$L(2,1)$-edge labeling of Infinite Triangular Grid*

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Abstract. An $L(h,k)$-edge labeling of a graph G is the assignment of labels $\{0,1,\cdots,n\}$ to the edges in such a way that two adjacent edges get labels with a difference at least h and the labels of distance two edges, i.e., two non-adjacent edges having a common edge connecting them get labels with a difference at least k, where h and k are two given non-negative integers. The span $\lambda'_{h,k}(G)$ is the minimum n such that G admits an $L(h,k)$-edge labeling. For three given integers n,h and k, an n-circular-$L(h,k)$-edge labeling $\mathbf{f}'$ of a graph G is an assignment from integers $\{0,1,\cdots,n-1\}$ in such a way that if two edges e and e' are adjacent then $|\mathbf{f}'(e)-\mathbf{f}'(e')|_n \geq h$ and if they are distance two edges then $|\mathbf{f}'(e)-\mathbf{f}'(e')|_n \geq k$, where $|x|_n = \min\{x, n-x\}$. The circular span $\sigma'_{h,k}(G)$ is the minimum n such that G admits an n-circular $L(h,k)$-edge labeling. Here, we focus on finding $\lambda'_{2,1}(G)$ and $\sigma'_{2,1}(G)$ where G is infinite regular triangular grid T_6. We work on the conjecture $\lambda'_{2,1}(T_6) = 16$ and the bound $\sigma'_{2,1}(T_6) \leq 18$ given by Lin and Wu [J. Comb. Opt., 2013]. We prove the conjecture and give a labeling function for T_6 such that $\sigma'_{2,1}(T_6) \leq 18$ as no labeling function was given by Lin and Wu in this case.

Keywords: Channel assignment problem · $L(2,1)$-labeling · infinite grids · lower bound · upper bound.

1 Introduction

In wireless communication, *Channel Assignment Problem* (CAP) is known as one of the fundamental and well-studied problems where the goal is to assign frequency channels to transmitters such that interference cannot occur. The target of this problem is to minimize the *span* of frequency spectrum, where the span is the difference between the lowest and highest frequencies used in the assignment. The formulation of CAP has been done in 1980 by Hale [1]. He formulated CAP as a vertex coloring problem. Then Roberts [2], in 1988, introduced the notion of $L(h,k)$-vertex labeling as defined below:

Definition 1. *For two non-negative integers h and k, an $L(h,k)$-vertex labeling of a graph $G(V,E)$ is a function $\mathbf{f}: V \to \{0,1,\cdots,n\}, \forall v \in V$ such that $|\mathbf{f}(u)-\mathbf{f}(v)| \geq h$ when $d(u,v)=1$ and $|\mathbf{f}(u)-\mathbf{f}(v)| \geq k$ when $d(u,v)=2$. Here, distance between vertices u and v, $d(u,v)$ is k' if at least k' edges are required to connect u and v.*

The *span* $\lambda_{h,k}(G)$ of $L(h,k)$-vertex labeling is the minimum n such that G admits an $L(h,k)$-vertex labeling. After some good years, Griggs and Yeh [9] extended the concept by introducing $L(k_1, k_2, \cdots, k_\ell)$-vertex labeling with separations $k_1, k_2, \cdots, k_\ell$ for $1, 2, \cdots, \ell$ distant vertices respectively and their main focus was on $L(h,k)$-vertex labeling for a special case $h = 2,\ k = 1$. Griggs and Jin [8] in 2007 studied $L(h,k)$-edge labeling, which can be formally defined as:

Definition 2. *For two non-negative integers h and k, an $L(h,k)$-edge labeling of a graph $G(V,E)$ is a function $\mathbf{f}' : E \to \{0, 1, \cdots, n\}, \forall e \in E$ such that $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq h$ when $d(e_1, e_2) = 1$ and $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq k$ when $d(e_1, e_2) = 2$. Here, for any two edges e_1 and e_2, the distance $d(e_1, e_2)$ is k' if at least $(k'-1)$ edges are required to connect e_1 and e_2.*

Like $L(h,k)$-vertex labeling, the *span* $\lambda'_{h,k}(G)$ of $L(h,k)$-edge labeling is the minimum n such that G admits an $L(h,k)$-edge labeling. In 2011, Calamoneri did a rigorous survey [7] on both vertex and edge labeling problems. A regular grid graph is a graph which is constructed by tessellation of regular two dimensional polygon in a two dimensional plane. Regular grids are a natural choice of modelling CAP for their symmetric geometric pattern and thus study of $L(h,k)$ labeling of regular grids have a relevance both in theory and in practice. Authors in [3–6] have studied $L(h,k)$-edge labeling of regular infinite hexagonal (T_3), square (T_4), triangular (T_6) and octagonal (T_8) grids for the special cases $h = 1,\ k = 2$ and $h = 2,\ k = 1$. They obtained some upper and lower bounds on $\lambda'_{1,2}(G)$ for T_3, T_4, T_6 and T_8 with a gap between them. Later on, Bandopadhyay et al. [10] improved some bounds on T_3, T_4 and T_6.

In the year 1998 Van den Heuvel [11] introduced another interesting variant of the $L(h,k)$-labeling called circular-$L(h,k)$-labeling. An n-circular-$L(h,k)$-labeling can be formally defined as follows:

Definition 3. *For three given non-negative integers n, h and k, n-circular-$L(h,k)$-labeling f of a graph G is an assignment from integers $\{0, 1, \cdots, n-1\}$ in a way that if two vertices u and v are adjacent, then $|\mathbf{f}(u) - \mathbf{f}(v)|_n \geq h$ and if they are distance two apart then $|\mathbf{f}(u) - \mathbf{f}(v)|_n \geq k$, where $|x|_n = \min\{x, n-x\}$. Moreover, the minimum n such that G admits an n-circular-$L(h,k)$-labeling is called the circular span $\sigma_{h,k}(G)$.*

Authors in [11] gave the circular span for T_3 and T_4 and also gave the following Lemma:

Lemma 1. *For a graph $G(V,E)$ and for two given non-negative integers h and k such that $h \geq k$, we can write, $\lambda_{h,k}(G) + 1 \leq \sigma_{h,k}(G) \leq \lambda_{h,k}(G) + h$.*

Naturally this problem as well as the Lemma have their corresponding an edge versions. As in this paper our main work in on edge labeling, we discuss those formally.

Definition 4. *For three given non-negative integers n, h and k, an n-circular-$L(h,k)$-edge labeling $\mathbf{f}'$ of graph G is an assignment from integers $\{0, 1, \cdots, n-1\}$ in a way that if two edges e and e' are adjacent, then $|\mathbf{f}'(e) - \mathbf{f}'(e')|_n \geq h$ and if they are distance two apart, then $|\mathbf{f}'(e) - \mathbf{f}'(e')|_n \geq k$, where $|x|_n = min\{x, n-x\}$. Moreover, the minimum n such that G admits an n-circular-$L(h,k)$-edge labeling is called the circular span $\sigma'_{h,k}(G)$.*

The edge version of the Lemma 1 is

Lemma 2. *For a graph $G(V, E)$ and for two given non-negative integers h and k such that $h \geq k$, we can write, $\lambda'_{h,k}(G) + 1 \leq \sigma'_{h,k}(G) \leq \lambda'_{h,k}(G) + h$.*

Lin and Wu [3] studied regular infinite hexagonal (T_3), square (T_4) and triangular (T_6) grids for the edge versions of $L(h, k)$-labeling and circular-$L(h, k)$-labeling for $h = 2$ and $k = 1$. They conjectured on the triangular grids as given below:

Conjecture 1. $\lambda'_{2,1}(T_6) = 16$.

They also gave the bound $\sigma'_{2,1}(T_6) \leq 18$.

1.1 Our Contributions

In this paper we first focus on the Conjecture 1. We show that $\lambda'_{2,1}(T_6) = 16$. We also give a labeling function for T_6 such that $\sigma'_{2,1}(T_6) \leq 18$ as no labeling function was given by Lin and Wu [3] in this case.

Table 1: The result regarding $\lambda'_{2,1}(T_6)$. *The proof of lower bound for $\lambda'_{2,1}(T_6)$ stated in [12] is incorrect. They proved $\lambda'_{2,1}(H'') \geq 16$ where H'' is the subgraph of T_6 as shown by the bold edges in Figure 1. But we have shown that $\lambda'_{2,1}(G) \leq 15$ as shown in Figure 1. Note that H'' is a subgraph of G.

	$\lambda'_{2,1}(G)$	
Grid	Known	Ours
T_6	15-16 [3], (16-16) [12]*	16-16

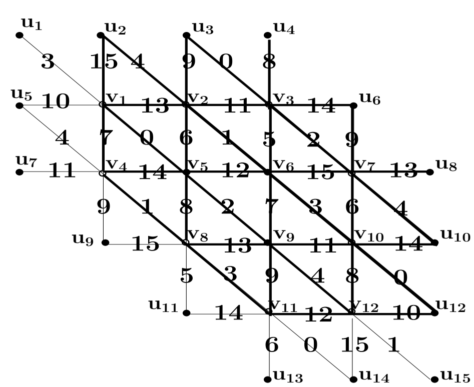

Fig. 1: Coloring of G with colors $\{0, 1, \cdots, 15\}$.

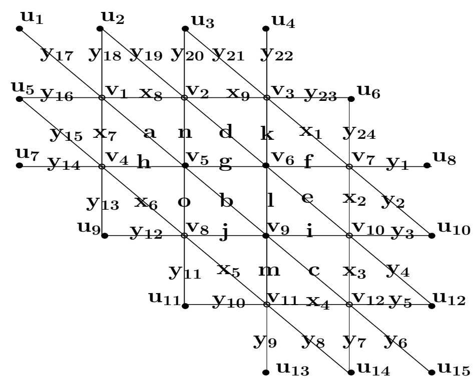

Fig. 2: Graph G

2 Results for $L(2,1)$-edge labeling

For a vertex v, let $N(v)$ denotes the set of neighbors of v and for a set of vertices S, let $N(S) = \underset{v \in S}{\cup} N(v)$. Consider the subgraph $G(V,E)$ of T_6 centering the triangle formed by $S = \{v_5, v_6, v_9\}$ as shown in Figure 2, where $V = S \cup N(S) \cup N(N(S))$ and E is the set of all edges which are incident to u where $u \in S \cup N(S)$. Now, we define the set of edges in G into five subsets of edges as following:

$S_1 = \{g, b, l\}$
$S_2 = \{d, e, n, o, i, j\}$
$S_3 = \{k, f, a, h, m, c\}$
$S_4 = \{x_1, x_2, x_3, \cdots, x_9\}$
$S_5 = \{y_1, y_2, y_3, \cdots, y_{24}\}$

Our proof approach is as follows. In first few observations and lemma (Observation 1, Observations 2, Observations 3, Observation 4 and Lemma 3) we first investigate if a color c' is used in S_1 or in S_2 or in S_3, then how many times at maximum it can be reused in G and repetition pattern of the colors such that edges of G can be colored with $15(0, 1, \cdots, 14)$ colors. In Observation 5, we determine the maximum number of times a color c' can be used in G if it is not used in $S1 \cup S2 \cup S3$. In subsequent observation and lemmas(Observation 6, Lemma 4, Lemma 5) we discuss the scenario when a color $c' \notin \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$ and two consecutive colors c' and $c' + 1$ are used in different types of edges. Based on the discussion and results of the mentioned observations and lemmas, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when a color $c_u \notin \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$ in Theorem 1, Theorem 2 and Theorem 3. In Theorem 4, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when a color $c_u \in \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$. In Theorem 5, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when all colors in $\{0, 1, \cdots, 14\}$ are used in $S_1 \cup S_2 \cup S_3$. In all the cases, derived lower bounds of $\lambda'_{2,1}(T_6)$ are identical.

Observation 1 *Let c' be any color used at an edge in S_1, then c' can be used at most once more in G.*

Proof. Without loss of generality, assume that c' is used in the edge g. Now, it is clear that c' can only be used in some edges incident to v_{11} and v_{12}. Since, edges incident to v_{11} and v_{12} are not mutually three distance apart, c' can only be used at most in one edge.

Observation 2 *Let c' be any color used in an edge in S_2, then c' can be used in at most two more times in G.*

Proof. As all the edges in S_2 are in symmetric position, without loss of generality we can assume that c' is used in the edge d. One can verify that some edges incident to v_4, v_8, v_{11}, v_{12} only are distant three from d. Note that v_4 and v_8 are adjacent to each other. Similarly v_{11} and v_{12} are also mutually adjacent. Hence c' can be used two times, once in an edge incident to v_4, v_8 and the other in an edge incident to v_{11}, v_{12}. But if c' is used in x_5, then c' cannot be used once more.

Observation 3 *Let c' be any color used at an edge in S_3, then c' can be used in at most three more edges in G.*

Proof. Observe that here also all the edges are symmetric, so, without loss of generality we can assume that c' is used at the edge k. Some edges adjacent to the vertices $v_1, v_4, v_8, v_{11}, v_{12}$ only are three distance apart from k. It is clear that c' can be used at the edges adjacent to alternate vertices in the sequence $v_1, v_4, v_8, v_{11}, v_{12}$. Moreover, it can be observed that each edge in S_4 where c' can be used is adjacent to two vertices in $v_1, v_4, v_8, v_{11}, v_{12}$. So, if we want to use c' three more times in G, then c' must be used at edges adjacent to v_1, v_8, v_{12}, and the color must be used at edges in S_5 only.

From Observations 1, 2 and 3 it is clear that colors used to color the edges in S_1, S_2 and S_3 can be used at most two, three and four times in G, respectively. Note that G has 48 edges in total. We need all distinct colors to color the edges in $S_1 \cup S_2 \cup S_3$ as they are mutually at most two distance apart. So, we need 3, 6 and 6 distinct colors to color the edges in S_1, S_2 and S_3, respectively. If we can repeat all these color with their maximum potential, then it is possible to color the graph G using 15 colors. Here we state Observation 4 and give the unique repetition pattern of the colors, to color G using 15 colors in Lemma 3. Let H be the subgraph of G induced by the edges in $S_1 \cup S_2 \cup S_3$. We say two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ as a pair of *opposite edges* iff $d(e_1, e_2) = 3$, $d(u_1, u_2) = 2$, $d(u_1, v_2) = 2$, $d(v_1, u_2) = 2$ and $d(v_1, v_2) = 2$.

Observation 4 *If a color c' is used in an edge e_1 in T_6 and is not used at the opposite edge of e_1, then there exists a subgraph H' isomorphic to H where c' cannot be used.*

Proof. Without loss of generality let us consider the pair of opposite edges n and x_2 (Figure 2). Let $\mathbf{f}'(n) = c'$ and $\mathbf{f}'(x_2) \neq c'$. Consider the two triangles $\{v_3, v_6, v_7\}$ and $\{v_6, v_9, v_{10}\}$. Clearly, c' cannot be used in any edge incident to v_3, v_6 and v_9. Therefore c' can be assigned to either an edge incident to v_7 or an edge incident to v_{10} but not both. So, c' cannot be used either in the subgraph H' isomorphic to H with $S_1' = \{v_3, v_6, v_7\}$ or in the subgraph H'' isomorphic to H with $S_1'' = \{v_6, v_9, v_{10}\}$. Hence the proof.

Lemma 3. *If G is colored with 15 colors only, then there is the following unique repetition of the colors used in subgraph H: (a) each color used at the edges in S_3 must be repeated three times more in the edges of S_5, (b) each color used at the edges in S_2 must be repeated two times more, once in its opposite edge in S_4 and the other in an edge in S_5, (c) each color used at the edges in S_1 must be repeated once more in its opposite edge in S_4.*

Proof. Observe that the first condition for G to be colored with 15 colors is the colors used in S_1, S_2 and S_3 have to be used two times, three times and four times in G respectively. From the proof of Observation 3 it directly follows that if we want to reuse three times a color c_3 that has been used in an edge of S_3, then c_3 has to be reused at the edges of S_5 only. Now assume that c_2 be a color used in an edge of S_2 then c_2 cannot be reused at two edges in S_4 as there does not exists two mutually three distant edges in S_4 such that c_2 can be used. From Observation 4 it follows that c_2 should be reused at the corresponding opposite edge in S_4 and another edge in S_5. Observe that only three edges in S_4 remain uncolored now. Again from Observation 4 we can say that if c_1 be a color used in an edge of S_1, then c_1 has to be reused at the corresponding opposite edge in S_4 only.

We aim to prove $\lambda'_{2,1}(T_6) = 16$. We already know that $\lambda'_{2,1}(H) = 14$ and $\lambda'_{2,1}(T_6) \geq 15$ [3]. So, without loss of generality, we can assume that one color is *unused* at H. In Observations $1, 2, 3$ we showed the re-usability of the colors used at S_1, S_2 and S_3, respectively. Now we focus on the color which is not used in H.

Observation 5 *Let c' be any color not used in H, then c' can be used at most four edges in G.*

Proof. Clearly, c' is used at edges in $S = S_4 \cup S_5$. All edges in S are incident to one or two vertices in $Q = \{v_1, v_2, v_3, v_7, v_{10}, v_{12}, v_{11}, v_8, v_4\}$. So, c' can be used at edges adjacent to alternative vertices in the sequence Q. There are nine vertices in Q. If we pick every alternate vertices in the sequence starting from v_1, then we end up with a set Q_1 of five vertices $\{v_1, v_3, v_{10}, v_{11}, v_4\}$. Since, v_1 and v_4 are adjacent in G, we can give the color c' to edges incident to either v_1 or v_4. So, we have only four vertices whose adjacent edges can be colored with c', and note that that no two edge adjacent to same vertices can be colored with same color. Hence there are at most four edges which can be colored with the unused color. Similarly, if we pick alternate vertices in the sequence starting from v_2, then we end up with picking a set Q_2 of four vertices $\{v_2, v_7, v_{12}, v_8\}$. Again, we similarly argue that at most in four edges we can use the unused color in G.

From Figure 2 it is clear that the subgraph H consists of five vertical edges, five horizontal edges and five slanting edges. Let us denote these three types of edges as T_v, T_h and T_s, respectively. Now, we prove some properties of colors used in these three types of edges.

Observation 6 *Let $\mathbf{f}'(p) = c'$ and $\mathbf{f}'(q) = c' + 1$ where $p, q \in E(H)$ and they are different types of edges. Then there exists a H' isomorphic to H in T_6 where either c' or $c' + 1$ cannot be used.*

Proof. Let us consider $p = f, q = a$ (Figure 2). Note that $\mathbf{f'}(f) = c'$ cannot be used at any edge incident to v_2 as $d(v_2, v_6) = 1$. Again, c' cannot be used at any edge incident to v_1 or v_5 as $\mathbf{f'}(a) = c' + 1$. Hence c' cannot be used in the subgraph H' isomorphic to H centering the triangle $S'_1 = \{v_1, v_2, v_5\}$. In general, for any (p, q) pair in H, such a triangle can be found by considering the two triangles with common edge p and the two triangles with common edge q. Hence the proof.

Lemma 4. *In the subgraph H, let the unused color $c' \notin \{5, 10\}$. Then, there must exist two disjoint pair of different typed edges (p, q) and (r, s) such that consecutive colors have been used in each pair of edges and $p \in T_h, q \in T_s$; $r \in T_s, s \in T_v$.*

Proof. Since, distance between any two edges in H is at most two, no color can be repeated. We need three sets of five different colors to color each type of edges in H. That means we need to divide the set of colors into three sets of equal size (i.e., five colors in each set) in such a way that we can maximize the number of sets that contain consecutive colors. We can say that the unused color divides the color set into two parts. By pigeon hole principle, we can show that either one part contains more than 10 consecutive colors or both of them contains more than five consecutive colors as the unused color c' is neither 5 nor 10. In the former case, all three types of edges get color from the larger part of the colors as it contains more than 10 colors. In the later case, at least one type of edges get color from both the parts. So, in both the cases we get at least two such pairs.

Lemma 5. *For every (p, q) pair of edges in H where p and q are different types of edges and consecutive colors are assigned to p and q, at least 2 edges of $E(G) \setminus E(H)$ cannot be colored with the colors used in H.*

Proof. We prove this Lemma using case analysis and the cases are based on where the edges p and q are present. Without loss of generality, we assume $p \in T_h$ and $q \in T_s$ are colored with z and $z + 1$ respectively. We can have the following cases:

- $p, q \in S_3$: Without loss of generality, let $p = f$ and $q = c$. From Lemma 3 it follows that if z is to be reused for 3 more times, then it has to be reused at edges of S_5 incident to v_1, v_8 and v_{12}, which is not possible since c is adjacent to v_{12}. Similarly, if $z + 1$ is to be used for 3 more times, then it has to be reused at edges of S_5 incident to v_7, v_2 and v_4, which is also not possible since f is adjacent to v_7. So, there are two edges in $E(G) \setminus E(H)$ which remain uncolored with the colors used in H.
- $p \in S_3, q \in S_2$: Without loss of generality, let $p = h$ and $q = e$. From Lemma 3, if z is to be used for 3 more times, then it has to be reused at edges of S_5 incident to v_3, v_{10} and v_{11}. But it is not possible since e is adjacent to v_{10}. Similarly, if $z + 1$ is to be used for 2 more times, then it has to be reused at $x_5 = (v_8, v_{11})$ of S_4 and an edge of S_5 incident to v_1. It is not possible to use $z + 1$ at x_5 as z is reused at an edge of S_5 adjacent to v_{11}. So, there are two edges in $E(G) \setminus E(H)$ which cannot be colored with the colors used in H.
- $p, q \in S_2$: Without loss of generality, let $p = j$ and $q = d$. From Lemma 3, z is to be reused at $x_8 = (v_1, v_2)$ which is not possible here as edge d is adjacent to

(v_1, v_2). Similarly, $z+1$ is to be reused at $x_6 = (v_4, v_8)$ which is also not possible as edge j is adjacent to (v_4, v_8). Therefore, two edges at $E(G) \setminus E(H)$ cannot be colored with the colors used in H.

- $p \in S_1, q \in S_3$: Without loss of generality, let $p = g$ and $q = c$. From Lemma 3 it follows that z has to be reused at x_4. But here it is not possible as $z+1$ used at c. If x_4 is not left uncolored, then a color of H in $\{h, a, n, d, k, f\}$ can only be used at that edge. If $\mathbf{f}'(h)$ is used at x_4, then $\mathbf{f}'(h)$ can only be used at an edge of S_5 incident to v_3 but not at the edges of S_5 incident to v_{10} and v_{11}. So, in that case two edges will remain uncolored. If $\mathbf{f}'(n)$ is used at x_4, then $\mathbf{f}'(n)$ can neither be used at x_2 nor be used at an edge of S_5 incident to v_{11}. So, in that case too, two edges will remain uncolored. All other possibilities are symmetric to one of these two cases.
  We now consider the case when the unused color, say u of H is used at x_4. Since u cannot be used at v_8 and v_{10}, colors of $\{d, k, f, e\}$ must be reused at the edges $\{x_6, y_{12}, y_{11}, x_5\}$ and the colors of $\{n, a, h, o\}$ must be reused at the edges $\{x_2, y_3, y_4, x_3\}$. If $\mathbf{f}'(i)$ and $\mathbf{f}'(j)$ are reused at x_9 and x_8 respectively, then the edges where colors $\mathbf{f}'(i) \pm 1$ and $\mathbf{f}'(j) \pm 1$ can be used are $\{f, g\}$ and $\{g, h\}$ respectively. Therefore, if both $\mathbf{f}'(i)$ and $\mathbf{f}'(j)$ are reused at x_9 and x_8 respectively, then $\mathbf{f}'(g) \pm 1 = z \pm 1$ can only be used at $\{i, j\}$. Since $z+1$ is used at c, either $\mathbf{f}'(i)$ cannot be reused at x_9 or $\mathbf{f}'(j)$ cannot be reused at x_8. Hence the unused color u of H has to be used at x_9 or x_8. Hence two edges at $E(G) \setminus E(H)$ cannot be colored with the colors used in H.

Now we will look at the difference of colors in the edges incident to same vertex. Initially we investigate the case when the difference is at least three for every pair. Then we consider the case when there exists at least a pair of edges with difference exactly two. We classify the six edges incident to the same vertex as follows. We say two such edges are at $60°$ if one is the immediate next edge of the other in clockwise or anticlockwise direction. They are said to be at $120°$ and $180°$ if exactly one and two edges respectively is/are there in between them. Now we subdivide the second case into three more cases depending on the angle between them.

Lemma 6. *If* $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq 3$ *for every pair of consecutive edges* $e_1, e_2 \in E(T_6)$*, then* $\lambda'_{2,1}(T_6) \geq 16$.

Proof. Let us consider the edge $g = (v_5, v_6)$ and without loss of generality assume $\mathbf{f}'(g) = 0$. To keep $\lambda'_{2,1}(T_6)$ below 16, the colors that can be used at the remaining five incident edges of v_5 are 3, 6, 9, 12 and 15. Now the least color that can be used at any of the five edges incident to v_6 is 4. Therefore, the colors that can be used to the remaining four edges incident to v_6 are 7, 10, 13 and 16 respectively. Hence $\lambda'_{2,1}(T_6) \geq 16$.

Therefore there exists at least two adjacent edges in T_6 having color c_1 and c_2 with $|c_1 - c_2| = 2$.

Theorem 1. *If two colors* c' *and* $c'+2$ *have been assigned in any two adjacent edges at an angle* $60°$ *in* T_6*, then* $\lambda'_{2,1}(T_6) \geq 16$.

Proof. Without loss of generality, assume $\mathbf{f}'(b) = c'$ and $\mathbf{f}'(g) = c' + 2$. Observe that $c'+1$ must remain unused in H as $\forall e_1 \in E(H)\setminus\{b, g\}$ either $d(e_1, b) = 1$ or $d(e_1, g) = 1$. From Lemma 4 and Lemma 5, there are 4 edges in $E(G) \setminus E(H)$ which cannot be colored with colors used in H. To make $\lambda'_{2,1}(G)$ below 16, $c' + 1$ is to be used in those 4 edges. Without loss of generality, assume $c' + 1$ has been used at the edges incident to v_1, v_3, v_8 and v_{12} respectively. Note that $\mathbf{f}'(g) = c' + 2$ cannot be used at x_4 as $c' + 1$ is used at an edge incident to v_{12}. So, $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n), \mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$. Consider the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)\}$ as the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$ can be proved similarly. Assume $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. There are four edges of $S_4 \cup S_5$ incident to v_{10} where $\mathbf{f}'(a)$, $\mathbf{f}'(h)$, $\mathbf{f}'(n)$ and $\mathbf{f}'(o)$ can only be used. But $\mathbf{f}'(a)$ cannot be used there as $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. To make $\lambda'_{2,1}(G)$ below 16, $\mathbf{f}'(x_3) = c' + 1$ and $\mathbf{f}'(n)$, $\mathbf{f}'(h)$ and $\mathbf{f}'(o)$ are assigned to the other three edges. Similarly, there are four edges of $S_4 \cup S_5$ incident to v_2 where $\mathbf{f}'(i)$, $\mathbf{f}'(c)$, $\mathbf{f}'(m)$ and $\mathbf{f}'(j)$ can only be used. Now observe that $\mathbf{f}'(b) = c'$ cannot be used at x_1 as $c' + 1$ is used at an edge incident to v_3. Again, $\{\mathbf{f}'(n), \mathbf{f}'(h), \mathbf{f}'(o)\}$ and $\{\mathbf{f}'(i), \mathbf{f}'(c), \mathbf{f}'(m), \mathbf{f}'(j)\}$ cannot be used at x_1 as they are used at edges incident to v_{10} and v_2 respectively. Hence $\mathbf{f}'(x_1) = \mathbf{f}'(a)$. Proceeding similarly we can show that $\mathbf{f}'(x_9) = \mathbf{f}'(i)$, $\mathbf{f}'(x_8) = \mathbf{f}'(j)$, $\mathbf{f}'(x_7) = \mathbf{f}'(l)$ and $\mathbf{f}'(x_5) = \mathbf{f}'(e)$. Now observe that $\mathbf{f}'(a)$ and $\mathbf{f}'(x)$ are used at two adjacent vertices for all $x \in E(H) \setminus \{d\}$. Therefore one of $\mathbf{f}'(a) \pm 1$ cannot be used in H. Moreover, none of $\mathbf{f}'(a) \pm 1$ can be the unused color $c' + 1$ as c' and $c' + 2$ are used at b and g respectively. Hence $\lambda'_{2,1}(G) \geq 16$.

Theorem 2. *If two colors c' and $c' + 2$ have been assigned in any two adjacent edges at an angle $120°$ in T_6, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof. Without loss of generality, assume $\mathbf{f}'(g) = c'$ and $\mathbf{f}'(e) = c' + 2$ (Figure 2). There may be two cases, when $c' + 1$ is used in H and when $c' + 1$ is not used in H. First consider the second case. From Lemma 4, there are 4 edges in $E(G) \setminus E(H)$ which cannot be colored with the colors used in H. To make $\lambda'_{2,1}(G)$ below 16, $c' + 1$ is to be used in those 4 edges. Without loss of generality, assume $c' + 1$ is used in edges in $E(G) \setminus E(H)$ adjacent to vertices v_1, v_3, v_8 and v_{12} respectively. Note that $\mathbf{f}'(x_4) \neq \mathbf{f}'(g) = c'$ as $c' + 1$ is used in an edge adjacent to v_{12}. Therefore $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n), \mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$. Let $\mathbf{f}'(x_4) = \mathbf{f}'(d)$. The color of four edges in $S_4 \cup S_5$ incident to v_8 are $c' + 1$ and any three of $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$. But $\mathbf{f}'(e)$ cannot be used there as $\mathbf{f}'(e) = c' + 2$; $\mathbf{f}'(d)$ cannot be used there as $\mathbf{f}'(x_4) = \mathbf{f}'(d)$. Hence a new color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar result holds when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(f), \mathbf{f}'(k)\}$. Now let us consider the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)\}$. Let us consider $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. Here the colors of four edges in $S_4 \cup S_5$ incident to v_{12} are $c'+1, \mathbf{f}'(h), \mathbf{f}'(n)$ and $\mathbf{f}'(o)$. Therefore $\mathbf{f}'(d), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ must be used at three edges incident to v_{12}. As, $\mathbf{f}'(e) = c' + 2$ cannot be used at any edge incident to v_8 due to usage of $c'+1$ at v_8, any one of $\mathbf{f}'(d), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ must be used in x_5. But it is not possible as $\mathbf{f}'(d), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ are used in edges adjacent to v_{12}. Hence another color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar argument holds when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(h), \mathbf{f}'(n)\}$. Hence the proof for this case.

Now we consider the case when $\mathbf{f}'(g) = c'$, $\mathbf{f}'(e) = c' + 2$ and $c' + 1$ is used in H. Let us consider a color $c'' \in C = \{0, 1, \cdots, 15\} \setminus \{c', c' + 1, c' + 2\}$ is unused in H. Without loss of generality, say $c'' = c' + 5$. For any other color c'' in C we can prove the same result with similar arguments. From Lemma 4, $c' + 5$ must be used in 4 edges of $S_4 \cup S_5$ in $E(G) \setminus E(H)$. Here we assume $c' + 5$ is used in 4 edges adjacent to the vertices v_1, v_3, v_8 and v_{12} respectively. Since $f(g) = c'$, $c' \pm 1$ can only be used in $\{c, i, j, m\}$. Let us consider $\mathbf{f}'(j) = c' - 1$. The three edges h, f and i in T_h are to be colored. Here we assume exactly two distinct pair of different types of edges (p, q) and (r, s) are assigned consecutive colors such that $p \in T_h, q \in T_s$ and $r \in T_h, s \in T_v$. In that case $c' - 2$, $c' - 3$ and $c' - 4$ must be assigned to edges in T_h in H. Let us consider $\mathbf{f}'(j) = c' - 1$. Now if $\mathbf{f}'(m) = c' + 1$, then two edges j and m at $60°$ have colors $c' - 1$ and $c' + 1$ and from theorem 1, $\lambda'_{2,1}(T_6) \geq 16$. Again $\mathbf{f}'(i) \neq c' + 1$ as $\mathbf{f}'(e) = c' + 2$. Therefore $\mathbf{f}'(c) = c' + 1$. If $\mathbf{f}'(b) = c' + 3$, then neither $c' + 4$ nor $c' + 6$ can be used at the edge a as $c' + 5$ is used at an edge incident to v_1. Hence $\mathbf{f}'(a) = c' + 3$. Similarly we can show that $\mathbf{f}'(d) = c' + 4$ and $\mathbf{f}'(b) = c' + 6$. As $c' + 5$ is used at an edge in $S_4 \cup S_5$ incident to v_{12}, at least any three among $\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)$ and $\mathbf{f}'(o)$ must be assigned to the edges of $S_4 \cup S_5$ incident to v_{10}. Note that $\mathbf{f}'(a) = c' + 3$ and $\mathbf{f}'(e) = c' + 2$. Therefore $\mathbf{f}'(a)$ cannot be assigned to an edge incident to v_{10}. In that case $\mathbf{f}'(h)$ and $\mathbf{f}'(i)$ cannot be consecutive. So we get $\mathbf{f}'(h) = c' - 2$, $\mathbf{f}'(f) = c' - 3$ and $\mathbf{f}'(i) = c' - 4$. With similar argument we can show that $\mathbf{f}'(o) = c' - 5, \mathbf{f}'(m) = c' - 6, \mathbf{f}'(k) = c' - 7, \mathbf{f}'(n) = c' - 8$ and $\mathbf{f}'(l) = c' - 9$. Now consider the edges of $S_4 \cup S_5$ incident to v_8. Only $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ can be used at edges of $S_4 \cup S_5$ incident to v_8. Note that $\mathbf{f}'(d) = c' + 4$ cannot be used there because $c' + 5$ is used at an edge of S_5 incident to v_8. Therefore $\mathbf{f}'(k) = c' - 7, \mathbf{f}'(f) = c' - 3$ and $\mathbf{f}'(e) = c' + 2$ must be used at edges in $S_4 \cup S_5$ incident to v_8. Now $\mathbf{f}'(x_5) \neq \mathbf{f}'(k) = c' - 7$ as $\mathbf{f}'(m) = c' - 6$. If $\mathbf{f}'(x_5) = \mathbf{f}'(f) = c' - 3$, then two edges j and x_5 at $60°$ have colors $c' - 1$ and $c' - 3$ and from theorem 1, $\lambda'_{2,1}(T_6) \geq 16$. Hence $\mathbf{f}'(x_5) = \mathbf{f}'(e) = c' + 2$. Since either $\mathbf{f}'(y_{11}) = c' + 5$ or $\mathbf{f}'(y_{12}) = c' + 5$ we get either $\mathbf{f}'(x_6) = c' - 3$ or $\mathbf{f}'(x_6) = c' - 7$. In both cases two edges o and x_6 residing at an angle $60°$ have colors $(c' - 5, c' - 3)$ or $(c' - 5, c' - 7)$. Hence from theorem 1 $\lambda'_{2,1}(T_6) \geq 16$. When more than two distinct pair of different types of edges are assigned consecutive colors, arguing similarly, we can prove the same result. Hence the proof.

Theorem 3. *If two colors c' and $c' + 2$ have been assigned in any two adjacent edges at an angle $180°$ in T_6, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof. Without loss of generality, assume $\mathbf{f}'(g) = c'$ and $\mathbf{f}'(h) = c' + 2$ (Figure 2). There may be two cases, when $c' + 1$ is used in H and when $c' + 1$ is not used in H. First assume $c' + 1$ is used in H. Let us consider a color $c'' \in C = \{0, 1, \cdots, 15\} \setminus \{c', c' + 1, c' + 2\}$ is unused in H and it is used at the edges of $S_4 \cup S_5$ in $E(G) \setminus E(H)$ adjacent to v_1, v_3, v_8 and v_{12}. Without loss of generality, say $c'' = c' + 7$. For any other color c'' in C we can prove the same result with similar arguments. Both the colors $c' \pm 1$ must be used at edges in H incident to v_9. To make $\lambda'_{2,1}(G)$ less than 16, the two colors must be used at two edges at $180°$ incident to v_9 otherwise from theorem 1 or theorem 2, $\lambda'_{2,1}(T_6) \geq 16$. So, we assume $\mathbf{f}'(j) = c' + 1$ and $\mathbf{f}'(i) = c' - 1$. Similarly, $\mathbf{f}'(f) = c' - 2$. The color $c' - 3$ can be used at an edge adjacent to v_5. Let us consider $\mathbf{f}'(n) = c' - 3$. Therefore, $\mathbf{f}'(m) = c' - 4, \mathbf{f}'(k) = c' - 5, \mathbf{f}'(o) = c' - 6$ and

$\mathbf{f}'(l) = c' - 7$. Similarly, it can be shown that $\mathbf{f}'(d) = c' + 3, \mathbf{f}'(c) = c' + 4, \mathbf{f}'(a) = c' + 5, \mathbf{f}'(e) = c' + 6$ and $\mathbf{f}'(b) = c' + 8$. Now observe that $\mathbf{f}'(c)$, $\mathbf{f}'(m)$, $\mathbf{f}'(i) = c' - 1$ and $\mathbf{f}'(j) = c' + 1$ must be used at the edges at $S_4 \cup S_5$ incident to v_2 as the unused color c'' is not used here. This implies $\mathbf{f}'(j) = c' + 1$ and $\mathbf{f}'(i) = c' - 1$ must be used at two edges in $S_4 \cup S_5$ incident to v_2 which are at $180°$, as otherwise from theorem 1 or theorem 2, $\lambda'_{2,1}(T_6) \geq 16$. Hence $c' + 1$ and $c' - 1$ must be used at x_8 and x_9. Now notice that $\mathbf{f}'(d) = c' + 3$ and the edge d is at $60°$ and $120°$ with x_9 and x_8 respectively. Hence at v_2, $(c' + 1, c' + 3)$ must be used at two edges either at $60°$ or at $120°$ resulting $\lambda'_{2,1}(T_6) \geq 16$ from theorem 1 or theorem 2.

Now consider the case when $\mathbf{f}'(g) = c', \mathbf{f}'(h) = c' + 2$ and $c' + 1$ is not used in H. Assume $c' + 1$ is used at four edges incident to v_1, v_3, v_8 and v_{12}. Note that $c' - 1$ must be used at an edge incident to v_9 in H. Here we assume exactly two distinct pair of different types of edges (p, q) and (r, s) are assigned consecutive colors such that $p \in T_h, q \in T_s$ and $r \in T_h, s \in T_v$. Consider $\mathbf{f}'(j) = c' - 1$. So, $\mathbf{f}'(f) = c' - 2$ otherwise c' and $c' - 2$ must be at two adjacent edges at an angle $60°$ or $120°$. Note that either $\mathbf{f}'(i) = c' - 3$ or $\mathbf{f}'(i) = c' + 3$. First we assume $\mathbf{f}'(i) = c' + 3$. In that case $c' + 4$ can only be used at an edge incident to v_6 in H, as otherwise $c' + 2$ and $c' + 4$ will be at two edges at an angle $60°$ or $120°$. So, $\mathbf{f}'(d) = c' + 4$. Therefore, $\mathbf{f}'(a) = c' + 5, \mathbf{f}'(c) = c' + 6, \mathbf{f}'(e) = c' + 7$ and $\mathbf{f}'(b) = c' + 8$. Similarly, $\mathbf{f}'(n) = c' - 3, \mathbf{f}'(m) = c' - 4, \mathbf{f}'(k) = c' - 5, \mathbf{f}'(o) = c' - 6$ and $\mathbf{f}'(l) = c' - 7$. Note that any three among $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ must be used at $S_4 \cup S_5$ incident to v_8 as $c' + 1$ is used here. But $\mathbf{f}'(f) = c' - 2$ and $\mathbf{f}'(k) = c' - 5$ cannot be used there as $\mathbf{f}'(j) = c' - 1$ and $\mathbf{f}'(o) = c' - 6$. Hence one more color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar argument holds when $\mathbf{f}'(i) = c' - 3$. Hence the proof.

Till now we have considered the case when in H, the unused color $c_u \notin \{5, 10\}$. Now we consider the case when $c_u \in \{5, 10\}$ and the case when there is no unused color in H.

Theorem 4. *If a color $c_u \in \{5, 10\}$ is unused at H, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof. Without loss of generality let us assume color 5 is unused in H. The set of colors $\{0, 1, \cdots, 4\}$ must be used in same type of edges and same holds for the sets $\{6, 7, \cdots, 10\}$ and $\{11, 12, \cdots, 15\}$ otherwise there exists at least two disjoint pair of edges (p, q) and (r, s) where consecutive colors are used in each pair and there we can prove $\lambda'_{2,1}(T_6) \geq 16$ using similar argument as depicted in Lemma 6 and Theorem 1 or 2 or 3. Let us consider $p = f, q = a, \mathbf{f}'(p) = c'$ and $\mathbf{f}'(q) = c' + 1$. From Observation 6, c' cannot be used in the subgraph H' isomorphic to H centering $S'_1 = \{v_1, v_2, v_5\}$. If $c' \neq 10$, then from Theorem 1 or 2 or 3 $\lambda'_{2,1}(T_6) \geq 16$. If $c' = 10$ then $\mathbf{f}'(q) = c' + 1 = 11$. In that case the colors $\{0, 1, \cdots, 4\}$ must be used in T_v, $\{6, 7, \cdots, 10\}$ must be used in T_h and $\{11, 12, \cdots, 15\}$ must be used in T_s. From Observation 4, the edges for reusing $\mathbf{f}'(f) = c' = 10$ are y_5, y_{12}, y_{16} and (u_3, u_4). If $\mathbf{f}'(f) = c' = 10$ is used at (u_3, u_4), then $\mathbf{f}'(a) = c' + 1 = 11$ cannot be used at its opposite edge y_{21} and hence from Observation 4, Lemma 6 and Theorem 1 or 2 or 3 $\lambda'_{2,1}(T_6) \geq 16$. If $\mathbf{f}'(f) = c' = 10$ is not used at (u_3, u_4), then either color 5 or a color $c'' \neq \{5, 10\}$ used in H must be used there. If a color c'' is used there, then there exists a pair of opposite edges where c'' cannot be used and again from Observation 4, Lemma 6 and Theorem 1 or 2 or 3 $\lambda'_{2,1}(T_6) \geq 16$. So, to keep $\lambda'_{2,1}(T_6)$ below 16, color 5 must be used at (u_3, u_4). With exactly same argument we can show that color 5 must also be used at y_5, y_{12}, y_{16} otherwise $\lambda'_{2,1}(T_6) \geq 16$. Remember that the color 4 is used in T_v and it must also be used at its opposite edges. Note that for any edges $e_1 \in T_v \setminus \{m\}$, either e_1 or its opposite edge is adjacent to an edge e_2 where $f(e_2) = 5$. Hence $\mathbf{f}'(m) = 4$. But y_{13}, the opposite edge of m, cannot have color 4 as color 5 is used at y_{12}. Hence from Observation 4, Lemma 6 and Theorem 1 or 2 or 3 $\lambda'_{2,1}(T_6) \geq 16$.

Theorem 5. *If all colors $c_1 \in \{0, \cdots, 14\}$ are used at H, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof. In this case there exists a pair of different types of edges in H where $(c', c'+1)$ are used. From Observation 6, there exists a H' isomorphic to H where c' or $c'+1$ cannot be used. Hence either from Theorem 4 or from Lemma 6 and Theorem 1 or 2 or 3, it follows that $\lambda'_{2,1}(T_6) \geq 16$.

3 Result for circular-$L(2,1)$-edge labeling

Lemma 7. $\sigma'_{2,1}(T_6) \leq 18$

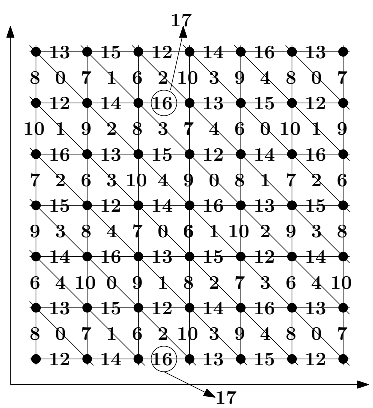

Fig. 3: Circular $L(2,1)$-edge labeling of T_6

Proof. In this grid there are three types of edges- horizontal, vertical and slanted. In order to proof the Lemma, we now consider the labeling shown in Figure 3. Assuming left bottom corner point as origin, the labeling functions corresponding to horizontal, vertical and slanted edges can be stated as:

$$\mathbf{f}'((x,y),(x+1,y)) = (7x+y) \; mod \; 5 + 12$$

$$\mathbf{f}'((x,y),(x,y+1)) = (3y - x + 2) \; mod \; 5 + 6$$

$$\mathbf{f}'((x-1,y+1),(x,y)) = (x + 4y - 1) \; mod \; 5$$

. Observe that different types of edges get different colors in the coloring. Horizontal edges get color $12, 13, 14, 15, 16$; vertical edges get $6, 7, 8, 9, 10$ and slanted edges get $0, 1, 2, 3, 4$. By the pattern it is clear that two adjacent edges of same type do not get two consecutive colors. By the definition of n-circular-$L(2,1)$-edge labeling 0 and n are two consecutive colors. In our coloring there are many places where two adjacent edges get 0 and 16. So, the color 16 cannot be the circular span of the grid. That's why we introduce a new color 17, which is used at edges as a replacement for 16 such that 0 and 16 become two non-consecutive colors. Two such 16 colored edges are shown in Figure 3 whose colors can be replaced by 17. Putting 17 at any one such edge is sufficient. The main goal behind introducing a new color 17 was to make 0 and 16 non-consecutive. As the colors 5 and 11 are unused in the graph, we can conclude that no two adjacent edges of different types get two consecutive colors. Now we have to show that no two edges at distance two get the same color. For the same type of edges it can be easily followed from the pattern of repetition. In case of different type of edges, observe that distant two edges of two different types get color with difference at least two. Hence this labeling can be extended to infinite grid.

4 Conclusion

Here we prove the conjecture $\lambda'_{2,1}(T_6) = 16$ given by Lin and Wu [3]. We prove that $\lambda'_{2,1}(T_6) \geq 16$ and as $\lambda'_{2,1}(T_6) \leq 16$ [3], it immediately follows that $\lambda'_{2,1}(T_6) = 16$. We also show that $\sigma'_{2,1}(T_6) \leq 18$ by giving a labeling function. Determining the value of $\sigma'_{2,1}(T_6)$ is an open problem and can be done as a future work.

References

1. W. K. Hale, Frequency assignment: Theory and applications, Proceedings of the IEEE 68 (12) (1980) 1497-1514. doi:10.1109/PROC.1980.11899.
2. F. Roberts, Working group agenda, In: DIMACS/DIMATIA/Renyi Working Group on Graph Colorings and their Generalizations (2003).
3. W. Lin, J. Wu, Distance two edge labelings of lattices, J. Comb. Optim. 25 (4) (2013) 661-679. doi:10.1007/ s10878-012-9508-5.
4. Q. Chen, W. Lin, $L(j,k)$-labelings and $L(j,k)$-edge-labelings of graphs, Ars Comb. 106 (2012) 161-172.
5. D. He, W. Lin, L(1, 2)-edge-labelings for lattices, Applied Mathematics-A Journal of Chinese Universities 29 (2014) 230-240. doi:10.1007/s11766-014-3176-4.
6. T. Calamoneri, Optimal $L(j,k)$-edge-labeling of regular grids, Int. J. Found. Comput. Sci. 26 (4) (2015) 523-535. doi:10.1142/S012905411550029X.
7. T. Calamoneri, The $L(h,k)$-labelling problem: An updated survey and annotated bibliography, Comput. J. 54 (8) (2011) 1344-1371. doi:10.1093/comjnl/bxr037.
8. J. R. Griggs, X. T. Jin, Real number labelings for paths and cycles, Internet Mathematics 4 (1) (2007) 65-86. doi:10.1080/15427951.2007.10129140.
9. J. R. Griggs, R. K. Yeh, Labelling graphs with a condition at distance 2, SIAM J. Discrete Math. 5 (4) (1992) 586-595. doi:10.1137/0405048.
10. S. Bandopadhyay, S. C. Ghosh, S. Koley, Improved bounds on the span of $L(1,2)$-edge labeling of some infinite regular grids, in: 18th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW 2020), Springer, accepted, September 14-16, 2020.

11. J. van den Heuvel, R. A. Leese, M. A. Shepherd, Graph labeling and radio channel assignment, Journal of Graph Theory 29 (4)(1998) 263-283.
12. D. D, J. V. Kureethara, On L(2,1)-edge coloring number of regular grids, An. St Univ. Ovidius Constanta 27 (3) (2019) 65-81. doi:10.2478/auom-2019-0034.