General Case of Wavelet Transform with Reducible Rational Dilation Factor

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Abstract. Datasets of different nature can be effectively analyzed with the help of wavelet analysis. There are two types of discrete wavelet transforms – dyadic and non-dyadic. The latter one allows for more accurate detection and separation of features that are present in analyzed data. A lot of methods use the irreducible fractions as a dilation factor for rational wavelet transform. In this paper the general case of reducible rational dilation factor will be considered. The procedure for building filters will be shown as well as the perfect reconstruction condition. Also, an approach for selecting the best reducible rational dilation factor will be proposed.

Keywords: Wavelet Transform, Non-Dyadic Wavelet, Dilation Factor.

1 Introduction

Wavelet transform (WT) is widely used to analyze datasets of various types. WT has proved its efficiency in analysis of medical signals [1], processing of multimedia data [2], speech and image recognition, etc.

According to the value of dilation factor discrete WT are classified into dyadic, where dilation factor equals 2, and non-dyadic in other cases. Dyadic WT are often used, but non-dyadic wavelet transform can be more suitable for precise localization of signal singularities and similar tasks.

Various authors proposed their own approaches to non-dyadic WT. Their properties and main features are shortly described in previous work [3].

Rational multiresolution analysis [4] looks like the simplest, but the most effective method among all others.

2 Problem Formulation

Usually, the irreducible dilation factor is used in rational wavelet transforms. But in some cases, using the reducible dilation factor can improve the quality of wavelet analysis.

The conditions for building filters for the dilation factor that equals 6/4 were shown earlier [3, 5]. The purpose of the current work is to generalize these conditions to the arbitrary reducible fraction. Also, authors will generalize the perfect reconstruction condition for this case.

The criteria for selecting the best value of rational dilation factor from a set of proportional fractions will be introduced.

3 Problem Solution

3.1 Conditions for Filters Coefficients

Let's take arbitrary reducible fraction as a dilation factor of rational wavelet transform

$$N = \frac{p}{q}, \quad p = \alpha \cdot p', \quad q = \alpha \cdot q', \quad \alpha \in \mathbb{N}, \quad \alpha > 2$$

where fraction

$$\frac{p'}{q'}$$

is irreducible.

We have function

$$\varphi \in V_0 \subset V_{-1} = \overline{Span\left\{\varphi\left(\frac{p}{q} \cdot - n\right)\right\}}$$

that can be represented as

$$\varphi(x) = \sqrt{\frac{p}{q}} \sum_n h_n^0 \varphi\left(\frac{p}{q}x - n\right)$$

Let's define set of functions

$$\varphi(x - ql) = \sqrt{\frac{p}{q}} \sum_n h_n^0 \varphi\left(\frac{p}{q}(x - ql) - n\right) = \sqrt{\frac{p}{q}} \sum_n h_n^0 \varphi\left(\frac{p}{q}x - pl - n\right) =$$

$$= \sqrt{\frac{p}{q}} \sum_k h_{k-pl}^0 \varphi\left(\frac{p}{q}x - k\right)$$

This set of functions has to be orthonormal, then

$$\delta_{0l} = \langle \varphi(x), \varphi(x-ql) \rangle = \int_{\mathbb{R}} \varphi(x) \cdot \overline{\varphi(x-ql)}\, dx =$$

$$= \int_{\mathbb{R}} \left(\sqrt{\frac{p}{q}} \sum_n h_n^0 \varphi\left(\frac{p}{q}x - n\right) \right) \left(\sqrt{\frac{p}{q}} \sum_k \overline{h_{k-pl}^0 \varphi\left(\frac{p}{q}x - k\right)} \right) dx$$

$$= \frac{p}{q} \sum_n \sum_k h_n^0 \overline{h_{k-\mathrm{p}l}^0} \int_{\mathbb{R}} \varphi\left(\frac{p}{q}x - n\right) \cdot \overline{\varphi\left(\frac{p}{q}x - k\right)} dx = \sum_n h_n^0 \overline{h_{n-pl}^0}$$

So, orthonormality of the set of functions $\{\varphi(\cdot - ql)\}$ implies the condition

$$\sum_n h_n^0 \overline{h_{n-pl}^0} = \delta_{0l}$$

Let's take functions

$$\varphi(x-1), \quad \varphi(x-2), \quad \dots, \quad \varphi\big(x-(q-2)\big), \quad \varphi\big(x-(q-1)\big)$$

from the same V_0. Due to this fact we can write each of these functions as

$$\varphi(x-j) = \sqrt{\frac{p}{q}} \sum_n h_n^j \cdot \varphi\left(\frac{p}{q}x - n\right), \quad j = \overline{1 \dots q-1}$$

Then for each function $\varphi(x-j)$ we define a set of functions $\{\varphi(\cdot - ql - j)\}$ as

$$\varphi(x-ql-j) = \sqrt{\frac{p}{q}} \sum_n h_n^j \varphi\left(\frac{p}{q}(x-ql) - n\right) =$$

$$= \sqrt{\frac{p}{q}} \sum_n h_n^j \varphi\left(\frac{p}{q}x - pl - n\right) = \sqrt{\frac{p}{q}} \sum_k h_{k-pl}^{\mathrm{j}} \varphi\left(\frac{p}{q}x - k\right)$$

Each such set of functions has to be orthonormal, so

$$\delta_{0l} = \langle \varphi(x-j), \varphi(x-ql-j) \rangle = \int_{\mathbb{R}} \varphi(x-j) \cdot \overline{\varphi(x-ql-j)}\, dx =$$

$$= \int_{\mathbb{R}} \left(\sqrt{\frac{p}{q}} \sum_n h_n^j \varphi\left(\frac{p}{q}x - n\right) \right) \left(\sqrt{\frac{p}{q}} \sum_k \overline{h_{k-pl}^j \varphi\left(\frac{p}{q}x - k\right)} \right) dx$$

$$= \frac{p}{q}\sum_n\sum_k h_n^j \overline{h_{k-pl}^j} \int_{\mathbb{R}} \varphi\left(\frac{p}{q}x - n\right) \cdot \overline{\varphi\left(\frac{p}{q}x - k\right)}\,dx = \sum_n h_n^j \overline{h_{n-pl}^j}$$

This gives us *q-1* conditions for filter coefficients:

$$\sum_n h_n^j \cdot \overline{h_{n-pl}^j} = \delta_{0l}, \quad j = \overline{1 \dots q-1}$$

Also, all sets of functions

$$\{\varphi(\cdot - ql - j)\}, \quad j = \overline{0 \dots q-1}$$

have be mutually orthogonal

$$0 = \langle \varphi(x-j), \varphi(x - ql - \tilde{j})\rangle = \int_{\mathbb{R}} \varphi(x-j) \cdot \overline{\varphi(x - ql - \tilde{j})}\,dx =$$

$$= \int_{\mathbb{R}} \left(\sqrt{\frac{p}{q}}\sum_n h_n^j \varphi\left(\frac{p}{q}x - n\right)\right)\left(\sqrt{\frac{p}{q}}\sum_k \overline{h_{k-pl}^{\tilde{j}} \varphi\left(\frac{p}{q}x - k\right)}\right) dx$$

$$= \frac{p}{q}\sum_n\sum_k h_n^j \overline{h_{k-pl}^{\tilde{j}}} \int_{\mathbb{R}} \varphi\left(\frac{p}{q}x - n\right) \cdot \overline{\varphi\left(\frac{p}{q}x - k\right)}\,dx = \sum_n h_n^j \overline{h_{n-pl}^{\tilde{j}}}$$

where

$$j, \tilde{j} = \overline{0 \dots q-1}, \quad j \neq \tilde{j}$$

This gives us next conditions:

$$\sum_n h_n^j \cdot \overline{h_{n-pl}^{\tilde{j}}} = 0, \quad j, \tilde{j} = \overline{0 \dots q-1}, \quad j \neq \tilde{j}$$

Finally, we have next conditions for low-pass filter coefficients:

$$\begin{cases} \sum_n h_n^j \cdot \overline{h_{n-pl}^j} = \delta_{0l}, \quad j = \overline{1 \dots q-1} \\ \sum_n h_n^j \cdot \overline{h_{n-pl}^{\tilde{j}}} = 0, \quad j, \tilde{j} = \overline{0 \dots q-1}, \quad j \neq \tilde{j} \end{cases}$$

Now let's denote the Fourier transform of function $\varphi(x)$ as $\hat{\varphi}(\omega)$. Applying the Fourier transform to the

$$\varphi(x-j) = \sqrt{\frac{p}{q}}\sum_n h_n^j \cdot \varphi\left(\frac{p}{q}x - n\right), \quad j = \overline{0 \dots q-1}$$

we will get

$$\hat{\varphi}(\omega) \cdot e^{-ij\omega} = \sqrt{\frac{p}{q}} \sum_n h_n^j \cdot \frac{q}{p} \hat{\varphi}\left(\frac{q}{p}\omega\right) \cdot e^{-i\frac{q}{p}n\omega} = \left(\sqrt{\frac{q}{p}} \sum_n h_n^j \cdot e^{-i\frac{q}{p}n\omega}\right) \cdot \hat{\varphi}\left(\frac{q}{p}\omega\right)$$

that can be rewritten as

$$\hat{\varphi}(\omega) \cdot e^{-ij\omega} = m_0^j\left(\frac{q}{p}\omega\right) \cdot \hat{\varphi}\left(\frac{q}{p}\omega\right)$$

where functions $m_0^j(\omega)$ are defined according to

$$m_0^j(\omega) = \sqrt{\tfrac{q}{p}} \textstyle\sum_n h_n^j \cdot e^{-in\omega} \tag{1}$$

Next, we define *p-q* wavelet functions

$$\psi_j(x) = \sqrt{\frac{p}{q}} \sum_n g_n^j \cdot \varphi\left(\frac{p}{q}x - n\right), \quad j = \overline{1 \dots p - \mathrm{q}}$$

From the orthonormality of these functions and their orthogonality to the functions φ we get next conditions:

$$\begin{cases} \sum_n g_n^j \cdot \overline{g_{n-pl}^j} = \delta_{0l}, \quad j = \overline{1 \dots \mathrm{p} - q} \\ \sum_n g_n^j \cdot \overline{g_{n-pl}^{\tilde{j}}} = 0, \quad j, \tilde{j} = \overline{1 \dots \mathrm{p} - q}, \quad j \neq \tilde{j} \\ \sum_n g_n^j \cdot \overline{h_{n-pl}^{\tilde{j}}} = 0, \quad j = \overline{1 \dots \mathrm{p} - q}, \quad \tilde{j} = \overline{0 \dots q - 1} \end{cases}$$

Based on wavelet functions we build functions

$$m_j(\omega) = \sqrt{\tfrac{q}{p}} \textstyle\sum_n g_n^j \cdot e^{-in\omega}, \quad j = \overline{1 \dots p - q} \tag{2}$$

that satisfy

$$\hat{\psi}_j(\omega) = m_j\left(\frac{q}{p}\omega\right) \cdot \hat{\psi}_j\left(\frac{q}{p}\omega\right), \quad j = \overline{1 \dots p - q}$$

where $\hat{\psi}_j(\omega), \quad j = \overline{1 \dots p - q}$ are the Fourier transform of the corresponding wavelet functions.

3.2 Perfect Reconstruction Condition

Let's define matrix $\mathbf{M}(\omega)$ based on introduced functions $m_0^j(\omega)$ and $m_j(\omega)$:

$$\begin{pmatrix} m_0^0(\omega_0) & .. & m_0^{q-1}(\omega_0) & m_1(\omega_0) & .. & m_{p-q}(\omega_0) \\ m_0^0(\omega_1) & .. & m_0^{q-1}(\omega_1) & m_1(\omega_1) & .. & m_{p-q}(\omega_1) \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ m_0^0(\omega_{p-2}) & .. & m_0^{q-1}(\omega_{p-2}) & m_1(\omega_{p-2}) & .. & m_{p-q}(\omega_{p-2}) \\ m_0^0(\omega_{p-1}) & .. & m_0^{q-1}(\omega_{p-1}) & m_1(\omega_{p-1}) & .. & m_{p-q}(\omega_{p-1}) \end{pmatrix}$$

where arguments $\omega_k, \quad k = \overline{0 \dots n-1}$ are defined according to the formula:

$$\omega_k = \omega + 2\pi \frac{k}{p}, \quad k = \overline{0 \dots n-1}$$

Matrix **A** is defined as

$$\mathbf{A} = \mathbf{M}^*(\omega) \cdot \mathbf{M}(\omega)$$

where $\mathbf{M}^*(\omega)$ is a complex conjugate of the transposition of $\mathbf{M}(\omega)$.

Let's show that matrix **A** satisfies the condition

$$\mathbf{A} = q \cdot \mathbf{I}_{pxp} \tag{3}$$

where $\mathbf{I}_{pxp}$ is a unit matrix of dimension p. In [6] Li shows that for the irreducible dilation factor of rational wavelet transform such expression gives a necessary and sufficient condition for perfect reconstruction.

For this purpose, we will calculate elements of matrix **A**. Diagonal elements of this matrix can be written as

$$a_{jj} = \begin{cases} \sum_{l=0}^{p-1} \overline{m_0^{j-1}\left(\omega + l \cdot \frac{2\pi}{p}\right)} \cdot m_0^{j-1}\left(\omega + l \cdot \frac{2\pi}{p}\right), & j = \overline{1 \dots q} \\ \sum_{l=0}^{p-1} \overline{m_{j-q}\left(\omega + l \cdot \frac{2\pi}{p}\right)} \cdot m_{j-q}\left(\omega + l \cdot \frac{2\pi}{p}\right), & j = \overline{q+1 \dots p} \end{cases}$$

After substituting expressions (1) and (2), multiplying sums and grouping of similar elements we get

$$a_{jj} = \begin{cases} \frac{q}{p} \sum_k \sum_n \overline{h_k^{j-1}} \cdot h_n^{j-1} \cdot e^{-i(n-k)\omega} \cdot \sum_{l=0}^{p-1} e^{-i(n-k)\frac{2\pi l}{p}}, & j = \overline{1 \dots q} \\ \frac{q}{p} \sum_k \sum_n \overline{g_k^{j-q}} \cdot g_n^{j-q} \cdot e^{-i(n-k)\omega} \cdot \sum_{l=0}^{p-1} e^{-i(n-k)\frac{2\pi l}{p}}, & j = \overline{q+1 \dots p} \end{cases}$$

Let's take a look at the last multiplier. After substituting *n-k* by m it can be written as

$$u(m) = \sum_{l=0}^{p-1} e^{-im\frac{2\pi l}{p}} = 1 + e^{-im\frac{2\pi}{p}} + \cdots + e^{-im\frac{2\pi(p-2)}{p}} + e^{-im\frac{2\pi(p-1)}{p}}$$

After denoting $\rho \equiv e^{-im\frac{2\pi}{p}}$ we can write this expression as

$$u(m) = 1 + \rho + \cdots + \rho^{p-2} + \rho^{p-1}$$

If ρ does not equal to one, then we can multiply last expression by 1-ρ and get

$$u(m)(1-\rho) = (1 + \rho + \cdots + \rho^{p-2} + \rho^{p-1})(1-\rho) = 1 - \rho^{p}$$

Due to the fact that $\rho^{p} = e^{-im\frac{2\pi}{p}\cdot p} = e^{-2\pi mi} = 1$ we get that in this case

$$u(m) = \frac{1-\rho^{p}}{1-\rho} = 0$$

If ρ equals one, i.e. for the values of m that are multiplies of the numerator, then from the definition of the function it immediately follows that in this case

$$u(m) = 1$$

So, now we can write the values of the diagonal elements of matrix **A** as

$$a_{jj} = \begin{cases} q \sum_{k} \sum_{l} \overline{h_{n-pl}^{j-1}} \cdot h_{n}^{j-1} \cdot e^{-ipl\omega}, & j = \overline{1 \ldots q} \\ q \sum_{k} \sum_{n} \overline{g_{n-pl}^{j-q}} \cdot g_{n}^{j-q} \cdot e^{-ipl\omega}, & j = \overline{q+1 \ldots p} \end{cases}$$

that, after taking into account conditions for filters coefficients, gives us

$$a_{jj} = q, \quad j = \overline{1 \ldots p}$$

It can be easily shown in a similar way that all extradiagonal elements of the matrix **A** are zeros.

So, matrix **A** satisfies condition (3). This means that it can be looked as a condition for the perfect reconstruction in the case of reducible dilation factor of rational wavelet transform.

3.3 Dilation Factor Selecting Criterion

In order to select optimal value from the set of reducible fractions that all are the multipliers of the same irreducible rational number authors propose to use entropy-based criterion.

Entropy is usually understood as a measure of uncertainty or unpredictability of information [7]. In [7, 8] entropy was proposed to use in order to find the optimal signal decomposition. In their previous work [9] authors also used entropy for selecting an optimal irreducible dilation factor.

Calculation of entropy for the rational wavelet decomposition at the level *N* of the signal is based on relative energy. First, it is necessary to find the energy *Ej* at the level *j* of the decomposition

$$E_j = \frac{1}{N_j} \sum_k d_{j,k}^2$$

where $d_{j,k}$ are the detailed coefficients at this level and N_j is the whole number of the coefficients.

Next, the relative energy Pj that shows the distribution of energy by levels is calculated

$$P_j = \frac{E_j}{\sum_{j=1}^{N} E_j}$$

And at last entropy is calculated according to the expression

$$Entr = -\sum_{j=1}^{N} P_j \cdot ln(P_j)$$

3.4 Experimental Results

We will illustrate the process of selecting the reducible dilation factor by the decomposition of signal of the spontaneous electromagnetic emission of the Earth.

The whole set of data was measured by “Yugneftegazgeologiya” company during the experiments on the September, 2012 in Bazeliyivka, Ukraine.

Fig. 1 shows the first second of the X channel record made by one of the mobile measurement stations at the “zero” point.

Fourier spectrum of this signal is shown at the Fig. 2.

Due to the Fourier spectrum and nature of the measured signal a decision to analyze it by rational wavelet transform with the value of 5/3 for dilation factor was made first. Two other values – 10/6 and 15/9 – for the dilation factor were also considered.

In order to select the most optimal value entropy-based criterion was used. Results are shown in Table 1. The minimum value of entropy is bolded. Entropy for dyadic wavelet decomposition is also included for comparison.

Table 1. Entropy for selected values of dilation factor

Dilation factor value	Entropy
5/3	0.1071
10/6	**0.1066**
15/9	0.1071
2	0.3175

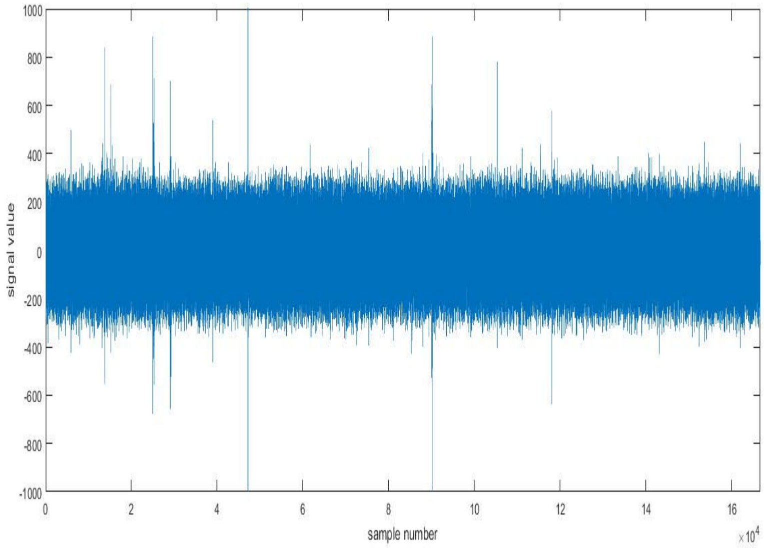

Fig. 1. Measured signal

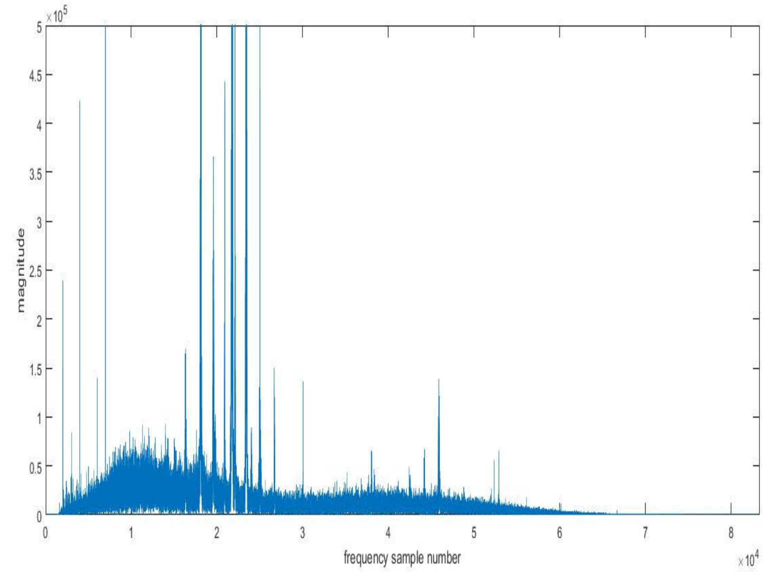

Fig. 2. Fourier spectrum of the signal

Fig. 3 illustrates the advantages of dilation factor 10/6 vs 5/3 in singularities separating.

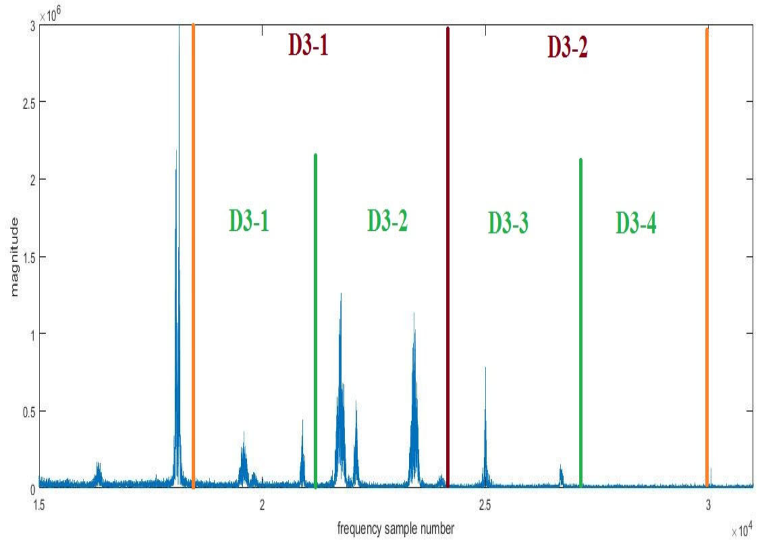

Fig. 3. Detailed components of third level on Fourier spectrum.

Orange vertical lines bound the frequency band of Fourier spectrum that corresponds to the detailed components on third level of wavelet decomposition.

Dark red vertical line separates the frequency intervals for the detailed components of decomposition with dilation factor 5/3. Green vertical lines show the spectrum bands for the detailed components of rational wavelet decomposition with dilation factor 10/6.

It can be easily seen that in the second case singularities of the signal are better separated from each other.

Further fragmentation of frequency band into smaller parts leads to the fact that some singularities will be placed at the boundaries of intervals, and, so, will not be separated. Increasing of wavelet entropy in Table 1 proofs this.

4 Conclusions

Authors have proposed to use arbitrary reducible rational fraction as a dilation factor for the rational wavelet transform. It has been shown that perfect reconstruction condition for such values of dilation factor is satisfied.

Criterion for selecting the optimal dilation factor from the set of reducible fractions that all are the multipliers of the same irreducible rational number has been introduced. Authors have proposed to use entropy-based criterion.

Process of selecting the optimal value for the dilation factor of rational wavelet transform is demonstrated on the signal of the spontaneous electromagnetic emission of the Earth.

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