Tensor Models for Data Extraction and Use of Hidden Knowledge in the Environment of Uncertainty Modeled by Fuzzy Sets of 1 and 2 Types

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Abstract

We consider the modeling of uncertainty, presented in the form of type-1 fuzzy sets and type-2 fuzzy sets, 2D and 3D tensors, which allows the use of matrix-tensor algebra (in particular, Kronecker algebra) to solve decision-making problems under uncertainty along with standard fuzzy mathematics; it is shown that the tensor decompositions of the formed models allow obtaining the closest (in the sense of Frobenius norm) subsets of ordered pairs and sequences, which can be used with limited possibilities of assignment of membership functions or as an alternative to fuzzy sets in solving fuzzy equations and fuzzy systems. An important type of hidden knowledge is the ability to obtain the values of matrix (tensor) invariants, presented in trace form, which significantly affects the quality of decision making.

Tensor models of fuzzy sets make it possible to expand the range of problems to be solved under conditions of uncertainty, in particular, the use of special matrices (tensors) - Toeplitz, Hankel, etc. allows to obtain for a given universal set an objective analog of a fuzzy set and to obtain a comparative assessment of the decision.

Keywords 1

Tensor, fuzzy set, uncertainty, data extraction, hidden knowledge, tensor decomposition, Kronecker product, matrix (tensor) invariants, Kronecker algebra

1. Introduction

Fuzzy set theory (FST) is now a practically universal apparatus that is used in almost all cases where there may be uncertainty. The circumstances that, in our opinion, contributed to this phenomenon are as follows:

- the subset of ordered pairs (SOP), which is the main element of the mathematical apparatus of FST, assumes its flexible modification: depending on the level of uncertainty type-1 FS can extend to type-2 FS, n-type in general and be a subset of ordered sequences (SOS);
- the presence of a component - membership function (MF), which requires virtually no mathematical constraints (except for convexity) and almost entirely depends on the opinion of the expert, allows you to adapt the mathematical apparatus to almost any type of real uncertainty problems.

In [1] it was shown that theoretically SOP can be most rationally used in the analysis of uncertainty in the form of fuzziness (vagueness) and inaccuracy, but real life does not support this thesis. Type-2 FS was introduced by Zadeh as a continuation of the concept of type-1 FS. Type-2 FS is rational to

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use to describe certain types of uncertainty formulated in [2] because the membership function of the type-2 FS itself is fuzzy and corresponds to the nature and characteristics of particular uncertainty.

Although FST is currently the most common mathematical apparatus for solving uncertainty problems (this applies to both type-1 FS and type-2 FS), there are virtually no convincing examples of the effectiveness of type-2 FS compared to type-1 FS, recent studies have shown that in some cases the standard FST does not allow to solve a number of problems under uncertainty. This is due to the arithmetic and logical nature of FS, which does not allow the use of FS tensor-matrix analysis directly and in full. This means, in particular, solutions of large fuzzy equations and systems of fuzzy equations, where the parameters can be both type-1 FS and type-2 FS.

In [3] shows that fuzzy sets over the last fifty years have laid the foundation for a successful method of modeling uncertainty and inaccuracy in a way that no other technique has. The use of fuzzy sets in real computer systems is extremely wide and constantly increasing, which emphasizes the relevance of research related to the discovery of hidden knowledge, which contains uncertainty and its fuzzy set models, presented in tensor form. Note that tensors and tensor decompositions are very powerful and versatile tools that can model a wide variety of inhomogeneous, multi-aspect data. As a result of tensor decompositions, it is possible to extract useful hidden information from multi-aspect data tensors [4], including from data under uncertainty.

The object of research is the generalized processing of multidimensional (multi-aspects) and large-volume data under conditions of uncertainty, which is modeled by fuzzy sets.

The subject of research – tensor models of type-1 and type-2 fuzzy sets, hidden knowledge that can be extracted, tensor decompositions and the formation of the nearest fuzzy sets, the solution of fuzzy equations based on the concept of the nearest fuzzy sets.

The purpose of the work is to expand the class of solvable problems under conditions of uncertainty by extracting hidden knowledge by using tensor models, in particular, the concepts of nearest fuzzy sets and properties of Kronecker algebra, tensor decompositions

The tasks that need to be solved to achieve the goal of the work are the following:

- substantiate the necessity and expediency of representing type-1 FS and type-2 FS by tensor models, which are based on the use of tensor products of FS components in modeling uncertainty;
- show the equivalence of tensor models FS-1 and -2 type with SOP, obtained by singular decomposition of the tensor model FS;
- identify the possibility of 3D tensor representation of uncertainty and identify areas of rational application of 3D models;
- identify hidden knowledge that can be used in tensor modeling of uncertainty;
- to develop methods for solving fuzzy equations at the level of matrix equations by using Kronecker algebra.

2. Problem statement

2.1 List of main symbols and abbreviations

In table 1 are presented the main abbreviations, that are used in the article.

Table 1
Abbreviations

Abbreviation	Explanation of the meaning of abbreviations
EEG	Electroencephalo graphics
US	Universal set
TRS	Type Reduced System
TS	Time series
T2FS	Type-2 fuzzy set
SOS	A subset of ordered sequences

SOP	A subset of ordered pairs
NKP	The nearest Kronecker product
MF	Membership function
KP	Kronecker product
TP	Tensor product
KMIP	Iteration procedure Karnik - Mendel
IT2 FS	Interval type-2 fuzzy set
ISD	an initial set of data
HOSVD	High-order singular decomposition
FV	Fuzzy variable
FST	Fuzzy set theory
FIS	fuzzy interval systems
FS	Fuzzy set

In table 2 are presented the main nomenclatures that is used in the article.

Table 2
Nomenclature

Symbol	Definition
A, A, a, a	Tensor, matrix, (column) vector, scalar
$\mathbb{R}$	The set of real numbers
$\circ$	Outer product
Vec( )	Vectorization operator
$\otimes$	Kronecker product
$\times_n$	n-mode product
$A_{(n)}$	n-mode matricization of tensor **A**
$\mathbf{A}^{-1}$	Inverse of A
$\mathbf{A}^{\dagger}$ or $\mathbf{A}^{+}$	Moore-Penrose Pseudoinverse of **A**
$\lVert A \rVert_F$	Frobenius norm - $trace\left(\mathbf{A}^T \ \mathbf{A}\right)^{1/2}$
A(:,i)	Spans the entire *i* th column of **A** (same for tensors)
A(i,:)	Spans the entire *i* th row of **A** (same for tensors)
reshape ( )	Rearrange the entries of a given matrix or tensor to a given set of dimensions
$\tilde{a}$	Type-1 fuzzy set: $\tilde{a} = \left\{a / \mu(a)\right\}$ or $\left(a_1 \ \mu(a_1); \cdots ; a_n \ \mu(a_n)\right) \in \mathbb{R}^{n \times 2}, a \in A, \mu(a) \to [0,1]$

2.2 Main statement

Recently, a number of uncertainty problems have emerged, the solution of which by TFS methods, in particular, by fuzzy mathematics methods, is either extremely difficult or the result is not constructive. This is especially true for data processing, which belongs to the category of BIG DATA, where ultra-high dimensionality is combined with a large amount of data and thus necessitates working with 3D data, membership functions for which are not tabulated, defining these functions is also extremely difficult.

Problems related to decision-making based on fuzzy equations and systems of fuzzy equations with general data (the parameters of the equations can be set in the form of both type-1 FS and type-2 require the use of new methods and algorithms. Note that modern methods focused on this class of problems, usually work with fuzzy numbers and contain a large number of assumptions. On the basis of the stated requirements, the tasks are formulated as follows: representation of type-1 FS in the form of a 2D tensor (matrix) and, accordingly, the tensor product: $\tilde{x} \to T^{(\tilde{x})} = \left(x \otimes \mu^{(x)}\right) \in \mathbb{R}^{n \times n}$; singular decomposition $\left[u\ s\ v\right] = svd\left(T^{(\tilde{x})}\right)$ allows you to calculate a subset of ordered pairs of sigmoid-like shape $\tilde{y} = \left\{x / \mu^{(y)}\right\}, \mathrm{y} \in X, \quad \mu^{(y)} \to [0,1]$ such, that:

$$T^{(\tilde{y})} = \left(y \otimes v^{(y)}\right), \left\|T^{(\tilde{x})}\right\|_F \cong \left\|T^{(\tilde{y})}\right\|_F, \ def\left(T^{(\tilde{x})}\right) \cong def\left(T^{(\tilde{y})}\right). \quad (1)$$

This allows you to implement mathematical operations on fuzzy variables $\tilde{x}, \tilde{\mathrm{y}}, ..., \tilde{z}$ at the level of tensor variables: $\tilde{x} \to T^{(\tilde{x})}, \ \tilde{\mathrm{y}} \to T^{(\tilde{y})}, ..., \tilde{z} \to T^{(\tilde{z})}$ with the subsequent transformation of the result into the SOP. A similar algorithm with Kronecker products is implemented for type-2 FS, defined on US *X* and presented as

$$\tilde{\tilde{z}} = \left\{z / \tilde{\mu}^{(z)}\right\}, z \in X, \mu^{(z)} \to [0,1], \tilde{\mu}^{(z)} \overset{\Delta}{=} \left\{\mu^{(x)} / \mu^{(z)}\right\}; \quad (2)$$

- for a given FS $\tilde{x} = \left\{x / \mu^{(x)}\right\}, x \in X, \mu^{(x)} \to [0,1]$ find a subset of ordered pairs $\tilde{y} = \left\{y / \mu^{(y)}\right\}, y \in X, \mu^{(y)} \to [0,1]$, obtained as a result of tensor decompositions, called the fuzzy nearest set (with respect to FS) $\tilde{x}$; in turn the tensor model of type-1 FS $\tilde{x}$ has the appearance $\mathcal{X} = \left(x \otimes \mu^{(x)}\right)$, nearest is determined by the principle of the nearest Kronecker product $\left\|\mathcal{X} - y \otimes \mu^{(y)}\right\|_F^2 \to \min$; .
- type-1 FS presented as a tensor product of components $\tilde{x} \in \mathbb{R}^{n \times 2} \to T^{(x)} \in \mathbb{R}^{n \times n}$, by implementing the procedure $\mathcal{T}^{(x)} = reshape\left(T^{(x)}, p, q, w\right), p \cdot q \cdot w = n \cdot n$,where $\mathcal{T}^{(x)}$ - 3D tensor (Fig. 1), and high-order singular decomposition which allows you to obtain a subset of order sequences $T_X^{(multy)} = \left\{x / \langle \mu_1(x), \mu_2(x), ...\rangle\right\}$. Note that $T_X^{(multy)}$ is a multi-fuzzy set

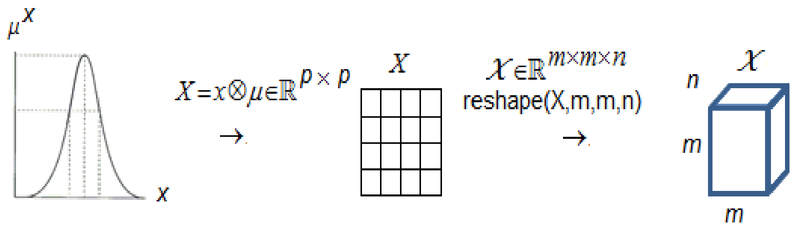


Figure 1: The sequence of conversion from standard FS $\tilde{x}$ at 2D and 3D tensor models

- Formation of 2D tensor model type-2 FS:

$$\tilde{x} = \underset{\substack{x \in X \\ \mu^x \to [0,1]}}{\{x / \mu^x\}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \mu^{x_1} \to \left(\upsilon_{x_1}^{(1)} \dots \ \upsilon_{x_1}^{(m)}\right) \\ \vdots \qquad Blurring \quad MF \\ \mu^{x_n} \to \left(\upsilon_{x_n}^{(1)} \dots \ \upsilon_{x_n}^{(m)}\right) \end{pmatrix}., \tag{3}$$

$$\tilde{\tilde{x}} = \left(x_1 \left(y_1 \quad \nu^{y_1}\right); \cdots ; x_n \left(y_n \quad \nu^{y_n}\right)\right), \tag{4}$$

method of forming a 2D tensor model - double kroneker product of type-2 FS

$$\tilde{\tilde{x}}, T^{(\tilde{\tilde{x}})} = \left(x \otimes \left(y \otimes \nu^y\right)\right), T^{(\tilde{\tilde{x}})} \in \mathbb{R}^{n \cdot (k \times k)}. \tag{5}$$

Separately, we pay attention to the possibility of presenting the type-1 FS in the form as proposed in [5] - $\tilde{\mathcal{X}} = x \otimes \mu^{(x)} \in \mathbb{R}^{n \times n}$ or $\mathbb{R}^{n \times 1 \times n}$.

3. Review of the literature

One of the directions of expanding the use of FS is *granular computing*, in [6] it was shown that granular computing is a new computational theory and paradigm that deals with the processing of information granules, which are defined as a set of information entities grouped together by their similarity, physical adjacency or indistinguishable ability. In most aspects of human reasoning, these granules have an uncertain formation, so the concept of detailing fuzzy information (and revealing hidden information) may be of particular interest for applications where FSs must be converted to crisp sets to avoid uncertainty.

In [7] a tensor granule formed as a tensor product of FS components is proposed, which allows to significantly expand the possibilities of FST and, accordingly, to expand the range of solvable problems under conditions of uncertainty. According to [8], the theory of rough sets is an important approach to granular calculations.

Tensor models of type-1 and type-2 fuzzy sets (the concept of tensorization) not only significantly strengthen the arsenal of methods of fuzzy mathematics but also be an additional channel for comparing the quality of the obtained solutions. The concept of tensorization, as shown in [9], refers to procedures for generating structured tensors of higher-order from lower-order data formats (vectors, matrices, or even low-order tensors) or representing very large system parameters in low-order tensor formats.

For any given source data format, the tensor procedure can affect the choice and efficiency of tensor decomposition in the next step. Records of such a tensor can be obtained using:

- a certain permutation, for example, the transformation of the original data into a tensor,
- alignment of data blocks or epochs, for example, slices of the third-order tensor are epochs of multichannel EEG signals, or

- increasing the data using, for example, Toeplitz matrices / tensors and G (H)ankel.

Let's pay attention to the last thesis. Procedures for the formation and subsequent deposition of Toeplitz or Hankel matrices (tensors) formed on universal sets allow us to solve problems under uncertainty by FST methods under limited conditions of MF assignment, proposed by the authors in [33]. Recall that the tensor in the general case can be represented by fibers or slices [30, 33].

Note the following. First, the theory of rough sets as a new mathematical tool for the implementation of procedures (fuzzy) data conclusions is proposed in [10]. In this regard, we note that the vast majority of works concerning the type-2 FS and their extensions, consider the procedures of fuzzy conclusions, ie the implementation of fuzzy rules "if A, then B otherwise C", although the number of problems under uncertainty, where required type-2 FS and their extensions, much larger, especially for fuzzy mathematics with type-2 FS.

Secondly, as shown in the paper [11]: "Information granules are intuitively attractive constructions that play a key role in human cognitive activity and decision-making. We perceive complex phenomena by organizing existing knowledge together with existing experimental evidence and structuring it in the form of some meaningful, semantically sound entities that are central to all subsequent processes of world description, environmental reasoning, and decision support."

According to [12], type-1 FS can directly and effectively model certain types of uncertainty (according to [1], these are fuzzy and inaccurate), because their MFs are absolutely crisp. On the other hand, type-2 FS, having fuzzy MF, can model wider classes of uncertainty. The membership functions of type-1 FS are two-dimensional, while the membership functions of the type-2 FS are three-dimensional. It is the new third dimension of the type-2 FS that provides additional degrees of freedom, which allows you to directly model the uncertainties. In type-1 FS membership values are between zero and one, while the values of fuzzy membership type-2 are considered as the value of type-1 fuzzy membership, $\tilde{\tilde{A}}$ as a total type-2 FS, is described as follows:

$$\tilde{\tilde{A}} = \int_X \mu_{\tilde{A}}(x)/x = \int_X \Big[\int_{J_x^u} f_x(u)/u\Big]/x, \quad J_x^u = \left\{(x,\mu): u \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]\right\} \subseteq [0,1], \tag{6}$$

$$\underline{\mu}_{\tilde{A}}(x) = \mu_{\tilde{A}}^{(\min)}(x), \overline{\mu}_{\tilde{A}}(x) = \mu_{\tilde{A}}^{(\max)}(x). \tag{7}$$

In [13], the possibility of decomposing an interval type-2 fuzzy logic system into two parallel type-1 fuzzy systems was considered. This decomposition avoids the problems associated with type reduction methods, which are usually required in type-2 fuzzy systems. Type-2 fuzzy set (T2 FS) - is a three-dimensional fuzzy set, in which the primary fuzzy set is characterized by classes of membership, which are not crisp numbers, and actually fuzzy sets - the so-called. secondary membership functions. In works [14,15] tensor models of type-1 FS are offered, which allow to apply matrix-tensor methods, to use tensor-matrix analysis for problems of fuzzy mathematics and if necessary to receive the result in the form standard for FST. Since type-2 FS is an extension of type-1 FS and this object has an effective representation in the matrix (tensor) basis, there is a logical desire to expand the application of tensor methods and models directly for type-2 FS, especially since type-2 FS - 3D measurable object, i.e. tensor.

Moreover, in [16] it was shown that the use of type-1 fuzzy sets for modeling words is scientifically incorrect. However, as shown in [30], most likely the reason lies in the fact that insufficient resources were spent by researchers to develop the actual theory of type-2 FS, as evidenced by the fact that the proposed operation for type-2 FS is not as effective and understandable as they need to be to satisfy real application developers, and the lack of real compelling examples of type-2 FS applications

It is known that FS $\tilde{A}$ on the universal set U is characterized by the membership function $\mu_{\tilde{A}}: U \to [0,1]$ and is recorded as $\tilde{A} = \sum_{u \in U} \mu_{\tilde{A}}(u)/u$ or $\tilde{A} = \int_{u \in U} \mu_{\tilde{A}}(u)/u$, when U is discrete or continuous respectively, abbreviated record $\tilde{A} \overset{\Delta}{=} \left\{u / \mu_{\tilde{A}}(u)\right\}, u \in U$, $\mu_{\tilde{A}}(u) \to [0,1]$. One of the greatest results of FST is the principle of fuzzy expansion, which allows to fuzzify any mathematical theory.

Recall the following:

- type-1 fuzzy sets are a special case of type-2 fuzzy sets, where for all $u \in U$ the set of degrees of primary membership, namely $\sum_{\mu_i^{(u)} \in J_u} \mu_i^{(u)}$ is a singleton (with a maximum value equal to one);
- type-2 FS in the future will be denoted as follows: $\tilde{\tilde{A}} = \sum_{x \in X} \mu_{\tilde{A}}(x)/x$, $\tilde{\tilde{B}} = \sum_{x \in X} \mu_{\tilde{B}}(x)/x$,

where $\mu_{\tilde{A}}(x) = \sum_{u \in J_u^A} f(u)/u$, $\mu_{\tilde{B}}(x) = \sum_{u \in J_u^B} g(u)/u$.

Type-2 FS has features that not only complicate its use but do not positively affect its prevalence, especially in the problems of fuzzy mathematics, in particular fuzzy equations and fuzzy equation systems, because the fuzzy mathematics apparatus designed mainly for type-1 FS and common to type-2 FS, was unable to solve such problems under uncertainty. This is especially true for the presentation of type-2 FS in a form suitable for computer implementation, and defuzzification procedures. One of the ways to solve these and other issues related to the use of FS-2 is to find new forms of representation, granular form of representation (discussed by the authors earlier) and the geometric approach, which is considered in [17, 18]. Note that the practical majority of algorithms designed to represent and defuzzify type-2 FS, developed by J. Mendel.

The authors believe that the uncertainty simulated by the type-2 FS can be more effectively represented, in particular for the implementation of mathematical operations, by a tensor granule. For the generalized case of type-2 FS, when the functions of the secondary membership - the third dimension is of any type, there is a significant computational complexity that has limited their deployment. The complexity of the calculations in the general case of type-2 FS prevents their deployment.

Of course, type-2 fuzzy sets exist in three-dimensional environments, this additional dimension requires the introduction of additional notations. In particular, like type-1 FS, type-2 FS has a domain, in this case *X*. The membership level at one point in the domain is a type-1 fuzzy number, known as the secondary membership function. The domain of the secondary membership function in x, denoted by J_x, is known as the secondary domain or shared domain. Estimation of membership at point u in the function of secondary membership in x, denoted $\mu_{\tilde{A}}(x,u)$, is known as the average level of membership. In works [2, 3] the way of representation of type-2 FS under the name is resulted $Mod\tilde{\tilde{e}}rate$.

It is important to note that the use of type-2 FS requires a preliminary assessment of the quality of the solution obtained when using type-1 FS. In all cases, the type-2 FS is generated by the type-1 FS, MF which is called primary. The following questions arise:

- if based on IDS for modeling of uncertainty type-1 FS was offered $\tilde{x} = \{x/\mu(x)\}, x \in X$, $\mu^{(x)} \to [0,1]$, type-2 FS is formed by the erosion of the primary MF $\tilde{\tilde{x}} = \left\{(x,u)/\tilde{\mu}^{(u)}\right\}, x \in X$, $\left\{(x,u)/\tilde{\mu}^{(u)}\right\}, x \in X, u \in [0,1], \mu^{(u)} \to [0,1]$, the output of the FLS is a reduced type FS $\tilde{y} = \left\{y/\mu^{(y)}\right\}, \mu^{(y)} \to [0,1], \mathrm{y} \in X$, and defuzzyfied value $y = def(\tilde{y})$, then how the quantities are related $y = def(\tilde{y})$ and $x = def(\tilde{x})$;
- if the quality criteria of FS of all types are not defined, deffuzyfied values, for example, type-2 FS and type-1 FS, if they affect one object of uncertainty, are not considered criteria.

On this basis, we can assume that the task of finding type-1 FS, obtained as a result of the transformation of some type-2 FS, which have close (or coinciding) defuzzyfied values, is relevant. In addition, many researchers consider an important problem of accuracy in the application of FST methods, although accuracy under uncertainty is a conditional concept. The possibility of replacing the type-2 FS with an equivalent (from the point of view of defaced value) type-1 FS is relevant.

It should be added that this problem is not fundamentally new for the theory and practice of the type-2 FS. The above cited article [19] proposes ... a new approach to the defuzzification of interval fuzzy sets of type-2 based on the convolution method, which converts the interval type- 2 FS into an *embedded representative se*t of type- 1 (RES), the defuzzyfied value of which approaches the corresponding the value of the type-2 set, it is known that RES as a type-1 set, can be defuzzyfied quite easily.

Available methods of defuzzyfication for discrete type-2 sets, first of all, provide the so-called comprehensive defuzzyfication. For example, for fuzzy interval type-2 systems (FIS), the defuzzification stage consists of two parts - the actual type reduction and defuzzification. This type reduction algorithm was proposed by J. Mendel:

1. All possible built-in type-2 sets must be considered;
2. Minimum average membership found for each built-in set;
3. For each embedded set, the value of the domain of the centroid of type-1 of the embedded set of type-2 is calculated;
4. For each embedded set, the value of the domain of the centroid of type-1 of the embedded set of type-2 is calculated (x, z), it is possible that for some values x will be more than one corresponding value z;
5. For each value of the domain the maximum average estimation is chosen, it creates a subset of ordered pairs (x, z_{max}), such that between x and z_{max} there is an unambiguous correspondence. This completes the reduction of the type-2 set to (Type Reduced Type) type-1.

The obtained TRS - as a type-1 fuzzy set, is easily defuzzyfied by finding its centroid value. Thus, the reduction of the type involves the processing of all embedded sets in the type-2 FS, which is what makes the algorithm to be called "exhaustive defuzzification ". Naturally, there are a lot of built-in sets. For example, when in the above example type-2 FIS implemented inference using sets that were sampled on 51 slices on the *x* and *y* axes, the number of embedded sets in the aggregate set was calculated as a value of the order of 2.9·1063

Although embedded sets are generally easy to process, they create a bottleneck during processing due to their high dimensionality. As a result, exhaustive defuzzification is an impractical method to use, the most common method of reducing the type of fuzzy set interval – type-2 is the iterative procedure Karnik - Mendel (KMIP). The result of reducing the type of the interval of fuzzy sets - type 2 is the interval - type 1, where the centroid lies between two endpoints. An iterative procedure is an effective method of finding these endpoints. The center of this set – type-1 (i.e. defuzzyfied value of the set – type-2) is the center of this interval. Note that this procedure is extended to generalized fuzzy sets - 2 types [19].

In [19] T1 MF between the upper and lower uncertainty bands was found as a representative embedded set, but the method and concept used are not based entirely on the concept of the influence of uncertainty on certain data and degrees of affiliation. In [20] is proposed methods of overcoming difficulties in understanding and interpreting type-2 FS for FLC.

4. Materials and method.

Historically, FST has formed the main object of the theory - a subset of ordered pairs (type-1 FS) as a procedure for a heuristic blurring of a universal set and its representation as a set of α -levels. Further logic of development of the accepted concept naturally led to type-2 FS, formed as a procedure of blurring crisp values of membership function type-1 FS. Assuming that the number of α-levels in the representation of uncertainty using FS is large enough (the concept of BIG DATA provides for the use of FS as one of the possible models), we can show that initially selected by the expert FS (with heuristic FN) can be simultaneously represented as multi fuzzy set [21] or as type-2 FS in 3D space. The fuzzy set generalizes FS-1, -2 types, and intuitive sets.

Nearest FS. The ability to represent SOP as 2D and 3D objects requires an assessment of their proximity. Recall that the problem of finding the nearest (farthest) element, neighbor, etc. is not new to mathematics. In the last 20 years, it has been replenished with the so-called problem of the nearest Kronecker product, the solution of which is extremely important for modern mathematics, in particular, tensor (matrix) analysis. Unfortunately, the use of this powerful device to solve problems under uncertainty began only in the last 5-7 years [22].

Tensor models FS were first proposed in [23], we recall that the main apparatus TFS - a subset of ordered pairs - is a matrix in $\mathbb{R}^{n \times 2}$, which can be represented as a tensor y (n-number α-levels FS). In turn, the procedure $reshape\left(\mathcal{A},\left[m,p,q\right]\right)$ allows to represent the initial FS $\tilde{A}$ in space $\mathbb{R}^{m\times p\times q}$ and get a subset of ordered sequences (triplets), which allows fundamentally from new positions to implement the analysis of uncertainty in 3D space, in particular, this applies to type-2 FS and so on.

We present the problem of the nearest Kronecker product (NKP) using the paper [24]. It was shown in [24] that the solution of the NKP problem is associated with the procedure of singular decomposition of the permutation (vectorized) version of the matrix $\boldsymbol{A}$. This leads to the problem $\phi\left(\mathbf{B},\mathbf{C}\right)=\left\|R(\mathbf{A})-vec(\mathbf{B})\otimes vec(\mathbf{C})^T\right\|_F$, and the fact of minimization ϕ is the search for the nearest rank-1 matrix to $\mathcal{R}\left(\mathbf{A}\right)$. The nearest rank-1 matrix is a well-known problem of singular decomposition. In particular, if $\mathbf{U}^T\mathcal{R}\left(\mathbf{A}\right)\mathbf{V}=\Sigma$ - singular decomposition, the optimum is defined as:

$$vec\left(\mathbf{B}_{opt}\right)=\left(\sigma_1\right)^{1/2}\cdot\mathbf{U}\left(:,1\right),\ vec\left(\mathbf{C}_{opt}\right)=\left(\sigma_1\right)^{1/2}\cdot\mathbf{V}\left(:,1\right)\cdot \tag{8}$$

It is important to note that in this case the scaling is arbitrary. Indeed, if $\mathbf{B}_{opt}$ and $\mathbf{C}_{opt}$ is the solution of the NKP problem, and given α ≠0, then $\alpha\cdot\mathbf{B}_{opt}$ and $\left(1/\alpha\right)\cdot\mathbf{C}_{opt}$ $\left(1/\alpha\right)\cdot\mathbf{C}_{opt}$, then and are also optimal. It is accepted that $\alpha=\max\left(abs\left(\mathbf{V}\left(:,1\right)\right)\right)$, it allows to consider $\mathbf{B}_{opt}$ and $\mathbf{C}_{opt}$ as SOP, where one of components $\mathbf{C}_{opt}\in\left[0,1\right]$, - that gives the chance to apply the TFS device to the optimum decisions calculated as a result of singular decompositions.

Considering [31, 34] , the ordinary set A nearest to the fuzzy $\tilde{A}$ one is located at the smallest distance from the given fuzzy set, or in other words has the smallest norm. It is shown that this will be an ordinary set endowed with the following properties

$$\mu_{\tilde{A}}\left(x\right)=\begin{cases}0, if\ \mu_{\tilde{A}}\left(x_i\right)<0.5;\\1, if\ \mu_{\tilde{A}}\left(x_i\right)>0.5;\\0, if\ \mu_{\tilde{A}}\left(x_i\right)=0.5.\end{cases}$$

In turn, if FS is represented as a tensor model, as shown below,

$$\tilde{A}=\left\{a\ /\ \mu^{\left(a\right)}\right\}\rightarrow\left(a_1\ \mu^{\left(a_1\right)};\cdots;a_n\ \mu^{\left(a_n\right)}\right)\rightarrow\mathcal{A}=\left(\tilde{A}\left(:,1\right)\otimes\tilde{A}\left(:,2\right)\right)\in\mathbb{R}^{n\times n} \tag{9}$$

then the search for the nearest fuzzy set should be implemented by entering 2 prerequisites:

- fuzzy set is used to represent uncertainty, all subsequent mathematical procedures are performed on tensor models, the final result (if necessary) is converted into a subset of ordered pairs, which is analogous to fuzzy set, always has a sigmoid-like shape and is calculated as a result of tensor decompositions.

The main advantage of the concept of nearest fuzzy sets (or subsets of ordered sequences) is that:

- it is possible to use hidden knowledge, which is contained in the set of initial data (SID) and accumulated in the FS;
- there is an additional channel to obtain information for the formation of MF;
- there is a possibility of processing 3D data under conditions of uncertainty and the possibility of simplified analysis using type-2 FS;

- additional possibilities of expanding the classes of solvable problems under conditions of uncertainty, in particular, the solution of fuzzy equations and systems of fuzzy equations of type-2 FS and multi-fuzzy equations by using the methods of Kronecker algebra.

The specified model can be transformed into the following models:

- SOP, calculated on the basis of singular decomposition $T^{(\tilde{\tilde{x}})}$;
- 3D tensor $T_0^{(\tilde{\tilde{x}})} = [T_0^{(\tilde{\tilde{x}})}(:,:,1),...,T_0^{(\tilde{\tilde{x}})}(:,:,\mathrm{k})]$, presented in the form of frontal slices $T_0^{(\tilde{\tilde{x}})}(:,:,\mathrm{j}), j=1,k$;

$$T^{(\tilde{\tilde{x}})} \to \begin{cases} svd\left(T^{(\tilde{\tilde{x}})}\right) \to \left(z \ \ \lambda^{z}\right), z \in Z, \ ^{z}\!\to [0,1] \\ T_0^{(\tilde{\tilde{x}})} = \text{reshape}\left(T^{(\tilde{\tilde{x}})}, f, f, f\right), n \cdot k \cdot k = f \cdot f \cdot f \end{cases}, \tag{13}$$

$$T^{(\tilde{\tilde{x}})} = \left[T^{(\tilde{\tilde{x}})}(:,:,1),...,T^{(\tilde{\tilde{x}})}(:,:,\mathrm{k})\right] \tag{14}$$

The hidden knowledge that can be "extracted" from the tensor models of FS, includes the following, primarily matrix and tensor invariants. For 2D tensor $\mathbf{A} \in \mathbb{R}^{3\times 3}$ main invariants can be de-fined as $I_1 = tr(\mathbf{A}) = A_{11} + A_{22} + A_{33} = \lambda_1 + \lambda_2 + \lambda_3$; $I_2 = \frac{1}{2}\left(tr(\mathbf{A})^2 - tr\left(\mathbf{A}^2\right)\right) = \lambda_1\lambda_3 + {}_{+}$ $\lambda_1\lambda_2 + \lambda_2\lambda_3$; $I_3 = \det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3$.

Another possibility for obtaining new knowledge is that FS-1 and 2 types have fundamentally equivalent 2D tensor models (obtained on the basis of the tensor product of components), which allows solving fuzzy equations and systems of fuzzy equations, where all variables and coefficients are fuzzy sets (1 or 2 types) almost one algorithm, the concept of which is given below.

Solution of fuzzy equations $\tilde{a}\tilde{x}\tilde{b} = \tilde{c}$, where $\tilde{a}, \tilde{x}, \tilde{b}, \tilde{c}$ - fuzzy variables, $\tilde{a} \overset{\Delta}{=} \left\{a \,/\, \mu^{(a)}\right\}$, $a \in A$, $\mu^{(b)} \to [0,1]$; $\mu^{(a)} \to [0,1]$; $\tilde{x} \overset{\Delta}{=} \left\{x \,/\, \mu^{(x)}\right\}$, $x \in X$, $\mu^{(x)} \to [0,1]$; $\tilde{b} \overset{\Delta}{=} \left\{b \,/\, \mu^{(b)}\right\}, b \in B$, $\tilde{c} \overset{\Delta}{=} \left\{c \,/\, \mu^{(c)}\right\}, c \in C, \mu^{(c)} \to [0,1]$ based on 2 main principles:

- conversion of FV into 2D tensor (matrix)

$$\tilde{a} \to \mathbf{T}^{(a)} = \tilde{a}(:,1) \otimes \tilde{a}(:,2), \tilde{x} \to \mathbf{T}^{(x)} = \tilde{x}(:,1) \otimes \tilde{x}(:,2), \tag{15}$$

$$\tilde{b} \to \mathbf{T}^{(b)} = \tilde{b}(:,1) \otimes \tilde{b}(:,2), \tilde{c} \to \mathbf{T}^{(\tilde{c})} = \tilde{c}(:,1) \otimes \tilde{c}(:,2); \tag{16}$$

- formation of a matrix equation $\underbrace{\mathbf{T}^{(a)}\mathbf{T}^{(x)}\mathbf{T}^{(b)}}_{\mathbf{AXB}} = \underset{\mathbf{C}}{\mathbf{T}^{(c)}}$, its solution based on the vectorization procedure (Kronecker algebra) described in [26].

If we limit the case when all FV have the same number of (n) - α-levels, then the solution has the form $\mathbf{X} \in \mathbb{R}^{n\times n}$, using the procedure of singular decomposition $[u\ s\ v] = svd(\mathbf{X})$ we can obtain SOP $\tilde{x} \overset{\Delta}{=} \left\{x / \mu^{(x)}\right\}, x \in X, \mu^{(x)} \to [0,1]$, which is a concrete solution of the fuzzy equation.

The following fuzzy equation $\tilde{a}\tilde{x} + \tilde{x}\tilde{b} = \tilde{c}$, where $\tilde{a}, \tilde{x}, \tilde{b}, \tilde{c}$ - fuzzy variables is solved similarly. By converting the FV into a tensor variable, we obtain a matrix equation

$$\underbrace{\mathbf{T}^{(a)}\mathbf{T}^{(x)} + \mathbf{T}^{(x)}\mathbf{T}^{(b)}}_{\mathbf{AX}+\mathbf{XB}} = \underbrace{\mathbf{T}^{(c)}}_{\mathbf{C}}, \tag{17}$$

To solve this problem, we apply the operator *vec* to the left and right sides of the above equation. Thus, the equation can be written in the form $(\mathbf{I}_m \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I}_n)$ vec $(\mathbf{X})$ =vec $(\mathbf{C}) \to$ vec $(\mathbf{X})$= vec $(\mathbf{C})\cdot(\mathbf{I}_m \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I}_n)^{-1}$. Next steps: conversion vec $(\mathbf{X})$ into a matrix **X ,** singular decomposition of the matrix X, and determination of a subset of ordered pair $\tilde{x} \overset{\Delta}{=} \left\{x / \mu^{(x)}\right\}, x \in X, \mu^{(x)} \to [0,1]$.

The computer experiment contained specific tasks (algorithms, programs, interpretation of results) that must be implemented to achieve the goal: realize mathematical support using MatLab to represent FS-1 and 2 types of tensor models based on the use of tensor products of these components FS in order to justify the need and feasibility of the proposed approaches:

In fig. 2 presents a general scheme of a computer experiment: initial FS $\to$2D tensor model $\to$3D tensor model $\to$ tensor analysis.

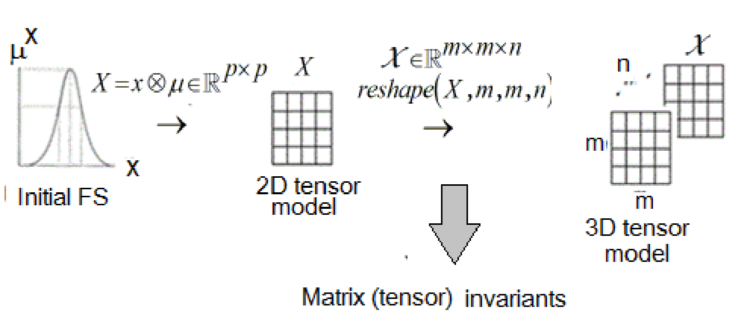


Figure 2: General scheme of computer simulation implementation main tasks.

1. Standard FS with triangular (or Gaussian) MF $\tilde{a}_{\text{trimf}} \in \mathbb{R}^{n\times 2}$, which is presented in matrix form, is transformed into a 2D tensor $\tilde{a}_{\text{trimf}} \underset{comp.\,FS}{\overset{}{\underset{TP}{\to}}} \mathbf{T}^{(\tilde{a}_{\text{trimf}})} \in \mathbb{R}^{n\times n}$, singular decomposition of which $svd\left(\mathbf{T}^{(\tilde{a}_{\text{trimf}})}\right)$ allows you to get SOP $^{(nev)}\tilde{a}_{\text{trimf}} \in \mathbb{R}^{n\times 2}$, whose properties: $\left\|^{(nev)}\tilde{a}_{\text{trimf}}\right\|_F \cong \left\|\tilde{a}_{\text{trimf}}\right\|_F$, def $\left(^{(nev)}\tilde{a}_{\text{trimf}}\right) \cong$ def $\left(\tilde{a}_{\text{trimf}}\right)$;

2. Transformation of a 2D tensor model $\mathbf{T}^{(\tilde{a}_{\text{trimf}})} \in \mathbb{R}^{n\times n}$ standard FS $\tilde{a}_{\text{trimf}}$ in the 3D tensor model: $\left(\mathbf{T}^{(\tilde{a}_{\text{trimf}})} \in \mathbb{R}^{n\times n}\right) \underset{reshape}{\to} \left(^{(new)}\mathbf{T}^{(\tilde{a}_{\text{trimf}})} \in \mathbb{R}^{p\times q\times m}\right), p\cdot q\cdot m = n\cdot n$, high-order

singular decomposition HOSVD$\left({}^{(new)}\mathbf{T}^{\left(\tilde{a}_{\text{trimf}}\right)}\right)$ allows you to get SOP $\tilde{b} \overset{\Delta}{=} \left(b, \left\langle \mu_1^{(b)}, \mu_2^{(b)} \right\rangle\right)$ $\in \mathbb{R}^{m\times 3}$, which in terms of the criteria of claim 1 is equivalent to SOP ${}^{(nev)}\tilde{a}_{\text{trimf}} \in \mathbb{R}^{n\times 2}$.

3. Given FS $\tilde{a}_{\text{trimf}} = \left\{a / \mu^{(a)}\right\}, a \in A, \mu^{(a)} \to [0,1]; \tilde{b}_{\text{trapmf}} = \left\{b / \mu^{(b)}\right\}, \quad b \in B,$ $\mu^{(b)} \to [0,1];$ defined results $\tilde{c} = \tilde{a} \circ_f \tilde{b} = \left\{c / \mu^{(c)}\right\}, c \in C, \mu^{(c)} \to [0,1]$, where $\circ_f \in \{+,-,*, \ ,/\}$, calculated tensor models FS $\tilde{a} \to T^{(a)}, \tilde{b} \to T^{(b)}$ accordingly, the calculated values $T^{(c)} = T^{(a)} \circ_f T^{(b)}$, which by means of singular decomposition are transformed into SOP:

$$T^{(c)} \to {}^{(new)}\tilde{c} \overset{\Delta}{=} \left\{{}^{(new)}c / {}^{(new)}\mu^{(c)}\right\}, {}^{(new)}c \in C, {}^{(new)}\mu^{(c)} \to [0,1].$$

Confirm equivalence $\left\| {}^{(new)}\tilde{c} \right\|_F \cong \|\tilde{c}\|_F$, def$\left({}^{(new)}\tilde{c}\right) \cong$ def$\left(\tilde{c}\right)$.

4. Modeling of 3D data processing using matrix algebra. Given a set of 3D unstructured data, the procedure reshape () allows you to structure a given set in the form of a 3D tensor - reshape (S, m, p, n) → {**A** (:,:, 1), **A** (:,:, 2) ,…, **A** (:,:, n)}, represented in the form of a set of frontal slices; in the following figure. this procedure is applied to a separate time series window.
According to Theorem 2.4.1 [27], blkdiag (A) is a block diagonal matrix, which is defined as follows:

$$\text{blkdiag}(\mathcal{A}) = \begin{pmatrix} \mathbf{A}^{(1)} & & & \\ & \mathbf{A}^{(2)} & & \\ & & \ddots & \\ & & & \mathbf{A}^{(n)} \end{pmatrix} \qquad (18)$$

where $\mathbf{A}^{(i)}$ – i-th frontal slice $\mathcal{A}$, i = 1, 2, ..., n_3.

Separately, we note that the proposed approach to the analysis of 3D data under uncertainty is applied to the analysis of 3D fuzzy time series

The next step is a singular decomposition of a block $\text{svd}\left(\text{blkdiag}(\mathcal{A})\right)$ diagonal matrix, which makes it possible to obtain a set of ordered pairs and process the resulting object by TFS methods. Note that the proposed procedure for converting 3D data to SOP can be used for fuzzy logic systems with 3D data.

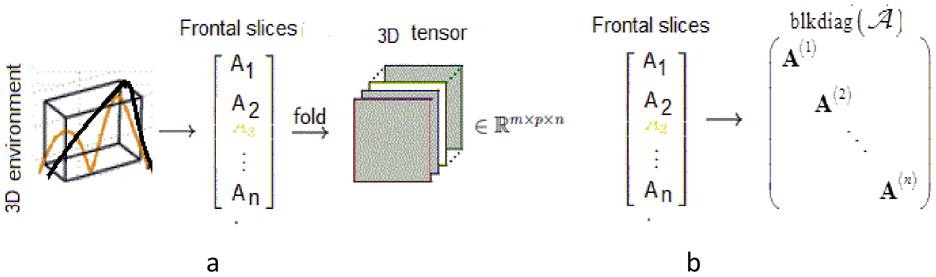


Figure 3: a-Example [28-29] representation of a separate window of a 3D time series in the form of a tensor model (a) - 3-time series, (b) - a tensor model of a window of the TS

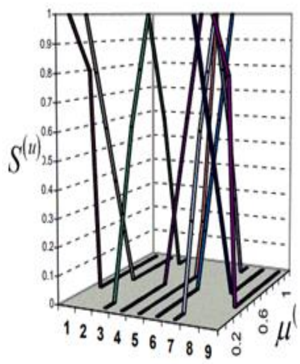
5. Modeling the solution of fuzzy equations and systems of fuzzy equations under conditions of uncertainty, if all parameters of the equation are type-1 FS or type-2 FS.

In [25, 29,32] an example of a real type-2 FS is given, the general form of which is presented in several formats, some of which are given below (Fig. 4):

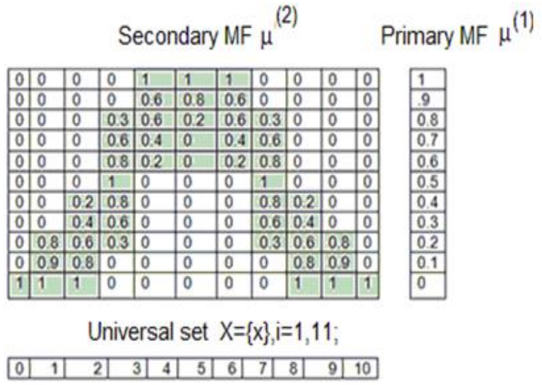
Consider this example in order to compare the proximity of defuzzyfied values and F-norms of type-1 FS and type-2 FS created from the initial FS by the fuzziness of MF. In addition, this example is important for 2 reasons: 1 - in [29] an example of type-2 FLS is given and it is shown that the yield of FLS-2 type has a defuzzyfied value $def(\tilde{y})$ for type-1 FS obtained as a reduction of the initial FS type obtained at the stage of the fuzzification.

According to the notation introduced in [30-33], this FS is also representative; 2 - tensor model type-2 FS allows you to calculate a subset of ordered pairs or a subset of ordered sequences (analogs), which allows you to have alternative solutions. For the purpose of transparency of calculations type-2 FS $\tilde{\tilde{A}}$ is transformed into a set of objects: matrices (secondary FN - $\mu^{(2)}$), vectors (primary FN - $\mu^{(1)}$ and US x respectively) are shown in Fig. 4 above.

Recall that the type-2 FS $\underline{\tilde{A}}, \underline{\tilde{B}}$ are considered as defined on U in the form $\underline{\tilde{A}} = \sum_{x \in X} \mu_{\tilde{A}}(x)/x$, $\underline{\tilde{B}} = \sum_{x \in X} \mu_{\tilde{B}}(x)/x$, where $\mu_{\tilde{A}}(x) = \sum_{u \in J_x^A} f_x(u)/u$, $\mu_{\tilde{B}}(x) = \sum_{w \in J_x^B} g_x(w)/w$.



a



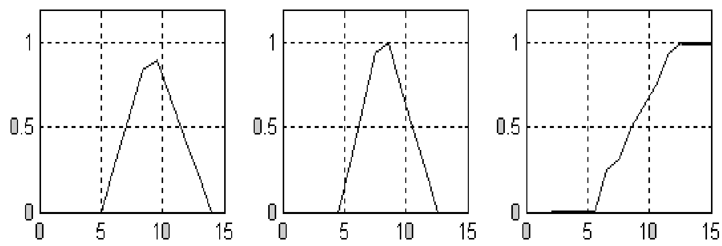
b

Figure 4: a - example type-2 FS is presented in the format Moderate [30],
b - 3D form of presentation type-2 FS in Moderate format [30]

In the [29] it is shown that FS $\underline{\tilde{B}} = \sum_{x \in X} \mu_{\tilde{B}}(x)/x$ can be represented as a subset of ordered pairs $\underline{\tilde{B}} = \begin{pmatrix} \mu_{\tilde{B}}(x) & x \end{pmatrix} \in \mathbb{R}^{n \times 2}$ and can in turn be converted to a 2D tensor $\mu_{\tilde{B}}(x) \otimes x \in \mathbb{R}^{n \times n}$, singular decomposition of which allows obtaining a new SOP (or SOS if necessary), endowed with the property of proximity to the original SOP.

5. Results

Below are FS with a triangular MF, which simulates the statement of **approximately 9.5,** and the nearest crisp set, their F-norms and defuzzyfied.



a b c

Figure 5.: a - the initial FS of approximately 9.5 with a triangular MF,
b, c - SOP, formed as a result of singular decomposition of the tensor model FS.

In fig.5. are shown the initial FS of **approximately 9.5** with a triangular MF: $\tilde{x} = \left\{ x \,/\, \mu^{(x)} \right\}$, $\mu^{(x)} \to [0,1]$ *X*=[5:9/8:14]; $\mu^{(x)} \overset{\Delta}{=}$ y=trimf(x, [5 9 14]); b, c - SOP, formed as a result of singular decomposition of the tensor model FS - $^{(x)}\mathbf{T} = \left(\tilde{x}(:,1) \otimes \tilde{x}(:,2) \right) \in \mathbb{R}^{9\times 9}$

FS 9.5 $_{trimf}$		crisp set	
5.00	0	5.00	0
6.13	0.28	6.13	0
7.25	0.56	7.25	1
8.38	0.84	8.38	1
9.50	0.90	9.50	1
10.63	0.68	10.63	1
11.75	0.55	11.75	1
12.88	0. 23	12.88	0
14.00	0	14.00	0

F-norm and defuzzyfied value

29.85	9.34	29.85	9.44

Tensor model FS 9.5 $_{trimf}$

0	1.41	2.81	4.22	4.50	3.38	2.25	1.13	0
0	1.72	3.45	5.17	5.51	4.13	2.76	1.38	0
0	2.04	4.08	6.12	6.53	4.89	3.26	1.63	0
0	2.36	4.71	7.07	7.54	5.65	3.77	1.88	0
0	2.67	5.34	8.02	8.55	6.41	4.28	2.14	0
0	2.99	5.98	8.96	9.56	7.17	4.78	2.39	0
0	3.30	6.61	9.91	10.58	7.93	5.29	2.64	0
0	3.62	7.24	10.86	11.59	8.69	5.79	2.90	0
0	3.94	7.88	11.81	12.60	9.45	6.30	3.15	0

Subset of ordered pairs

porp1 =		with sort porp =	
4.50	0	4.50	0
5.51	0.31	5.51	0
6.52	0.63	6.52	0.25
7.54	0.94	7.54	0.31
8.55	1.00	8.55	0.50
9.56	0.75	9.56	0.63
10.58	0.50	10.58	0.75
11.59	0.25	11.59	0.94
12.60	0	12.60	1.00

F-norm and defuzzyfied value

26.88 8.41 26.88 10.43

For comparison, we give similar parameters of the initial FS and SOP: F-norm: 29.85 ÷26.88; defuzzyfied value: 8.41 < (9.34÷ 9.44) < 10.43

Norm''s kron prod. of Subset of ordered pairs and Tensor model

$$\left\|\tilde{x}(:,1)\otimes\tilde{x}(:,1)\right\|_F = \left\|porp1(:,1)\otimes porp1(:,1)\right\|_F = \left\|porp(:,1)\otimes porp(:,1)\right\|_F = 48.30$$

This example confirms the main thesis - a subset of ordered pairs, the result obtained by the singular decomposition of the tensor model of the initial FS, adequately represents the initial FS.

Type-2 FS. Initial data - an object under uncertainty is represented as a data matrix

```
disp ('Initial data - object under uncertainty is represented as a data matrix')

        % Presented secondary FS – type-2 FS
mu2=
 [  0     0     0     0      1.00  1.00  1.00    0     0     0     0;
    0     0     0     0      0.20  0.60  0.80    0     0     0     0;
    0     0     0     0.30   0.60  0.20  0.60    0.30  0     0     0;
    0     0     0     0.60   0.40  0     0.40    0.60  0     0     0;
    0     0     0     0.80   0.20  0     0       1.00  0     0     0;
    0     0     0.20  0.80   0     0     0       0.80  0.20  0     0;
    0     0     0.40  0.60   0     0     0       0.60  0.30  0     0;
    0     .80   0.60  0.30   0     0     0       0.30  0.60  0.80  0;
    0     .90   0.80  0      0     0     0       0.80  0.90  0     0;
 1.00  1.00  1.00  0      0     0     0       0     1.00  1.00  1.00];

disp ('The specified object is approximated by a standard FS with a triangular MF')
disp ('Comparative evaluation - standard FS with triangular MF')

                                    Universal set
X = [0:10];                         Primary MF
mf = [0:0.1:1];                     standard triangular MF
y = trimf(X, [0 mean(X) 10]);       Standard FS with a triangular MF for comparison is presented
v = [X'  y'];                       as a matrix в ℝ^(11×2)

disp ('NORM and Defuzzyfied value of standard type-1 FS')
 [norm(v,'fro')          sum(v(:, 1).*v(:, 2))/sum(v(:, 2))]
     19.71                     5.00

n_kr_v = norm(kron(v(:,1),v(:,2)'),'fro')   -norm of the Kron product of the standard FS component
                                                                  36.18

disp('Implementation of the NORM calculation procedure and defuzzyfied value of type-2 FS')
used calculation formulas given in the work

for i=1:11
[b(:,i)]=mu2(:,i).*mf';
end
b;
bs=sum(b(:,1:11));
s=sum(mu2(:,1:11));
bss=bs./s;
vnew=[X'  (1-bss)'];
%**********************************************************************
```

disp('NORM and Defuzzyfied value of FS-2type(type reduction)')
[norm(vnew,'fro')sum(vnew(:,1).*vnew(:,2))/sum(vnew(:,2))]

Implementation of the NORM calculation procedure and defuzzyfied value of type-2 FS
F-NORM and Defuzzyfied value of FS-2 type (type reduction**)** **19.77** **5.02**

Universal set	Membership functions Type-2 FS trimf( )	Standard Type-1 FS
1	2	3
0	0.00	0.00
1.00	0.09	0.20
2.00	0.13	0.40
3.00	0.50	0.60
4.00	0.86	0.80
5.00	0.94	1.00
6.00	0.87	0.80
7.00	0.44	0.60
8.00	0.13	0.40
9.00	0.09	0.20
10.00	0.00	0.00

Note: 1. Universal set common to FS-1 and FS-2 type.
2.Col.2 –type-2 FS with triangular MF, blurring of MF forms type-1 FS.
3.Col.3 – type-1 FS, obtained as a result of the procedure reduced type:
type-2 FS $\rightarrow$ type-1 FS.

Note that the established criteria are -F-norm FS, presented in the form of a matrix with $\mathbb{R}^{n\times 2}$, and the defuzzyfied value for both cases of uncertainty representation practically coincide: (19.71, 5.00) and (19.77, 5.02), although the use of accuracy criteria in modeling uncertainty is a rather contradictory approach.

The tensor model FS-2 type in MatLab notation has the form:
fs1=[0:0.1:1];
z=kron(X, kron(fs1,mu2(:,1:11))); Tensor (Kroneker-product) model type-2 FS from $\mathbb{R}^{121\times 121}$
size(z) 11x11x121 – irrational form of representation

z1=reshape(z,121,121); Transformation of the initial KP model into a square matrix
n_kr_Ta 183.79

Formation of a subset of ordered pairs
[u s v]=svd(z1);
disp('1 variant -> Singular decomposition of the type-2 FS tensor model')
disp('Subset of ordered pairs -sort')
Tab_pup_x=sort([abs(u(:,1)*s(1,1))*max(abs(v(:,1))),abs(v(:,1))/max(abs(v(:,1)))]);
disp('F-norm and Defuzzyfied value of SOP')
[norm(Tab_pup_x,'fro') sum(Tab_pup_x(:,1).*Tab_pup_x(:,2))/sum(Tab_pup_x(:,2))]
singular decomposition of the type-2 FS tensor model
The subset of ordered pairs -sort
F-norm and Defuzzyfied value of SOP 47.89 5.75 (*)
NORM of kron product of SOP components
Comparison of norms 183.79 183.79
sparse SOP: NORM and Defazzifited value of sparse SOP 13.60 5.36 (**)
note that the case (*) concerns SOP from $\mathbb{R}^{121\times 2}$, the case (**)SOP from $\mathbb{R}^{11\times 2}$ (sparced set)

z3=reshape(z1,11,11,11,11); Transformation of the initial CD model into a tensor

CP4_ALSLS [32] CANDECOMP/PARAFAC decomposition of a fourth-order tensor(CP4).

[A1,A2,A3,A4]=cp4_alsls(X,R) computes a CANDECOMP/PARAFAC decomposition of a fourth-order tensor X in Rank-one terms, stored in the factor matrices A1, A2, A3, A4, belonging to the first, second, third and fourth mode, respectively.

[A1,A2,A3,A4] = cp4_alsls(z3,1)

CP (CANDECOMP/PARAFAC) decomposition (factorization) of the tensor $\mathrm{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ can be defined as $Y \cong \sum_{j=1}^{J} \lambda_j \circ \mathbf{a}_j^{(1)} \circ \mathbf{a}_j^{(2)} \circ \cdots \circ \mathbf{a}_j^{(N)} + \mathbf{E}$.

Recall that the matrix **Y** is a rank-1 matrix, if and only if **Y** = $\mathbf{uv}^{\mathrm{T}}$, where **u** and **v** are nonzero vectors.

Factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}_1^{(n)}, \mathbf{a}_2^{(n)}, \cdots, \mathbf{a}_j^{(n)}\right] \in \mathbb{R}^{I_n \times J}, n = 1, N$ contain latent components $\mathbf{a}_j^{(n)}$ as columns

The first columns of factor matrices

c=abs([A1 A2 A3 A4])=

0.17	0.66	0	0
0.09	1.44	0.17	0.17
0.31	1.44	0.34	0.34
0.50	0.97	0.51	0.51
0.61	0.22	0.68	0.68
0.73	0.09	0.85	0.85
0.77	0.24	1.02	1.02
0.76	1.36	1.19	1.19
1.38	1.47	1.36	1.36
1.52	1.00	1.53	1.53
2.09	0.66	1.70	1.70

Normalized factor matrices

c1=[c(:,1)*max(c(:,2))*max(c(:,3))*max(c(:,4)) c(:,2)/max(c(:,2)) c(:,3)/max(c(:,3)) c(:,4)/max(c(:,4))]

c1 =

0.72	0.45	0	0
0.38	0.98	0.10	0.10
1.31	0.98	0.20	0.20
2.09	0.66	0.30	0.30
2.59	0.15	0.40	0.40
3.09	0.06	0.50	0.50
3.24	0.16	0.60	0.60
3.20	0.92	0.70	0.70
5.83	1.00	0.80	0.80
6.44	0.68	0.90	0.90
8.84	0.45	1.00	1.00

F-norm of the result norm(c1,'fro') 14.50

Reduced type: matrix reduction $c1 \in \mathbb{R}^{11\times4}$ to SOP $c2 \in \mathbb{R}^{11\times2}$ (min(c(1:11,2:4))

c2=[c1(:,1) [0 0.1 0.2 0.3 0.15 0.06 0.16 0.7 0.8 0.68 0.45]']

c2 =[0.72 0; 0.38 0.10;1.31 0.20; 2.09 0.30;2.59 0.15; 3.09 0.06; 3.24 0.16; 3.20 0.70; 5.83 0.80; 6.44 0.68; 8.84 0.45]

Calculation of defuzzyfied value and F-norm of SOP

sum(c2(:,1).*c2(:,2))/sum(c2(:,2)) 4.90 14.50

The given object is approximated by standard FS with triangular FN

Comparative evaluation - standard FS with triangular FN

NORM and Defuzzyfied value of standard type-1 FS 19.71 5.00
Implementation of the procedure for calculating the NORM and Defuzzyfied value of type-2 FS
NORM and Defaulted value of type-2 FS (type reduction)
19.77 5.02
Sparsed SOP
NORM and Defuzzyfied value of Sparsed SOP 13.60 5.36

6. Conclusions

1. The theory of FS is now a practically universal apparatus, which is used in almost all cases where there may be uncertainty. However, the emergence of new problems requires continuous expansion of the standard theory of fuzzy sets, which is reproduced in the creation of new types of FS (rough FS, hesitate FS, multiFS, etc.), automation of FS formation processes, including MF, contradicts the ideology of TFS. However, the main object of TFS is the fuzzy set, which has not been studied to provide adequate answers to modern requirements, in particular, the urgent need to process BIG DATA.
2. One of the areas of possible research is tensor modeling of uncertainty, the basis of which is embedded in the nature of FS - a subset of ordered pairs. Objects that can represent tensors include vectors and scalars, as well as other tensors. Tensors can take several different forms, such as scalars and vectors (which are the simplest tensors), double vectors, multiline maps between vector spaces, and even some operations such as a point product. Tensors are defined independently of any basis, although their components are often called bases based on a particular coordinate system.
3. The representation of type-1 FS or type-2 FS as a tensor product of components is offered, the result is a 2D tensor (or 3D tensor in case of large dimension). This approach allows to use of the possibilities of tensor-matrix analysis to solve problems under uncertainty, along with the TFS apparatus, realizing the extraction of new knowledge (matrix-tensor invariants, matrix-tensor decompositions), which significantly expands the range of problems under uncertainty.
4. Based on the concept of extracting hidden knowledge, a method of automatically creating FS by structuring the initial data set with subsequent tensor decomposition is proposed, the obtained SOP is endowed with all the properties of MF. If it is impossible to implement the procedure of structuring IDS, it is proposed on the basis of calculating the values of creating the US vector in the form and blurring the latter by using special matrices (Toeplitz, Hankel, etc.), matrix (tensor) decomposition of which will create SOP - analog FS.
5. It is shown that tensor models of standard type-1 FS allow representing this object simultaneously as a 2D tensor, 3D tensor, type-2 FS with sparse US, and multiFS; in addition, the standard type-2 FS can be represented as type-1 FS, preserving the properties of the original object (F-norm, defuzzyfied value). This conclusion allows us to solve fuzzy equations in which the coefficients and the unknown are fuzzy variables of types 1 and/or 2, at the level of standard matrix equations, followed by the transformation of the matrix solution into the SOP.

7. References

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