Fuzzy orthopartitions and their logical entropy

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Abstract

In this paper, we present special families of intuitionistic fuzzy sets called *fuzzy orthopartitions*, which are generalizations of standard fuzzy partitions useful to model situations where both vagueness and uncertainty are involved. Moreover, we define and explain how to compute the *lower and upper entropy*, which measure the quantity of information contained in a fuzzy orthopartition.

Keywords

Entropy, Ruspini Partitions, Intuitionistic Fuzzy Sets, Orthopartitions

1. Introduction

The first purpose of the present work is to introduce the so-called *fuzzy orthopartitions*, which are generalized (fuzzy) partitions involving both vagueness and uncertainty. A fuzzy orthopartition is mathematically defined as a collection of intuitionistic fuzzy sets satisfying specific properties.

An intuitionistic fuzzy set A of a universe U is a pair (μ_A, ν_A), where μ_A and ν_A are functions from U to $[0, 1]$ such that $\mu_A(u) + \nu_A(u) \leq 1$ for each $u \in U$, and are respectively called the *membership* and *non-membership functions* of A. Moreover, the *hesitation margin* of an element u of U, given by $h_A(u) = 1 - (\mu_A(u) + \nu_A(u))$, expresses the degree of indeterminacy of u to A [1, 2].

Intuitionistic fuzzy sets can be seen as extensions of both orthopairs and fuzzy sets. More precisely, an orthopair (M, N) of U, i.e. a pair of disjoint subsets of U [3], is viewed as an intuitionistic fuzzy set (μ, ν) such that $\mu(u) = 1$ if $u \in M$, and $\mu(u) = 0$ otherwise, while $\nu(u) = 1$ if $u \in N$, and $\nu(u) = 0$ otherwise. Furthermore, each fuzzy set π on U, i.e. $\pi : U \to [0, 1]$, is identified with an intuitionistic fuzzy set $(\pi, 1 - \pi)$ such that $(1 - \pi)(u) = 1 - \pi(u)$ for each $u \in U$.

Analogously, fuzzy orthopartitions are considered generalizations of both classical *orthopartitions* and *Ruspini (fuzzy) partitions*.

Orthopartitions are generalized partitions where the membership class of some elements of the initial universe is not precisely known [4]. Mathematically, an orthopartition of U is a family $\mathcal{O} = \{(M_1, N_1), \ldots, (M_n, N_n)\}$ of orthopairs of U satisfying the following properties: they are mutually disjoint, i.e. $M_i \cap M_j = \emptyset$ and $M_i \cap B_j = \emptyset$ for each $i \neq j$, where $B_j = U \setminus (M_j \cup N_j)$; the union of $P_1 \cup \ldots \cup P_n$ and $B_1 \cup \ldots B_n$ covers U; for each $u \in U$, if $u \in B_i$ then there exits

$j \neq i$ such that $u \in B_j$; the number of orthopairs of $\mathcal{O}$ is less than or equal to the cardinality of U [1].

Ruspini partitions are generalized partitions where elements of the initial universe belong to the blocks with a membership degree of [0,1] [5]. Mathematically, a Ruspini partition of U is a family $\pi = \{\pi_1, \ldots, \pi_n\}$ of fuzzy sets on U such that $\pi_1(u) + \ldots + \pi_n(u) = 1$ for each $u \in U$.

The second purpose of this work is to present the concepts of *logical entropy of a Ruspini partition*, and *lower and upper entropy of a fuzzy orthopartition*. All of them are extensions of the logical entropy of a classical partition introduced in [6].

The logical entropy measures the quantity of information associated with a partition of a finite universe, and its definition is based on the notion of *distinction* (or *dit*) considered the atom of information. Given a partition π of U, a distinction is a pair of elements of U that belong to different blocks of π. Therefore, the logical entropy of a partition is mathematically defined as the number of distinctions normalized by the number of ordered pairs of U.

We point out that our approach is different from that shown in [7]. More precisely, our measure of entropy in the intuitionistic fuzzy case is based on the generalized notion of distinction, while the authors in [7] provide the logical entropy definition of special collections of intuitionistic fuzzy sets representing the results of a random experiment and so, characterized by a probability distribution.

Finally, we notice that we decided to generalize the logical entropy instead of the Shannon one, since already in the crisp orthopartition case, calculating the Shannon entropy is computationally intractable [4].

In the sequel, we will provide all concepts and notions by assuming that our universe is finite.

2. Fuzzy orthopartitions

This section principally presents the notion of fuzzy orthopartitions. A fuzzy orthopartition is a generalized partition, where an element u of the initial universe belongs to a block represented by the intuitionistic fuzzy set (μ_i, ν_i) with a degree of the interval $[\mu_i(u), \mu_i(u) + h_i(u)]$ that is not precisely known.

Definition 1. *Let $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ be a finite family of intuitionistic fuzzy sets of U. Then, $\mathcal{O}$ is a* fuzzy orthopartition *of U if and only if the followings hold for each $u \in U$:*

(a) $\sum_{i=1}^{n} \mu_i(u) \leq 1$,
(b) $\mu_i(u) + h_j(u) \leq 1$, *for each $i \neq j$,*
(c) $\sum_{i=1}^{n} \mu_i(u) + h_i(u) \geq 1$,
(d) *for each $i \in \{1, \ldots, n\}$ with $h_i(u) > 0$, there exists $j \in \{1, \ldots, n\} \setminus \{i\}$ such that $h_j(u) > 0$.*

Axioms (a) and (b) capture the idea that intuitionistic fuzzy sets of $\mathcal{O}$ must represent mutually disjoints blocks of U, and Axiom (c) corresponds to the covering condition.

The following theorem shows that Axiom (d) allows fuzzy orthopartitions to be extensions of crisp orthopartitions.

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[1] In this paper, we deal with more general orthopartitions, where the last property can be unsatisfied.

Theorem 1. *Let $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ be a fuzzy orthopartition of U such that $\mu_i(u)$, $\nu_i(u) \in \{0, 1\}$, for each $u \in U$ and $i \in \{1, \ldots, n\}$. Then, $\{(M_1, N_1), \ldots, (M_n, N_n)\}$ is an orthopartition of U, where*

$$M_i = \{u \in U \mid \mu_i(u) = 1\} \quad and \quad N_i = \{u \in U \mid \nu_i(u) = 1\},$$

for each $i \in \{1, \ldots, n\}$.

Proof. Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. If $u \in M_i$, then u cannot belong neither to M_j from Axiom (a) or to $U \setminus (M_j \cup N_j)$ from Axiom (b). Therefore, (M_i, N_i) and (M_j, N_j) are disjoint.

Suppose that there exists $u \in U$ such that $u \in N_i$ for each $i \in \{1, \ldots, n\}$. Then, since $(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)$ are intuitionistic fuzzy sets, we get $\sum_{i=1}^{n} \mu_i(u) + h_i(u) = 0$. But, the latter contradicts Axiom (c). Consequently, we can conclude $\bigcup_{i=1}^{n}(M_i \cup B_i) = U$.

Lastly, given $u \in U$ and $i \in \{1, \ldots, n\}$, $h_i(u) > 0$ if and only if $h_i(u) = 1$ (i.e. $u \in B_i$), from hypothesis. Thus, Axiom (d) is equivalent to the third property of classical orthopartitions. $\square$

Additionally, Ruspini partitions can be viewed as special fuzzy orthopartitions. More precisely, the following theorem holds.

Theorem 2. *Let $\pi_1, \ldots, \pi_n$ be some fuzzy sets on U. Then, $\{\pi_1, \ldots, \pi_n\}$ is a Ruspini partition if and only if $\{(\pi_1, 1 - \pi_1), \ldots, (\pi_n, 1 - \pi_n)\}$ is a fuzzy orthopartition.*

Proof. The thesis clearly follows from Definition 1. $\square$

By the previous theorem, each Ruspini partition is identified with a fuzzy orthopartition. Vice-versa, a fuzzy orthopartition corresponds to a collection of Ruspini partitions. Their definition is formulated by thinking that fuzzy orthopartitions approximate Ruspini partitions when the truth degree of elements of the initial universe is not exactly known.

Definition 2. *A Ruspini partition $\pi = \{\pi_1, \ldots, \pi_n\}$ of U is compatible with a fuzzy orthopartition $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ of U if and only if $\mu_i(u) \leq \pi_i(u) \leq \mu_i(u) + h_i(u)$ for each $u \in U$ and $i \in \{1, \ldots, n\}$.*

Moreover, we indicate with $\Pi_{\mathcal{O}}$ the set of all Ruspini partitions compatible with $\mathcal{O}$.

Example 1. *Let $U = \{u\}$. We consider an orthopartition $\mathcal{O} = \{(\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\}$ of U, where $\mu_1(u) = 0.2, \nu_1(u) = 0.6, \mu_2(u) = 0.4, \nu_2(u) = 0.3, \mu_3(u) = 0.2$, and $\nu_3(u) = 0.3$. Then, $\pi = \{\pi_1, \pi_2, \pi_3\}$ such that $\pi_1(u) = 0.2, \pi_2(u) = 0.4$ and $\pi_3(u) = 0.4$ is an example of Ruspini partition compatible with $\mathcal{O}$.*

3. Lower and upper entropy of fuzzy orthopartitions

This section aims to generalize the logical entropy of standard partitions to Ruspini partitions and fuzzy orthopartitions.

3.1. Logical entropy of Ruspini partitions

In this subsection, we provide the notion of logical entropy of a Ruspini partition.

Given a Ruspini partition π of U, since the total membership degree of each element of U is distributed among all blocks of π, a pair of elements (u, u') of $U \times U$ forms a distinction of π with a certain degree in $[0, 1]$. Therefore, we can generalize the concept of distinction to the case of Ruspini partitions by defining a fuzzy relation that assigns to each pair $(u, u') \in U \times U$ the maximum of the distances between u and u' w.r.t. all blocks of π.

Definition 3. *Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be a Ruspini partition of U. Then, we consider a fuzzy relation $dit\pi : U \times U \to [0, 1]$ such that let $u, u' \in U$,*

$$dit\pi(u, u') = \max\{|\pi_i(u) - \pi_i(u')| \text{ such that } i \in \{1, \ldots, n\}\}. \tag{1}$$

We call $dit\pi(u, u')$ the degree of distinction *of (u, u'), and it is interpreted as the capacity of π to distinguish u and u' by means of its fuzzy sets $\pi_1, \ldots, \pi_n$.*

A Ruspini partition $\pi = \{\pi_1, \ldots, \pi_n\}$ of U such that $\pi_i(u) \in \{0, 1\}$ for each $u \in U$ and $i \in \{1, \ldots, n\}$, corresponds to a crisp partition. Thus, it is easy to prove that a pair of elements is a distinction in the classical sense, i.e. it is made by elements of different blocks, if and only if the degree of distinction given by (1) is equal to 1.

Proposition 1. *Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be Ruspini partition of U such that $\pi_i(u) \in \{0, 1\}$ for each $u \in U$ and $i \in \{1, \ldots, n\}$. Then, $dit\pi(u, u') = 1$ if and only if there exist $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $\pi_i(u) = 1$ and $\pi_j(u') = 1$.*

Proof. The thesis clearly follows from Definition 3. $\qquad\square$

We now define the logical entropy of a Ruspini partition as the normalized sum of the degrees of distinctions of all pairs of the initial universe.

Definition 4. *Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be a Ruspini partition of U. Then, the logical entropy of π is given by*

$$h(\pi) = \frac{\sum_{(u,u') \in U \times U} dit\pi(u, u')}{|U \times U|}. \tag{2}$$

Finally, Proposition 1 implies $h(\pi)$, given by (2), coincides with the standard logical entropy when $\pi = \{\pi_1, \ldots, \pi_n\}$ and $\pi_i(u) \in \{0, 1\}$, for each $u \in U$ and $i \in \{1, \ldots, n\}$.

3.2. Lower and upper entropy of fuzzy orthopartitions

In this subsection, we define and explain how to compute the lower and upper entropy of a fuzzy orthopartition.

Definition 5. *Let $\mathcal{O}$ be a fuzzy orthopartition of U, the* lower *and* upper entropy *are respectively given by*

$$h_*(\mathcal{O}) = \min\{h(\pi) \mid \pi \in \Pi_{\mathcal{O}}\} \ \text{ and } \ h^*(\mathcal{O}) = \max\{h(\pi) \mid \pi \in \Pi_{\mathcal{O}}\}.^{2}$$

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[2]Let us recall that $\Pi_{\mathcal{O}}$ is the set of all Ruspini partitions compatible with $\mathcal{O}$ (see Definition 2).

By Definition 5, $h_*(\mathcal{O}) \leq h^*(\mathcal{O})$ for each fuzzy orthopartition $\mathcal{O}$.

Moreover, it is easy to verify that if $\pi = \{\pi_1, \ldots, \pi_n\}$ is a Ruspini partition, then $h(\pi)$ given by (2) equals both the lower and upper entropy of $\{(\pi_1, 1 - \pi_1), \ldots, (\pi_n, 1 - \pi_n)\}$.

We now intend to show how to compute the lower and upper entropy of a fuzzy orthopartition $\mathcal{O} = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ by using Definition 5. To achieve this goal, we need to find the Ruspini partitions π_* and π^* of $\Pi_{\mathcal{O}}$ such that $h_*(\mathcal{O}) = h(\pi_*)$ and $h^*(\mathcal{O}) = h(\pi^*)$.

By Definitions 3 and 4, π_* and π^* are respectively the minimum and the maximum points of the function $f : \Pi_{\mathcal{O}} \to R^+$ such that

$$f(\pi) = \sum_{(u,u') \in U \times U} \max\{ |\pi_i(u) - \pi_i(u')| \text{ such that } i \in \{1, \ldots, n\} \},$$

for each $\pi = \{\pi_1, \ldots, \pi_n\} \in \Pi_{\mathcal{O}}$.

Let us notice that π^* and π_* exist because $\Pi_{\mathcal{O}}$ is always non-empty from Definition 2.

Then, the problem to find π_* and π^* can be transformed in a *constrained optimization problem* as follows.

Let $\{\pi_1, \ldots, \pi_n\} \in \Pi_{\mathcal{O}}$, and let $U = \{u_1, \ldots, u_m\}$, we put $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$. Then, $\mu_i(u_j) \leq \pi_i(u_j)$ for each $(i, j) \in I \times J$ from Definition 2, and so, we can write

$$\pi_i(u_j) = \mu_i(u_j) + x_{ij}. \tag{3}$$

where $x_{ij} \geq 0$. Consequently, since every $\mu_i(u_j)$ is known, each Ruspini partition π of $\Pi_{\mathcal{O}}$ can be identified with a matrix $(x_{ij}) \in \mathbb{R}^{n \times m}$ having the elements determined by (3).

Hence, our aim becomes to find the maximum and the minimum points of a new function $g : \mathbb{R}^{n \times m} \to \mathbb{R}^+$ such that let $m \in \mathbb{R}^{n \times m}$,

$$g(m) = \sum_{(h,k) \in J \times J} \max\{|(\mu_i(u_h) + x_{ih}) - (\mu_i(u_k) + x_{ik})| \text{ such that } i \in I\}$$

subject to the constraints

$$\begin{cases} \sum_{i=1}^{n} (\mu_i(u_j) + x_{ij}) = 1 \text{ for each } j \in J, \\ 0 \leq x_{ij} \leq h_i(u_j) \text{ for each } (i, j) \in I \times J. \end{cases} \tag{4}$$

We can observe constraint 1 assures us that π, corresponding to m through (3), is a Ruspini partition, and constraint 2 is necessary so that π is also compatible with $\mathcal{O}$.

Finally, this problem can be converted into a linear programming problem by introducing $\binom{n}{2}$ additional variables. More precisely, let $\mathcal{X}_2 = \{X \subseteq U \text{ such that } |X| = 2\}$, we consider a new variable y_X for each $X \in \mathcal{X}_2$, and we aim to minimize or maximize a new function $\sum_{X \in \mathcal{X}_2} y_X$ subject to the previous constraints, plus the followings:

$$\begin{cases} y_X \geq (\mu_i(u_h) + x_{ih}) - (\mu_i(u_k) + x_{ik}), \text{ for each } X \in \mathcal{X}_2, i \in I \text{ and } (h, k) \in J \times J, \\ y_X \geq (\mu_i(u_k) + x_{ik}) - (\mu_i(u_h) + x_{ih}), \text{ for each } X \in \mathcal{X}_2, i \in I \text{ and } (h, k) \in J \times J. \end{cases}$$

$$\tag{5}$$

Thus, we can compute the optimal solutions, i.e. the matrices of $\mathbb{R}^{n \times m}$ representing π^* and π_*, by using one of the standard techniques in linear programming like the Simplex method [8].

Once obtained π^* and π_*, we can compute $h(\pi^*)$ and $h(\pi_*)$ by using (3), which correspond to the upper and lower entropy of $\mathcal{O}$, respectively.

Example 2. *Let $U = \{u, v, z\}$, consider a fuzzy orthopartition $\mathcal{O} = \{(\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\}$ according to the definition in Table 1.*

	μ_1	ν_1	μ_2	ν_2	μ_3	ν_3
u	0.3	0.2	0.4	0.3	0	0.7
v	0.2	0.4	0.3	0.2	0.3	0.3
z	0	0.5	0.3	0.4	0.6	0.2

Table 1
Definition of the elements of the orthopartition.

We suppose that $(\mu_1, \nu_1), (\mu_2, \nu_2)$, and (μ_3, ν_3) express the interest in three topics T_1, T_2, and T_3 of three different groups u, v, and z of users of a social network. For instance, by Table 1, the users of u are interested in the topic T_2 with a degree from $\mu_2(u) = 0.4$ to $1 - \nu_2(u) = 0.7$. Therefore, in this case, the upper and lower entropy of $\mathcal{O}$ measure how much the interests of u, v, and z (valued w.r.t. $\{T_1, T_2, T_3\}$) diverge from each other.

Then, in order to find the upper and lower entropy of $\mathcal{O}$, we have to maximize and minimize the function $y_{\{u,v\}} + y_{\{u,z\}} + y_{\{v,z\}}$ subject to the constraints given by equations (4) and (5). The maximum and the minimum points of this function are respectively the matrices m^* and m_* of $\mathbb{R}^{3 \times 3}$, where

$$m^* = \begin{bmatrix} 0 & 0 & 0.3 \\ 0.1 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix} \quad and \quad m_* = \begin{bmatrix} 0 & 0.3 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

By (3), their corresponding Ruspini partitions are $\pi^* = \{\pi_1^*, \pi_2^*, \pi_3^*\}$ and $\pi_* = \{\pi_{*1}, \pi_{*2}, \pi_{*3}\}$ as defined in Table 2.

	π_{*1}	π_{*2}	π_{*3}	π_1^*	π_2^*	π_3^*
u	0.3	0.7	0	0.3	0.4	0.3
v	0.2	0.5	0.3	0.3	0.4	0.3
z	0	0.3	0.7	0.1	0.3	0.6

Table 2
Definition of the obtained Ruspini partitions.

Lastly, $h(\pi^*) = 0.31$ and $h(\pi_*) = 0.13$ from Definition 4, which are the upper and lower entropy of $\mathcal{O}$, respectively.

4. Future directions

In the future, we intend to extend the contents of this paper as follows. Firstly, we want to verify that $h^*(\mathcal{O})$ and $h_*(\mathcal{O})$ given by Definition 5 respectively coincide with the upper and

lower entropy defined in [4], when $\mathcal{O}$ is a crisp orthopartition. Then, in order to provide a new measure of entropy, given a fuzzy orthopartition $\mathcal{O}$, we will find a Ruspini partition $\tilde{\pi}$ of $\Pi_{\mathcal{O}}$ such that $h(\tilde{\pi})$ is the arithmetic mean of $h^*(\mathcal{O})$ and $h_*(\mathcal{O})$. On the long term, we plan to apply our results to the machine learning field (similarly as it has been done with orthopartitions [4, 9]) in order to define new algorithms with the ability to handle both uncertainty and vagueness.

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