ with a parameter $\lambda=0{,}01\,\text{h}^{-1}$;
- density of distribution of operating time to failure $f(t)=F'(t)=\lambda^2 t\exp(-\lambda t)$;
- the rate of object failures $\lambda(t)=\frac{f(t)}{1-F(t)}=\frac{\lambda^2 t}{1+\lambda t}$;
- the intensity of restoration of the facility's operability $\mu=1/\bar{t}_R=1/6\ \text{h}^{-1}$ and performance of maintenance $\theta=1/\bar{t}_m=1/4\ \text{h}^{-1}$;
- the intensity of the transfer of the object to the main (working) mode $\gamma_z=1/\bar{t}_z=1{,}0\,\text{h}^{-1}$;
- the intensity of the receipt of tasks $\gamma_f=0{,}5\,\text{h}^{-1}$;
- values $\gamma=1/\bar{\tau}_a=0{,}5\,\text{h}^{-1}$ and $\gamma_1=1/\bar{\tau}_{д1}=0{,}5\,\text{h}^{-1}$;
- average costs per unit of time when restoring operability $c_R=50\,\text{у.е./ч}$, when performing maintenance $c_m=20\,\text{c.u./h}$ and when transferring an object to an operating mode $c_z=10\,\text{c.u./h}$.

For these initial data, we will determine the optimal values of the frequency of maintenance and the corresponding extreme values of reliability indicators $K_{tu}(T^*)$ and average unit costs $\bar{C}(T_1^*)$.

Solution. We substitute the initial data into the equation (26) and get:

$$1{,}25=V(T), \tag{33}$$

where $V(T)$ denotes the function

$$V(T) = e^{-0.01T}(1+0{,}01T) + \frac{10^{-4}T}{1+0{,}01T}\left[200\left(1-e^{-0{,}01T}\right) - Te^{-0{,}01T} + 6\right]. \tag{34}$$

Since for the accepted initial data $\lim_{T\to\infty} \lambda(T) < \infty$, we check the sufficient condition for the existence of the extremum of the function (12):

$$1{,}25 < \lim_{T\to\infty} V(T) = 2{,}06.$$

Consequently, equation (33) has a single finite root. The root of the equation (33) is determined from the graph of the function $V(T)$ on the right side of the equation, which is shown in Figure 4.4. For the accepted initial data, the optimal value of the maintenance frequency $T^* = 100$ h.

Next, we find the maximum value of the coefficient of technical utilization by the formula (27):

$$K_{tu}(T^*) = \frac{1+\lambda(T^*)\left[\dfrac{\mu+\gamma+\gamma_f}{(\mu+\gamma)(\mu+\gamma_f)} - \dfrac{\gamma_z+\gamma_1+\gamma_f}{\theta(\gamma_z+\gamma_1)}\right]}{1+\lambda(T^*)\left[\dfrac{\theta\gamma_z - \mu(\gamma_z+\gamma_f)}{\theta\mu\gamma_z}\right]} =$$

$$= \frac{1+\dfrac{10^{-4}\cdot 100}{1+10^{-2}\cdot 100}\left[\dfrac{0{,}17+0{,}5+0{,}5}{(0{,}17+0{,}5)(0{,}17+0{,}5)} - \dfrac{1{,}0+0{,}5+0{,}5}{0{,}25(1{,}0+0{,}5)}\right]}{1+\dfrac{10^{-4}\cdot 100}{1+10^{-2}\cdot 100}\dfrac{1{,}0\cdot 0{,}25 - 0{,}17(1{,}0+0{,}5)}{0{,}17\cdot 1{,}0\cdot 0{,}25}} = 0{,}992.$$

For comparison, we present the value of the coefficient of technical use for the case when maintenance is not carried out $(T^* = \infty)$:

$$K_{tu}(\infty) = K_h = \frac{\bar{t}_0 + \dfrac{\mu+\gamma+\gamma_f}{(\mu+\gamma)(\mu+\gamma_f)}}{\bar{t}_0 + \dfrac{1}{\mu}} = \frac{200 + \dfrac{0{,}17+0{,}5+0{,}5}{(0{,}17+0{,}5)(0{,}17+0{,}5)}}{200+6} = 0{,}988\ .$$

minimum value of the average unit costs. Substituting the initial data into equation (30), we get:

$$1{,}5 = V_1(T), \tag{35}$$

where $V_1(T) = e^{-0{,}01T}(1+0{,}01T) + \dfrac{10^{-4}T}{1+0{,}01T}\left[200\left(1-e^{-0{,}01T}\right) - Te^{-0{,}01T} + 6{,}69\right].$

Further check that the sufficient condition of an extremum of the function is executed (20)

$$1{,}5 < \lim_{T\to\infty} V_1(T) = 2{,}0669\ . \tag{36}$$

Let us now determine the optimal value of the frequency of maintenance, which ensures the

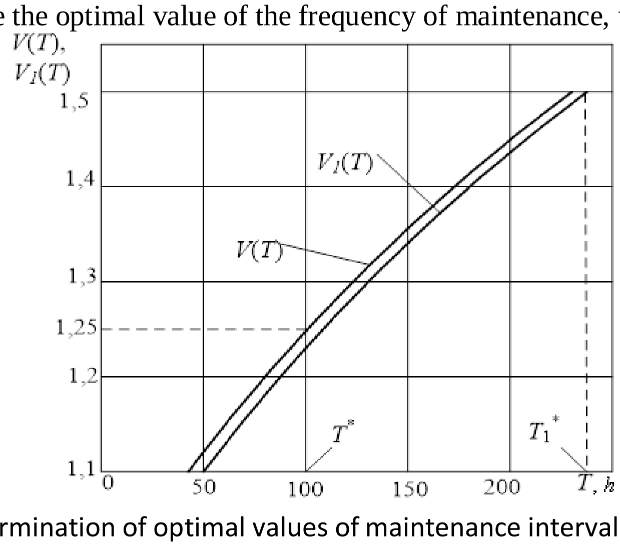

Figure 2: Graphical determination of optimal values of maintenance intervals

From inequality (36) it follows that equation (35) has one root, which can be determined from the graph of the function $V_1(T)$ (Fig. 2). For the accepted initial data, the optimal value of the frequency of maintenance $T_1^* = 240\ \text{h}$. To calculate the minimum value of the average unit costs $\overline{C}(T_1^*)$, we use the formula (31), into which we substitute the value

$$\lambda(T_1^*) = \frac{\lambda^2 T_1^*}{(1+\lambda T_1^*)}.$$

As a result,

$$\overline{C}(T_1^*) = \frac{\dfrac{\lambda^2 T_1^*}{1+\lambda T_1^*}\left[\dfrac{c_R}{\mu} - \left(\dfrac{c_m}{\theta} + \dfrac{c_z \gamma_f}{\theta \gamma_z}\right)\right]}{1 + \dfrac{\lambda^2 T_1^*}{1+\lambda T_1^*}\left[\dfrac{\mu+\gamma+\gamma_f}{(\mu+\gamma)(\mu+\gamma_f)} - \dfrac{\gamma_z+\gamma_1+\gamma_f}{\theta(\gamma_z+\gamma_1)}\right]} =$$

$$= \frac{\dfrac{10^{-4}\cdot 240}{1+10^{-2}\cdot 240}\left[50\cdot 6 - \left(\dfrac{20}{0{,}25} + \dfrac{10\cdot 0{,}5}{0{,}25\cdot 10}\right)\right]}{1 + \dfrac{10^{-4}\cdot 240}{1+10^{-2}\cdot 240}\left[\dfrac{0{,}17+0{,}5+0{,}5}{(0{,}17+0{,}5)(0{,}17+0{,}5)} - \dfrac{1{,}0+0{,}5+0{,}5}{0{,}25(1{,}0+0{,}5)}\right]} = 1{,}46\ \text{c.u./h}$$

For comparison, we present the value of the average unit costs for the case when the system is not serviced $(T_1^* = \infty)$:

$$\overline{C}(\infty) = \frac{\dfrac{c_R}{\mu}}{\bar{t}_0 + \dfrac{\mu+\gamma+\gamma_f}{(\mu+\gamma)(\mu+\gamma_f)}} = \frac{50\cdot 6}{200 + \dfrac{0{,}17+0{,}5+0{,}5}{(0{,}17+0{,}5)(0{,}17+0{,}5)}} = 1{,}481\ \text{c.u./h.}$$

Let us analyze some of the results of a theoretical study of the technical utilization factor $K_{tu}(T)$ of the considered short-term system with service interruption, which provides for temporary redundancy and periodic maintenance. In fig. 3 and fig.4 show graphs of the dependence of the maintenance factor on the frequency of maintenance T at different values $\gamma_z t_{a1}$ (Fig. 3) and at different values of the failure rate λ (Fig. 4). The solid curves correspond to the case when the operating time of the object to failure t_0 is distributed according to the second order Erlang law, and the dotted curves correspond to the case of the exponential distribution of a random variable t_0 (the calculations were carried out on the same mathematical expectations of the operating time to failure).

4. Summary

The studies carried out and the results obtained allow us to draw the following conclusions:

For the same values of the time for performing maintenance and restoring the object's operability $\mu/\theta = 1$, the considered maintenance strategy provides a sufficiently high efficiency of the types of restoration work used, and the value of the technical utilization factor depends significantly on the ratio of parameters $\gamma_z t_{a1}$, $\dfrac{\gamma_f}{\mu}$, $\dfrac{\gamma_f}{\gamma_z}$, that determine the amount of time reserve and the efficiency of its use. In particular, from Fig. 3 it can be seen that an increase in the intensity γ_z of the transfer of the object from the maintenance mode to the main mode (with an increase in $\gamma_z t_{a1}$ and $t_{a1} = \text{const}$) by only two times leads to a significant increase in the efficiency of maintenance. In this case, the optimal value of the service frequency is shifted to the left along the abscissa axis. This is due to the fact that with an increase in $\gamma_z t_{a1}$ system downtime due to maintenance, they are more and more compensated for by the time reserve existing in the system, while the amount of downtime compensation when restoring the object's operability does not change.

In the case when the value $\lambda T \to \infty$ (at $\lambda = \text{const}$), in the system, only the restoration of operability after failures is carried out and all six curves (Fig. 3) asymptotically tend to the value of the stationary availability factor of the system K_h in which maintenance is not carried out.

Fig. 4. it can be seen that the value of the technical utilization factor is also significantly influenced by the failure rate λ of the object under consideration. With a decrease λ, the optimal service frequency shifts to the right along the abscissa, while the curves in the region of maximum values $K_{tu}(T)$ become flatter. Of considerable interest is the study of the influence of the type of the law of distribution of operating time to failure $F(t)$ on the value of the coefficient of technical use. The graphs shown in Fig. 3 and Fig. 4 show that in the case of the distribution of a random variable t_0 according to the exponential law (dotted curves) and according to the second-order Erlang law (solid curves) at the same values of the parameters, the shape of the curves $K_{tu}(T)$ is different. At the same time, for the Erlang law of the second order, at certain values of the parameters, there is an extreme value of the coefficient of technical utilization when changing the values of the service frequency within the interval $[0, \infty)$, while the extreme value in the case of an exponential distribution is reached at the edges of this interval. The exponential distribution of a random variable t_0 gives the lower bound for the considered complex reliability indicator.

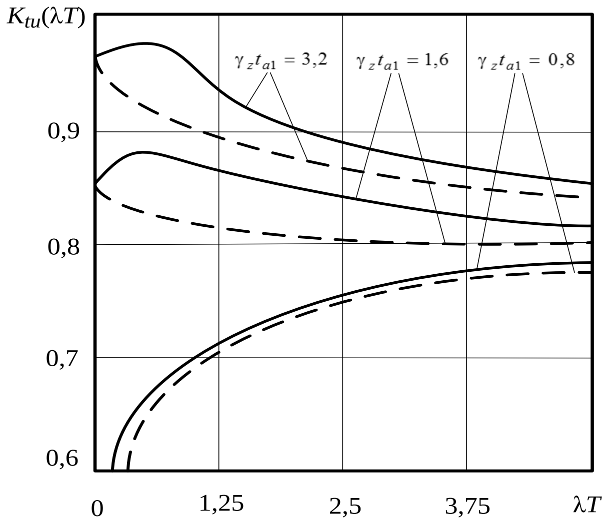

Figure 3: Graphs of the dependence of the coefficient of technical use on λT at various values $\gamma_z t_{a1}$ at $\mu/\theta = 1$, $\gamma_f \theta = 1$

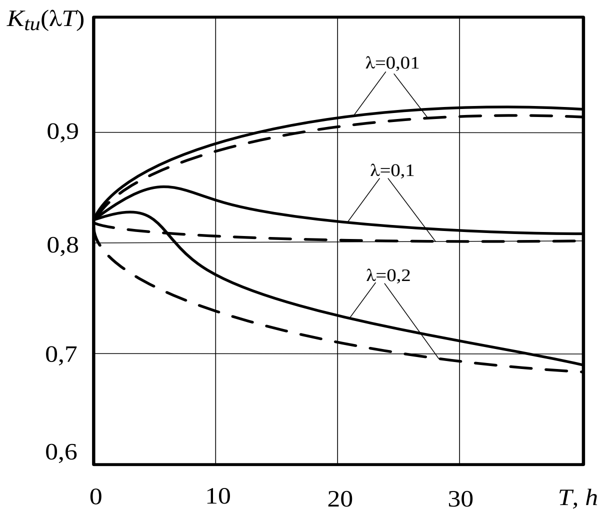

Figure 4: Graphs of the dependence of the coefficient of technical use on the frequency of maintenance at various values λ at $\mu/\theta = 1$, $\gamma_z t_{a1} = 1{,}6$, $\gamma_f \theta = 1$ (where 1 – $\lambda = 0{,}01$; 2 – $\lambda = 0{,}1$; 3 – $\lambda = 0{,}2$)

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