On Reduction of Cycloids

Rüdiger Valk[1], Daniel Moldt[2]

[1]University of Hamburg, Department of Informatics, Hamburg, Germany
[2]University of Hamburg, Department of Informatics, Hamburg, Germany

Abstract
Cycloids are particular Petri nets for modelling processes of actions and events, belonging to the fundaments of Petri's general systems theory. Defined by four parameters they provide an algebraic formalism to describe strongly synchronized sequential processes. To further investigate their structure, reduction systems of cycloids are studied. They allow for new synthesis approaches by deducing the parameters from the net structure.

Keywords
Structure of Petri Nets, Cycloids, Reduction, Cycloid Isomorphism, Cycloid Algebra, Synthesis,

1. Introduction

Cycloids have been introduced by C.A. Petri in [1] in the section on physical spaces, using as examples firemen carrying the buckets with water to extinguish a fire, the shift from Galilei to Lorentz transformation and the representation of elementary logical gates like Quine-transfers. Besides the far-sighted work of Petri we got insight in his concepts of cycloids by numerous seminars he hold at the University of Hamburg [2]. Based on formal descriptions of cycloids in [3] and [4] a more elaborate formalization is given in [5], where the most important contribution is a *Synthesis Theorem* computing the parameters of a cycloid from its pure graphical properties like number of nodes and minimal cycle length. Semantical extensions to include more elaborate features of traffic systems have been presented in [6]. The Synthesis Theorem [5] allows for a procedure to calculate from the Petri net parameters μ_0, τ, A and cyc of a cycloid the parameters α, β, γ and δ of a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ with the same net parameters. However, the solution was not unique, but all solutions were isomorphic with respect to particular transformation operations. In this paper we formulate these transformations as reduction rules and consider their reduced forms. It is proved that two cycloid systems are cycloid isomorphic if they are reducible from each other. This follows from the cycloid isomorphism of their reduced equivalents and shows the Synthesis Theorem to be complete in the case of cycloid isomorphic cycloids.

To give an application for the theory, as presented in this article, consider a distributed system of a finite number of circular and sequential processes. The processes are synchronized by uni-directional one-bit channels in such a way that they behave like a circular traffic queue when folded together. To give an example, Figure 1a) shows three such sequential circular

PNSE'22, International Workshop on Petri Nets and Software Engineering, Bergen, Norway, 2022
ruediger.valk@uni-hamburg.de (R. Valk); daniel.moldt@uni-hamburg.de (D. Moldt)

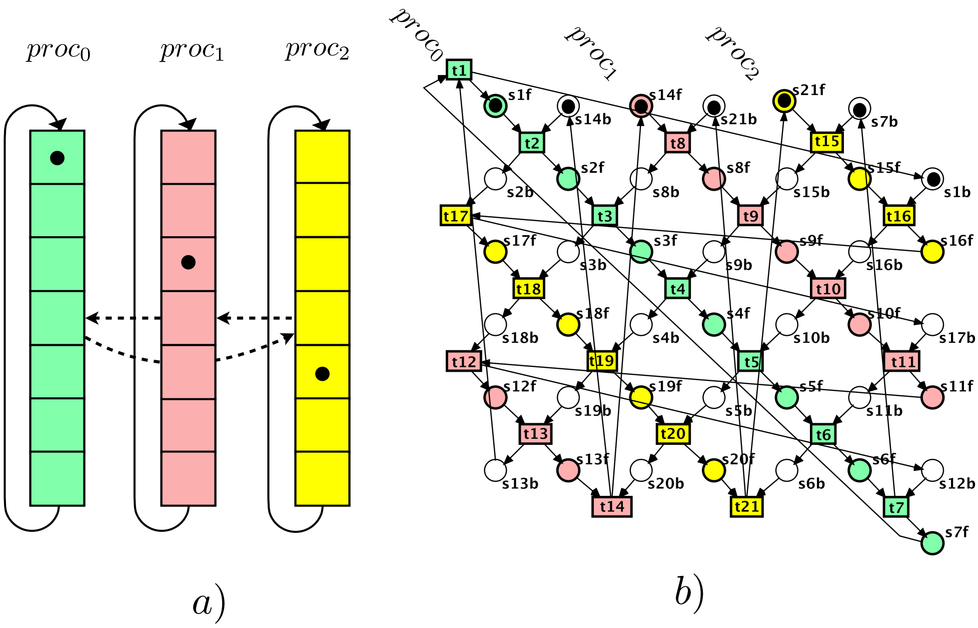

Figure 1: Three sequential processes synchronized by single-bit channels,

processes, each of length 7. In the initial state the control is in position 1, 3 and 5, respectively. The synchronization, realized by the connecting channels, should be such as the three processes would be folded together. This means, that the controls of $proc_0$ and $proc_1$ can make only one step until the next process makes a step itself, while the control of $proc_2$ can make two steps until $proc_0$ makes a step. Following [7] this behaviour is realized by the cycloid of Figure 1b) modelling the three processes by the transition sequences $proc_0$ = [**t1 t2** $\cdots$ **t7**], as well as $proc_1$ = [**t8 t9** $\cdots$ **t14**] and $proc_2$ = [**t15 t16** $\cdots$ **t21**]. The channels are represented by the safe places connecting these processes. By this example the power of the presented theory is shown, since the rather complex net is unambiguously determined by the parameters $\mathcal{C}(\alpha, \beta, \gamma, \delta) = \mathcal{C}(4, 3, 3, 3)$. A next question could be, how to change the cycloid when the parameters of $\beta = 3$ processes of process length $p = 7$ should be changed to a different value, say the double $p = 14$. As will be explained below, the theory returns even three cycloids, namely $\mathcal{C}_1(4, 3, 10, 3)$, $\mathcal{C}_2(4, 3, 6, 6)$ and $\mathcal{C}_3(4, 3, 2, 9)$. However, we will prove in this article that these three solutions are isomorphic and are related by a reduction calculus. The flexibilty of the model is also shown by the following additional example. By doubling in $\mathcal{C}(4, 3, 3, 3)$ the value of β we obtain the cycloid $\mathcal{C}(4, 6, 3, 3)$, which models a distributed system of three circular sequential processes, each of length $p = 10$. However, different to the examples above, each process contains **two** control tokens. Translated to the distributed model, in the initial state each of the three sequential processes contains two items, particularly $proc_0$ in positions 0 and 5 in the circular queue of length 10, $proc_1$ in positions 1 and 6 and $proc_2$ in positions 3 and 8. The present article is part of a general project to investigate all such features of cycloids to make them available for Software Engineering.

We recall some standard notations for set theoretical relations. If $R \subseteq A \times B$ is a relation and $U \subseteq A$ then $R[U] := \{b \mid \exists u \in U : (u, b) \in R\}$ is the *image* of U and $R[a]$ stands for $R[\{a\}]$. R^{-1} is the *inverse relation* and R^+ is the *transitive closure* of R if $A = B$. Also, if $R \subseteq A \times A$ is an equivalence relation then $[\![a]\!]_R$ is the *equivalence class* of the quotient A/R containing a. Furthermore $\mathbb{N}_+$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of positive integer, integer and real numbers, respectively. For integers: $a|b$ if a is a factor of b. The $modulo$-function is used in the form $a \, mod \, b = a - b \cdot \lfloor \frac{a}{b} \rfloor$, which also holds for negative integers $a \in \mathbb{Z}$. In particular, $-a \, mod \, b = b - a$ for $0 < a \leq b$.

2. Petri Space and Cycloids

We define (Petri) nets as they will be used in this article.

Definition 1 ([5]). *As usual, a net $\mathcal{N} = (S, T, F)$ is defined by non-empty, disjoint sets S of places and T of transitions, connected by a flow relation $F \subseteq (S \times T) \cup (T \times S)$ and $X := S \cup T$. A transition $t \in T$ is* active *or* enabled *in a marking $M \subseteq S$ if ${}^\bullet t \subseteq M \, \wedge \, t^\bullet \cap M = \emptyset$[1]. In this case we obtain $M \xrightarrow{t} M'$ if $M' = M \setminus {}^\bullet t \cup t^\bullet$, where ${}^\bullet x := F^{-1}[x],\ x^\bullet := F[x]$ denotes the input and output elements of an element $x \in X$, respectively. $\xrightarrow{*}$ is the reflexive and transitive closure of $\rightarrow$. A net together with an initial marking $M_0 \subseteq S$ is called a* net system $(\mathcal{N}, M_0)$. *Given two net systems $\mathcal{N}_1 = (S_1, T_1, F_1, M_0^1)$ and $\mathcal{N}_2 = (S_2, T_2, F_2, M_0^2)$ a mapping $f : X_1 \to X_2$ ($X_i = S_i \cup T_i$) is a* net morphism *([8]) if $f(F_1 \cap (S_1 \times T_1)) \subseteq (F_2 \cap (S_2 \times T_2)) \cup id$ and $f(F_1 \cap (T_1 \times S_1)) \subseteq (F_2 \cap (T_2 \times S_2)) \cup id$ and $f(M_0^1) = M_0^2$. It is an* isomorphism *if it is bijective and the inverse mapping f^{-1} is also a net morphism. $\mathcal{N}_1 \simeq \mathcal{N}_2$ denotes isomorphic nets. Omitting the initial states the definitions apply also to nets.*

Petri started with an event-oriented version of the Minkowski space which is called Petri space now. Contrary to the Minkowski space, the Petri space is independent of an embedding into $\mathbb{Z} \times \mathbb{Z}$. It is therefore suitable for the modelling in transformed coordinates as in non-Euclidian space models. However, the reader will wonder that we will apply linear algebra, for instance using equations of lines. This is done only to determine the relative position of points. It can be understood by first topologically transforming and embedding the space into $\mathbb{R} \times \mathbb{R}$, calculating the position and then transforming back into the Petri space. Distances, however, are not computed with respect to the Euclidean metric, but by counting steps in the grid of the Petri space, like Manhattan distance or taxicab geometry.

For instance, the transitions of the Petri space might model the moving of items in time and space in an unlimited way. To be concrete a coordination system is introduced with arbitrary origin (see Figure 2 a). The occurrence of transition $t_{1,0}$ in this figure, for instance, can be interpreted as a step of a traffic item (the token in the left input-place) in both space and time

[1] With the condition $t^\bullet \cap M = \emptyset$ we follow Petri's definition, but with no impacts in this article.

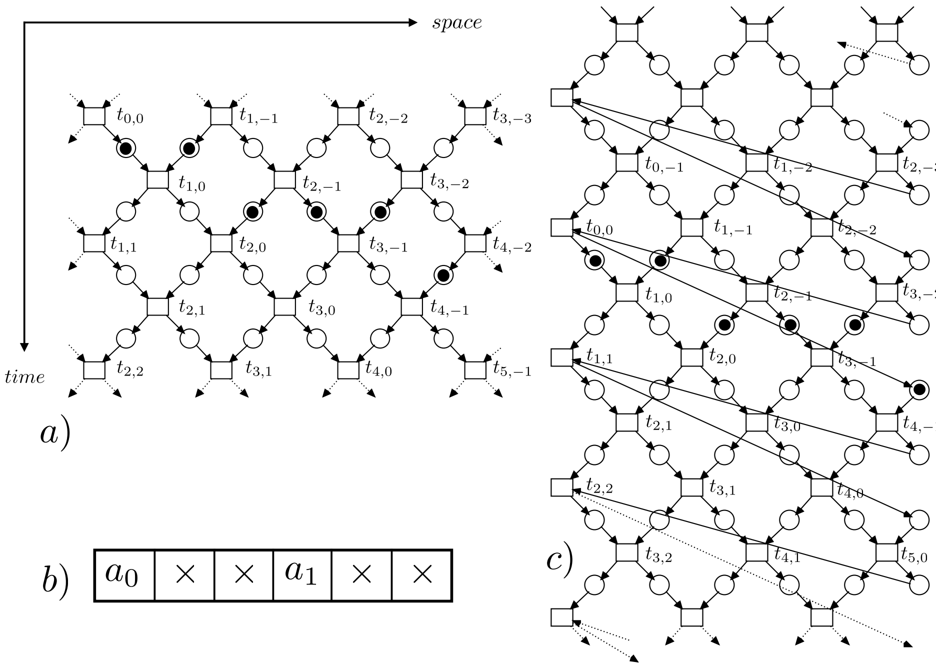

Figure 2: a) Petri space, b) circular traffic queue and c) time orthoid.

direction. It is enabled by a gap or co-item (the token in the right input-place), which is enabling a next traffic item after occurrence of $t_{2,0}$. By the following definition the places obtain their names by their input transitions (see Figure 3 a).

Definition 2 ([5]). *A* $Petri$ $space$ *is defined by the net* $\mathcal{PS}_1 := (S_1, T_1, F_1)$ *where* $S_1 = S_1^{\rightarrow} \cup S_1^{\leftarrow}$, $S_1^{\rightarrow} = \{s^{\rightarrow}_{\xi,\eta} \mid \xi, \eta \in \mathbb{Z}\}$, $S_1^{\leftarrow} = \{s^{\leftarrow}_{\xi,\eta} \mid \xi, \eta \in \mathbb{Z}\}$, $S_1^{\rightarrow} \cap S_1^{\leftarrow} = \emptyset$, $T_1 = \{t_{\xi,\eta} \mid \xi, \eta \in \mathbb{Z}\}$, $F_1 = \{(t_{\xi,\eta}, s^{\rightarrow}_{\xi,\eta}) \mid \xi, \eta \in \mathbb{Z}\} \cup \{(s^{\rightarrow}_{\xi,\eta}, t_{\xi+1,\eta}) \mid \xi, \eta \in \mathbb{Z}\} \cup \{(t_{\xi,\eta}, s^{\leftarrow}_{\xi,\eta}) \mid \xi, \eta \in \mathbb{Z}\} \cup \{(s^{\leftarrow}_{\xi,\eta}, t_{\xi,\eta+1}) \mid \xi, \eta \in \mathbb{Z}\}$ *(cutout in Figure 3 a).* $S_1^{\rightarrow}$ *is the set of* forward places *and* $S_1^{\leftarrow}$ *the set of* backward places. ${}^{\overset{\bullet}{\rightarrow}}t_{\xi,\eta} := s^{\rightarrow}_{\xi-1,\eta}$ *is the forward input place of* $t_{\xi,\eta}$ *and in the same way* ${}^{\overset{\bullet}{\leftarrow}}t_{\xi,\eta} := s^{\leftarrow}_{\xi,\eta-1}$, $t^{\overset{\bullet}{\rightarrow}}_{\xi,\eta} := s^{\rightarrow}_{\xi,\eta}$ *and* $t^{\overset{\bullet}{\leftarrow}}_{\xi,\eta} := s^{\leftarrow}_{\xi,\eta}$ *(Figure 3 a).*

In two steps, by a twofold folding with respect to time and space, Petri defined the cyclic structure of a cycloid. One of these steps is a folding f with respect to space with $f(i,k) = f(i+\alpha, k-\beta)$, fusing all points (i,k) of the Petri space with $(i+\alpha, k-\beta)$ where $i, k \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}_+$ ([1], page 37). While Petri gave a general motivation, oriented in physical spaces, we interpret the choice of α and β by our model of traffic queues.

We assume that our model of a circular traffic queues has six slots containing two items a_0 and a_1 as shown in Figure 2 b). These are modelled in Figure 2 a) by the tokens in the forward input places of $t_{1,0}$ and $t_{3,-1}$. The four co-items are represented by the tokens in the backward input places of $t_{1,0}, t_{2,0}$ and $t_{3,-1}, t_{4,-1}$. By the occurrence of $t_{1,0}$ and $t_{2,0}$ the first item can

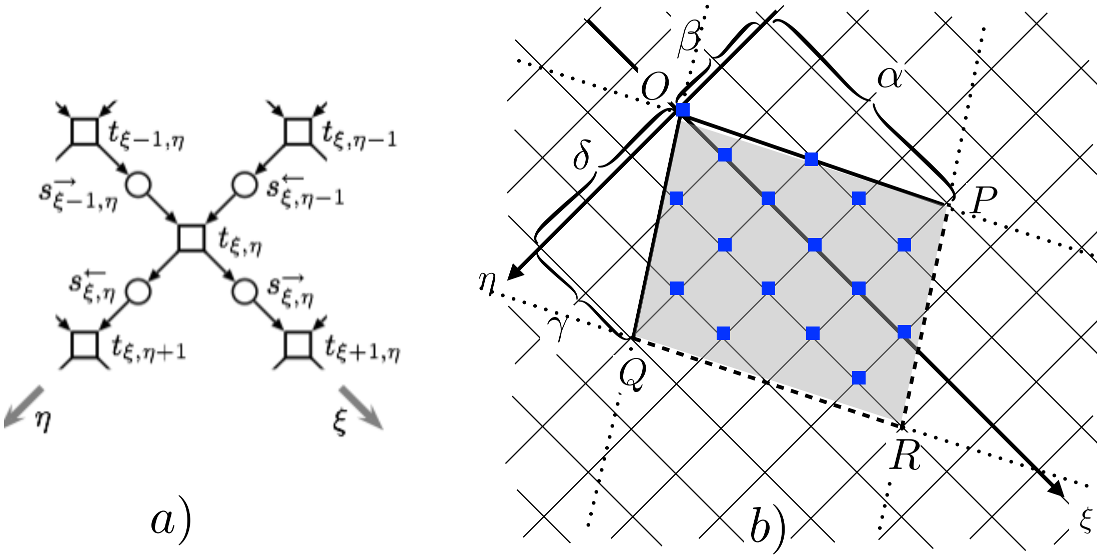

Figure 3: a) Petri space, b) Fundamental parallelogram of $\mathcal{C}(\alpha, \beta, \gamma, \delta) = \mathcal{C}(4, 2, 2, 3)$.

make two steps, as well as the second item by the transitions $t_{3,-1}$ and $t_{4,-1}$, respectively. Then a_1 has reached the end of the queue and has to wait until the first item is leaving its position. Hence, we have to introduce a precedence restriction between the transitions $t_{1,0}$ and $t_{5,-1}$. This is done by fusing the transitions $t_{5,-1}$ and the left-hand follower $t_{1,1}$ of $t_{1,0}$. To determinate α and β we set $(5, -1) = (1 + \alpha, 1 - \beta)$ which gives $\alpha = 4$ and $\beta = 2$. By the equivalence relation $t_{\xi,\eta} \equiv t_{\xi+4,\eta-2}$ we obtain the structure in Figure 2 c). The resulting still infinite net is called a *time orthoid* ([1], page 37), as it extends infinitely in temporal future and past. The second step is a folding with $f(i, k) = f(i + \gamma, k + \delta)$ with $\gamma, \delta \in \mathbb{N}_+$ reducing the system to a cyclic structure also in time direction. As shown in [7] an equivalent cycloid for the traffic queue of Figure 2 b) has the parameters $(\alpha, \beta, \gamma, \delta) = (4, 2, 2, 2)$. To keep the example more general, in Figure 3 b) the values $(\alpha, \beta, \gamma, \delta) = (4, 2, 2, 3)$ are chosen. In this representation of a cycloid, called *fundamental parallelogram*, the squares of the transitions as well as the circles of the places are omitted. All transitions with coordinates within the parallelogram belong to the cycloid including those on the lines between O, Q and O, P, but excluding those of the points Q, R, P and those on the dotted edges between them. All parallelograms of the same shape, as indicated by dotted lines outside the fundamental parallelogram are fused with it.

Definition 3 ([5]). *A cycloid is a net* $\mathcal{C}(\alpha, \beta, \gamma, \delta) = (S, T, F)$*, defined by parameters* $\alpha, \beta, \gamma, \delta \in \mathbb{N}_+$*, by a quotient [8] of the Petri space* $\mathcal{PS}_1 := (S_1, T_1, F_1)$ *with respect to the equivalence relation* $\equiv\, \subseteq X_1 \times X_1$ *with* $X_1 = S_1 \cup T_1$, $\equiv[S_1^{\rightarrow}] \subseteq S_1^{\rightarrow}, \equiv[S_1^{\leftarrow}] \subseteq S_1^{\leftarrow}, \equiv[T_1] \subseteq T_1$, $x_{\xi,\eta} \equiv x_{\xi+m\alpha+n\gamma,\,\eta-m\beta+n\delta}$ *for all* $\xi, \eta, m, n \in \mathbb{Z}$, $X = X_1/_{\equiv}$, $[\![x]\!]_{\equiv}\ F\ [\![y]\!]_{\equiv} \Leftrightarrow \exists\, x' \in [\![x]\!]_{\equiv}\, \exists\, y' \in [\![y]\!]_{\equiv} : x' F_1 y'$ *for all* $x, y \in X_1$*. The matrix* $\mathbf{A} = \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix}$ *is called* *the matrix of the cycloid. Petri denoted the number $|T|$ of transitions as the area A of the cycloid and proved in [1] its value to $|T| = A = \alpha\delta + \beta\gamma$ which equals the determinant $A = det(\mathbf{A})$. The embedding of a cycloid in the Petri space is called* fundamental parallelogram *(see Figure 3 b).*

Definition 4. *a) The net $\mathcal{N} = (S, T, F)$ from Definition 3 (without explicitly giving the parameters $\alpha, \beta, \gamma, \delta$) is called the underlying net of the cycloid. It is a T-net with $|^{\bullet}s| = |s^{\bullet}| = 1$ for all places $s \in S$.*
b) When the distinction between forward places $S^{\rightarrow}$ and backward places $S^{\leftarrow}$ is kept we denote it as the cycloid net *of the cycloid and represent it by $\mathcal{N} = (S^{\rightarrow}, S^{\leftarrow}, T, F)$.*

To give an example, Figure 8 shows a graphical representation of the cycloid net of the cycloid system $\mathcal{C}(5, 3, 2, 6, M_0)$[2]. The forward places $S^{\rightarrow}$ and the backward places $S^{\leftarrow}$ are labelled by the letter **f** and **b**, respectively. Note that the parameters are not visible in this representation, but will be deducible by the results of Sections 4 and 5. Also degenerate cycloids have been introduced by C.A. Petri [9] (page 46) and their properties are studied in [5]. In this article they are used within proofs only.

Definition 5 ([5]). *If in Definition 3 at least one of the parameters $\alpha, \beta, \gamma, \delta$ is zero we call $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ a* degenerate cycloid *when also the additional restriction $A > 0$ for the area $A = \alpha\delta + \beta\gamma$ holds.*

For proving the equivalence of two points in the Petri space the following procedure[3] is useful.

Theorem 2.1 ([7]). *Two points $\vec{x}_1, \vec{x}_2 \in X_1$ are equivalent $\vec{x}_1 \equiv \vec{x}_2$ if and only if for the difference $\vec{v} := \vec{x_2} - \vec{x_1}$ the parameter vector $\pi(\vec{v}) = \frac{1}{A} \cdot \mathbf{B} \cdot \vec{v}$ has integer values, where A is the area and $\mathbf{B} = \begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}$. Also, in analogy to Definition 3 we obtain $\vec{x}_1 \equiv \vec{x}_2 \Leftrightarrow \exists\, m, n \in \mathbb{Z}$: $\vec{x_2} - \vec{x_1} = \mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix}$.*

Since constructions of cycloids may result in different but isomorphic forms the following theorem is important. We give here a proof using the cycloid algebra from Theorem 2.1, which was not yet known when the article [5] had been published.

Theorem 2.2 ([5]). *The following cycloids are net isomorphic (Definition 1) to $\mathcal{C}(\alpha, \beta, \gamma, \delta)$:*
a) $\mathcal{C}(\alpha, \beta, \gamma - \alpha, \delta + \beta)$ if $\gamma > \alpha$,
b) $\mathcal{C}(\alpha, \beta, \gamma + \alpha, \delta - \beta)$ if $\delta > \beta$.

Proof. Let be $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ with matrix $\mathbf{A}$ (Definition 3) and the vector $\overrightarrow{mn} := (m, n) \in \mathbb{Z}^2$. By Theorem 2.1 with the matrix $\mathbf{A}_1 = \begin{pmatrix} \alpha & \gamma \pm \alpha \\ -\beta & \delta \mp \beta \end{pmatrix}$ of $\mathcal{C}_1 = \mathcal{C}_1(\alpha, \beta, \gamma \pm \alpha, \delta \mp \beta)$ we obtain

[2] The net is generated by the *Automatic Net Layout* of the RENEW tool.
[3] The algorithm is implemented under http://cycloids.de/home.

$\mathbf{A}_1 \cdot \overrightarrow{mn} = \mathbf{A} \cdot \overrightarrow{mn} + \begin{pmatrix} 0 & \pm\alpha \\ 0 & \mp\beta \end{pmatrix} \cdot \overrightarrow{mn} = \mathbf{A} \cdot \overrightarrow{mn} + \begin{pmatrix} \pm n \cdot \alpha \\ \mp n \cdot \beta \end{pmatrix} = \mathbf{A} \cdot \overrightarrow{mn} + \mathbf{A} \cdot \begin{pmatrix} \pm n \\ 0 \end{pmatrix} =$
$\mathbf{A} \cdot \begin{pmatrix} m \pm n \\ n \end{pmatrix}$. Hence, the equivalence relations of $\mathcal{C}$ and $\mathcal{C}_1$ are the same. □

In plane geometry, a shear mapping is a linear map that displaces each point in a fixed direction, by an amount proportional to its signed distance from the line that is parallel to that direction and goes through the origin[4]. For a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ the corners of its fundamental parallelogram have the coordinates $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$, $R = \begin{pmatrix} \alpha + \gamma \\ \delta - \beta \end{pmatrix}$ and $Q = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$. Comparing them with the corners O', P', R', Q' of the transformed cycloid $\mathcal{C}(\alpha, \beta, \gamma + \alpha, \delta - \beta)$ of Theorem 2.2 b) we observe $O' = O, P' = P, Q' = \begin{pmatrix} \gamma + \alpha \\ \delta - \beta \end{pmatrix} = R$ and the lines $\overline{Q, R}$ and $\overline{Q', R'}$ are the same. Therefore the second is a shearing of the first one. This is shown in Figure[5] 4 for the cycloids $\mathcal{C}(2, 3, 2, 8), \mathcal{C}(2, 3, 4, 5)$ and $\mathcal{C}(2, 3, 6, 2)$. When applying the equivalences of

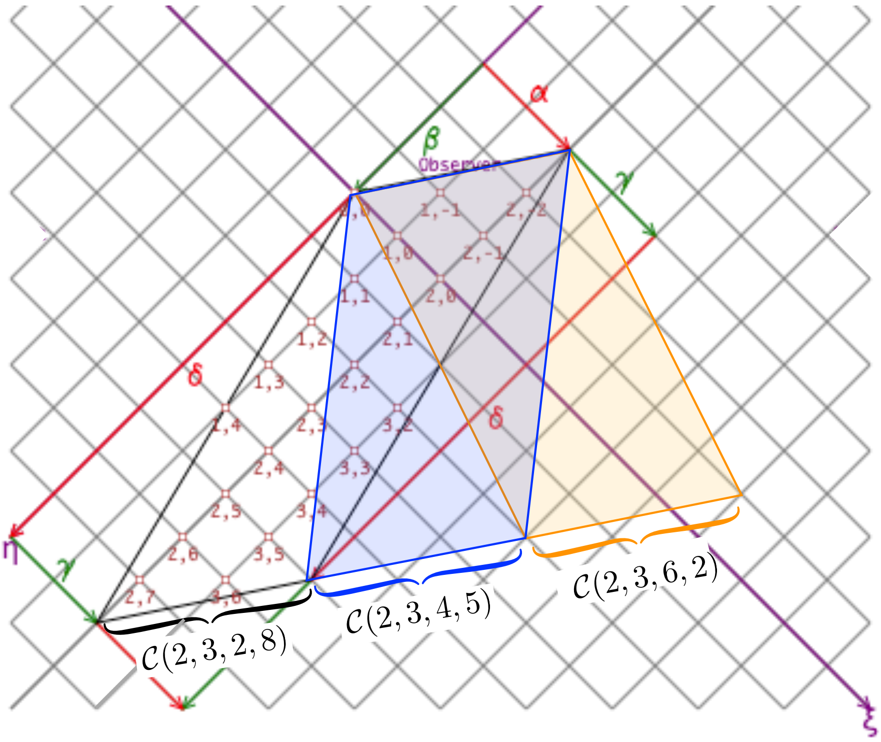

Figure 4: A shearing from $\mathcal{C}(2, 3, 2, 8)$ to $\mathcal{C}(2, 3, 6, 2)$.

[4]https://en.wikipedia.org/wiki/Shear_mapping
[5]The figure has been designed using the tool http://cycloids.adventas.de.

Theorem 2.2 the parameters γ and δ are changed which leads to the following definition of $\gamma\delta$-reduction equivalence.

Definition 6. *If a cycloid or cycloid system $\mathcal{C}_1$ can be obtained from a cycloid $\mathcal{C}_2$ by iterated applications of the transformations given in Theorem 2.2 then they are called $\gamma\delta$-*reduction equivalent*, denoted $\mathcal{C}_1 \simeq_{\gamma\delta} \mathcal{C}_2$.*

Lemma 1 ([5]). *For any cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ there is a minimal cycle containing the origin O in its fundamental parallelogram representation.*

For the next Theorem from [5], we give a proof which follows the same concept, but is more formal.

Theorem 2.3 ([5]). *The length of a minimal cycle of a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ is $cyc(\alpha, \beta, \gamma, \delta) =$*
$$cyc = \gamma + \delta + \begin{cases} \lfloor \frac{\delta}{\beta} \rfloor (\alpha - \beta) & \text{if } \alpha \leq \beta \\ -\lfloor \frac{\gamma}{\alpha} \rfloor (\alpha - \beta) & \text{if } \alpha > \beta \end{cases}$$
The length of a minimal cycle of a degenerate cycloid with $\alpha \leq \beta$ is also cyc if $\alpha > 0$ and $\beta > 0$.

Proof. a) We first consider the case $\alpha \leq \beta$. With respect to paths and cycles in the fundamental parallelogram and by Lemma 1 it is sufficient to consider paths starting in the origin O. Such a cycle of the cycloid corresponds to a path from O to an equivalent point $\vec{x}$ in the Petri space. Each such point has the form $\vec{x} = i \cdot \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + j \cdot \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ for $i, j \in \mathbb{N}$. The case $i = 0$ is to be excluded since no point (ξ, η) with $\eta < 0$ is reachable from O in the Petri space. Since a cycle of minimal length is searched, also the cases $i > 1$ are excluded. Therefore we consider the points $\vec{x} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + j \cdot \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ for $j \in \mathbb{N}$. Next we prove that increasing the value of j does not increase the distance to the origin (while the condition $\eta \geq 0$ is not violated when going β steps in direction $-\eta$). More precisely, for any $\xi \geq 0, \eta \geq 0$ we have to prove $d(O, \begin{pmatrix} \xi \\ \eta \end{pmatrix}) \geq d(O, \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \alpha \\ -\beta \end{pmatrix})$ under the condition $\eta - \beta \geq 0$. This follows from $\alpha \leq \beta$ by $0 \geq \alpha - \beta \;\Rightarrow\; \xi + \eta \geq \xi + \alpha + \eta - \beta \;\Rightarrow\; |\xi + \eta| \geq |\xi + \alpha| + |\eta - \beta| \;\Rightarrow\; d(O, \begin{pmatrix} \xi \\ \eta \end{pmatrix}) \geq d(O, \begin{pmatrix} \xi + \alpha \\ \eta - \beta \end{pmatrix})$ Again, since points (ξ, η) with $\eta < 0$ are not reachable, we obtain the condition $\delta + j \cdot (-\beta) \geq 0$, which is $j \leq \frac{\delta}{\beta}$. Hence, the maximal integer value for j is $j = \lfloor \frac{\delta}{\beta} \rfloor$. The length of this cycle is $\gamma + \delta + \lfloor \frac{\delta}{\beta} \rfloor \cdot (\alpha - \beta)$, which finishes the proof in this case.
b) For the alternative case we look at the cycloid $\mathcal{C}(\beta, \alpha, \delta, \gamma)$ (by interchanging α and β, as well as γ and δ), which is net isomorphic [5] and therefore has a minimal cycle of the same length, hence $cyc = \gamma + \delta + \lfloor \frac{\gamma}{\alpha} \rfloor \cdot (\beta - \alpha)$ in the case $\alpha > \beta$. Both cases together verify the theorem.
c) For the case of a degenerate cycloid we refer to [5]. □

Definition 7 ([10]). *A* forward-cycle *of a cycloid is an elementary*[6] *cycle containing only forward places of $S_1^{\rightarrow}$. A* backward-cycle *of a cycloid is an elementary cycle containing only backward places of $S_1^{\leftarrow}$ (Definition 2).*

[6] An elementary cycle is a cycle where all nodes are different.

Theorem 2.4 ([10]). *In a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ with area A the length of a forward-cycle is $p = \frac{A}{gcd(\beta,\delta)}$ and length of a backward-cycle is $p' = \frac{A}{gcd(\alpha,\gamma)}$. The cycloid contains $gcd(\beta, \delta)$ disjoint forward-cycles and $gcd(\alpha, \gamma)$ disjoint backward cycles. With respect to the standard initial marking (Definition 9) the number of tokens in a forward cycle is $\frac{\beta}{gcd(\beta,\delta)}$ and $\frac{\alpha}{gcd(\alpha,\gamma)}$ in a backward cycle.*

For the cycloids $\mathcal{C}(4, 3, 3, 3)$ and $\mathcal{C}(4, 6, 3, 3)$ from the introduction we obtain $p = 7$ and $p = 10$, respectively. The number of tokens in a forward-cycle of $\mathcal{C}(4, 6, 3, 3)$ is $\frac{6}{gcd(6,3)} = 2$. An important class of cycloids has the property to represent a number of sequential processes of the same length. Such a cycloid is called regular.

Definition 8. *A cycloid $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ is* regular *if β divides δ. It consists of a number β forward-cycles (called* processes*) of length $p = \frac{A}{\beta}$. $\mathcal{C}$ is called is* co-regular *if α divides γ. Then it consists of a number α backward-cycles (called* co-processes*) of length $p = \frac{A}{\alpha}$.*

The cycloid $\mathcal{C}(4, 3, 3, 3)$ from the introduction is regular, whereas and $\mathcal{C}(4, 6, 3, 3)$ is not. For the computation of the parameters γ and δ for given values of α, β and p we implicitly presume regular cycloids which leads to the equation $p = \frac{A}{\beta} = \frac{\alpha}{\beta} \cdot \delta + \gamma$ or $\gamma = -\frac{\alpha}{\beta} \cdot \delta + p$. For the values $\alpha = 4, \beta = 3, p = 14$, as given in the example of the introduction, the equation $\gamma = -\frac{4}{3} \cdot \delta + 14$ has three solutions for the pair (γ, δ), namely $(10, 3)$, $(6, 6)$ and $(2, 9)$, since only positive integer values are consistent. In different examples there is only one solution or even none (for instance with $(\alpha, \beta, p) = (5, 11, 4)$).

Definition 9 ([5]). *For a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ we define a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ or $\mathcal{C}(\mathcal{N}, M_0)$ by adding the standard initial marking:*
$M_0 = \{s^{\rightarrow}_{\xi,\eta} \in S^{\rightarrow}_1 \mid \beta\xi + \alpha\eta \leq 0 \wedge \beta(\xi + 1) + \alpha\eta > 0\} /_{\equiv} \cup$
$\{s^{\leftarrow}_{\xi,\eta} \in S^{\leftarrow}_1 \mid \beta\xi + \alpha\eta \leq 0 \wedge \beta\xi + \alpha(\eta + 1) > 0\} /_{\equiv}$.

Lemma 2 ([5]). *Given a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ with standard initial marking M_0 then $|M_0 \cap S^{\rightarrow}| = \beta$ and $|M_0 \cap S^{\leftarrow}| = \alpha$.*

See Figure 5 for an example. The following Synthesis Theorem allows for a cycloid system, given as a net without the parameters $\alpha, \beta, \gamma, \delta$, to compute these parameters. It does not necessarily give a unique result, but for $\alpha \neq \beta$ the resulting cycloids are isomorphic. In the theorem $\tau_0 := |\{t| \ \ |{}^{\bullet}t \cap M_0| \geq 1 \ \}|$ is the number of initially marked transitions and $\tau_a := |\{t| \ \ |{}^{\bullet}t \cap M_0| = 2 \ \}|$ is the number of initially active transitions. They are used to determine α and β. In this paper, however, we use Lemma 2, instead.

Theorem 2.5 (Synthesis Theorem [5]). *Cycloid systems with identical system parameters τ_0, τ_a, A and cyc are called σ-equivalent. Given a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ in its net representation (S, T, F, M_0) where the parameters τ_0, τ_a, A and cyc are known (but the parameters $\alpha, \beta, \gamma, \delta$ are not). Then a σ-equivalent cycloid $\mathcal{C}(\alpha', \beta', \gamma', \delta')$ can be computed by $\alpha' = \tau_0$, $\beta' = \tau_a$ and for γ', δ' by some positive integer solution of the following formulas using these settings of α' and β':*
a) case $\alpha' > \beta'$: $\gamma' \ mod \ \alpha' = \frac{\alpha' \cdot cyc - A}{\alpha' - \beta'}$ and $\delta' = \frac{1}{\alpha'}(A - \beta' \cdot \gamma')$,
b) case $\alpha' < \beta'$: $\delta' \ mod \ \beta' = \frac{\beta' \cdot cyc - A}{\beta' - \alpha'}$ and $\gamma' = \frac{1}{\beta'}(A - \alpha' \cdot \delta')$,

c) case $\alpha' = \beta'$: $\gamma' = \lceil \frac{cyc}{2} \rceil$ and $\delta' = \lfloor \frac{cyc}{2} \rfloor$.
These equations may result in different cycloid parameters, however the cycloids are isomorphic in the cases a) and b) as in Theorem 2.2. If the distinction between $S^{\rightarrow}$ and $S^{\leftarrow}$ is known Lemma 2 can be used in place of τ_0 and τ_a .

When working with cycloids it is sometimes important to find for a transition outside the fundamental parallelogram the equivalent element inside. In general, by enumerating all elements of the fundamental parallelogram (using Theorem 7 in [11]) and applying the equivalence test from Theorem 2.1 a runtime is obtained, which already fails for small cycloids. The following theorem allows for a better algorithm[7], which is linear with respect to the cycloid parameters.

Theorem 2.6 ([10]). *For any element $\vec{u} = (u, v)$ of the Petri space the (unique) equivalent element within the fundamental parallelogram is $\vec{x} = \vec{u} - \mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix}$ where $m = \lfloor \frac{1}{A}(u\delta - v\gamma) \rfloor$ and $n = \lfloor \frac{1}{A}(v\alpha + u\beta) \rfloor$.*

3. Reduction of Cycloid systems

Following Theorem 2.2 we introduce two reduction rules for cycloids keeping them isomorphic.

Definition 10. *For cycloids $\mathcal{C}_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ and $\mathcal{C}_2(\alpha_2, \beta_2, \gamma_2, \delta_2)$ the following conditional reduction rules are defined:*
R1: $\alpha_2 = \alpha_1, \beta_2 = \beta_1, \gamma_2 = \gamma_1 - \alpha_1$ and $\delta_2 = \delta_1 + \beta_1$ if $\gamma_1 > \alpha_1$. If this rule cannot be applied the cycloid system $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ is called γ-reduced. If $\mathcal{C}$ is γ-reduced and $\gamma < \alpha$ (resp. $\gamma = \alpha$) then $\mathcal{C}$ is called strongly γ-reduced (resp. weakly γ-reduced).
R2: $\alpha_2 = \alpha_1, \beta_2 = \beta_1, \gamma_2 = \gamma_1 + \alpha_1$ and $\delta_2 = \delta_1 - \beta_1$ if $\delta_1 > \beta_1$.
If this rule cannot be applied the cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ is called δ-reduced. If $\mathcal{C}$ is δ-reduced and $\delta < \beta$ (resp. $\delta = \beta$) then $\mathcal{C}$ is called strongly δ-reduced (resp. weakly δ-reduced).

In some cases for reduced cycloids the cycloid parameters γ and δ can be directly deduced from the parameters α and β and the properties cyc and A.

Theorem 3.1. *Let be $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ a cycloid with known values $\alpha \neq \beta$, area A, minimal cycle length cyc, $U := \frac{1}{\alpha - \beta} \cdot (\alpha \cdot cyc - A)$ and $V := \frac{1}{\alpha - \beta} \cdot (A - \beta \cdot cyc)$.*

a) If $\mathcal{C}$ is strongly δ-reduced and $\alpha \leq \beta$ or strongly γ-reduced and $\alpha > \beta$ then $\gamma = U$ and $\delta = V$.
b) If $\mathcal{C}$ is weakly δ-reduced and $\alpha \leq \beta$ then $\gamma = U - \alpha$ and $\delta = \beta$.
c) If $\mathcal{C}$ is weakly γ-reduced and $\alpha > \beta$ then $\gamma = \alpha$ and $\delta = V - \beta$.

Proof. Since in item a) of the theorem we have $\lfloor \frac{\gamma}{\alpha} \rfloor = 0$ or $\lfloor \frac{\delta}{\beta} \rfloor = 0$ by Theorem 2.3 we obtain $cyc = \gamma + \delta$. With the formula for A we have the equation $\begin{pmatrix} cyc \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$

[7]The algorithm is implemented under http://cycloids.de/home.

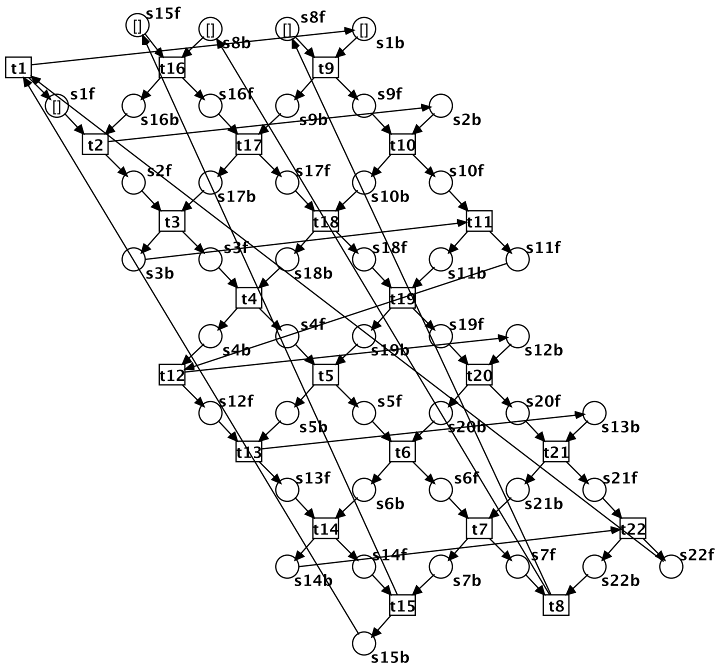

Figure 5: The cycloid system $\mathcal{C}(2, 3, 6, 2, M_0)$

to compute the solution $\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}1 & 1\\ \beta & \alpha\end{pmatrix}^{-1} \begin{pmatrix}cyc\\ A\end{pmatrix} = \frac{1}{\alpha-\beta}\begin{pmatrix}\alpha & -1\\ -\beta & 1\end{pmatrix}\begin{pmatrix}cyc\\ A\end{pmatrix} = \frac{1}{\alpha-\beta}\begin{pmatrix}\alpha\cdot cyc - A\\ -\beta\cdot cyc + A\end{pmatrix} = \begin{pmatrix}U\\ V\end{pmatrix}$. If $\beta = \delta$ in item b) then we make another step in the δ-reduction and obtain a degenerate cycloid with $\delta = 0$. Then again $\lfloor\frac{\delta}{\beta}\rfloor = 0$ and we proceed as before. By reversing the reduction from the degenerate cycloid it follows $\gamma = U - \alpha$ and $\delta = 0 + \beta$. The case for $\alpha > \beta$ is similar. $\square$

To give an example, for the strongly δ-reduced cycloid system $\mathcal{C}(2, 3, 6, 2, M_0)$ of Figure 5 with $cyc = 8$ and $A = 22$, we obtain $\gamma = \frac{1}{\alpha-\beta}\cdot(\alpha\cdot cyc - A) = 6$ and
$\delta = \frac{1}{\alpha-\beta}\cdot(A - \beta\cdot cyc) = 2$. Different to Theorem 3.1 the next result is not a special case of Theorem 2.2, which does not work in the case of $\alpha = \beta$. To distinguish cycloids also in this case we introduce the notion of an inclination. In this case a cycloid has transitions with coordinates $t_{0,0}, t_{1,-1}, \cdots, t_{\alpha-1,-(\alpha-1)}$, for instance the transitions $t_{0,0}$ = **t1**, $t_{1,-1}$ = **t19**, $t_{2,-2}$ = **t10** in $\mathcal{C}(3, 3, 1, 8, M_0^1)$ from Figure 6. A forward cycle of such a cycloid contains one of these transition repeatedly. The inclination is the index of the first such transition. If the cycloid is regular (Definition 8), i.e. β divides δ then this transition is $t_{0,0}$ and the inclination is $inc = 0$. The values of inc are bounded by $0 \leq inc < \alpha$.

Definition 11. *Let $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ be a cycloid with $\alpha = \beta$. A forward or backward cycle (Definition 7) starting in the origin $t_{0,0}$ contains one of the transitions $\{t_{j,-j} | 0 \leq j < \alpha\}$ for the first time, say $t_{i,-i}$.*

a) With respect to the forward cycle the forward inclination *of the cycloid is defined by this index* $inc := i \in \{0, \cdots, \alpha - 1\}$. *The path from* $t_{0,0}$ *to* $t_{i,-i}$ *is called* pseudo-process *and its length is denoted by* $\widetilde{p}$.
b) With respect to the backward cycle the backward inclination *of the cycloid is defined by this index* $inc' := i \in \{0, \cdots, \alpha - 1\}$. *In this case, the path from* $t_{0,0}$ *to* $t_{i,-i}$ *is called* pseudo-co-process *and its length is denoted by* $\widetilde{p}'$.

Theorem 3.2. *Let* $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ *be a cycloid with* $\alpha = \beta$.
a) The forward inclination inc *exists and has the values* $inc = \delta \, mod \, \alpha$ *and* $\widetilde{p} = \gamma + \delta$. *If* $\mathcal{C}$ *is* δ*-reduced form (Definition 10) then* $inc = \delta$ *and if* $\mathcal{C}$ *is regular then* $inc = 0$ *and* $\widetilde{p} = p$ *for the process length* p *(Definition 8).*
b) The backward inclination inc' *exists and has the value* $inc' = 0$ *if* $\mathcal{C}$ *is co-regular, else* $inc' = \alpha - \gamma \, mod \, \alpha$. *Moreover,* $\widetilde{p}' = \gamma + \delta$.

Proof. a) From the definition of inc we obtain $\begin{pmatrix} \widetilde{p} \\ 0 \end{pmatrix} \equiv \begin{pmatrix} i \\ -i \end{pmatrix}$ which is by Theorem 2.1 equivalent to $\mathbb{Z}^2 \ni \frac{1}{A} \begin{pmatrix} \delta & -\gamma \\ \alpha & \alpha \end{pmatrix} \begin{pmatrix} \widetilde{p} - i \\ i \end{pmatrix} = \frac{1}{\alpha \cdot (\gamma + \delta)} \begin{pmatrix} \delta(\widetilde{p} - i) - i \cdot \gamma \\ \alpha \cdot \widetilde{p} \end{pmatrix}$. The second row of this formula gives the solution $\widetilde{p} = \gamma + \delta$. Using this result from the first row we obtain $\mathbb{Z} \ni \frac{1}{A}(\delta(\gamma + \delta - i) - i \cdot \gamma) = \frac{(\delta - i) \cdot (\gamma + \delta)}{\alpha \cdot (\gamma + \delta)} = \frac{\delta - i}{\alpha}$ with a solution $i = \delta$. Therefore $t_{\delta,-\delta}$ is a transition of the form $t_{i,-i}$ as mentioned in Definition 11. However the condition $0 \leq i < \alpha$ may not be fulfilled as the transition may lie outside the fundamental parallelogram. To find the (unique) equivalent transition inside the fundamental parallelogram we apply Theorem 2.6. With $(u, v) = (\delta, -\delta)$ and $A = \alpha \cdot (\gamma + \delta)$ we obtain $m = \lfloor \frac{1}{A}(u \cdot \delta - v \cdot \gamma) \rfloor = \lfloor \frac{\delta \cdot (\gamma + \delta)}{\alpha \cdot (\gamma + \delta)} \rfloor = \lfloor \frac{\delta}{\alpha} \rfloor$ and $n = \lfloor \frac{1}{A}(v \cdot \alpha + u \cdot \beta) \rfloor = \lfloor \frac{1}{A}(-\delta \cdot \alpha + \delta \cdot \alpha) \rfloor = 0$. Using the parameters m and n the transition in question is computed by:

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \alpha & \gamma \\ -\alpha & \delta \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \delta \\ -\delta \end{pmatrix} - \begin{pmatrix} \alpha & \gamma \\ -\alpha & \delta \end{pmatrix} \begin{pmatrix} \lfloor \frac{\delta}{\alpha} \rfloor \\ 0 \end{pmatrix} = \begin{pmatrix} \delta - \alpha \lfloor \frac{\delta}{\alpha} \rfloor \\ -\delta + \alpha \lfloor \frac{\delta}{\alpha} \rfloor \end{pmatrix} = \begin{pmatrix} inc \\ -inc \end{pmatrix}.$$

Finally we conclude $inc = \delta - \alpha \lfloor \frac{\delta}{\alpha} \rfloor = \delta \, mod \, \alpha$. If $\mathcal{C}$ is regular then $\alpha = \beta \ \wedge \ \beta | \delta$ implies $inc = \delta \, mod \, \alpha = 0$. Applying the rule $R2$ from Definition 10 up to a δ-reduced cycloid, from arbitrary δ we obtain $\delta < \alpha$ and $inc = \delta \, mod \, \alpha = \delta$.

b) Similar to case a) from the definition of inc' we obtain $\begin{pmatrix} 0 \\ \widetilde{p}' \end{pmatrix} \equiv \begin{pmatrix} i \\ -i \end{pmatrix}$ which is by Theorem 2.1 equivalent to $\mathbb{Z}^2 \ni \frac{1}{A} \begin{pmatrix} \delta & -\gamma \\ \alpha & \alpha \end{pmatrix} \begin{pmatrix} -i \\ \widetilde{p}' + i \end{pmatrix} = \frac{1}{A} \begin{pmatrix} -i(\gamma + \delta) - \gamma \cdot \widetilde{p}' \\ \alpha \cdot \widetilde{p}' \end{pmatrix}$. The second row of this formula gives the solution $\widetilde{p}' = \gamma + \delta$. Using this result from the first row we obtain $\mathbb{Z} \ni \frac{-1}{A}(i(\gamma + \delta) + \gamma \cdot (\gamma + \delta)) = \frac{-(i + \gamma) \cdot (\gamma + \delta)}{\alpha \cdot (\gamma + \delta)} = \frac{-(i + \gamma)}{\alpha}$ with a solution $i = -\gamma$. Therefore $t_{-\gamma,\gamma}$ is a transition of the form $t_{i,-i}$ as mentioned in Definition 11. However the condition $0 \leq i < \alpha$ may not be fulfilled as the transition may lie outside the fundamental parallelogram. To find the (unique) equivalent transition inside the fundamental parallelogram we apply Theorem 2.6. With $(u, v) = (-\gamma, \gamma)$ and $A = \alpha \cdot (\gamma + \delta)$ we obtain $m = \lfloor \frac{1}{A}(u \cdot \delta - v \cdot \gamma) \rfloor = \lfloor \frac{-\gamma \cdot (\gamma + \delta)}{\alpha \cdot (\gamma + \delta)} \rfloor = \lfloor \frac{-\gamma}{\alpha} \rfloor$ and $n = \lfloor \frac{1}{A}(v \cdot \alpha + u \cdot \beta) \rfloor = \lfloor \frac{1}{A}(\gamma \cdot \alpha - \gamma \cdot \alpha) \rfloor = 0$. Using the parameters m and n the transition in question is computed by:

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \alpha & \gamma \\ -\alpha & \delta \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} -\gamma \\ \gamma \end{pmatrix} - \begin{pmatrix} \alpha & \gamma \\ -\alpha & \delta \end{pmatrix} \begin{pmatrix} \lfloor \frac{-\gamma}{\alpha} \rfloor \\ 0 \end{pmatrix} = \begin{pmatrix} -\gamma - \alpha \lfloor \frac{-\gamma}{\alpha} \rfloor \\ \gamma + \alpha \lfloor \frac{-\gamma}{\alpha} \rfloor \end{pmatrix} = \begin{pmatrix} inc' \\ -inc' \end{pmatrix}$$

and $inc' = -\gamma - \alpha \lfloor \frac{-\gamma}{\alpha} \rfloor$. Using $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \in \mathbb{Z} \\ -\lfloor x \rfloor - 1 & \text{if } x \notin \mathbb{Z} \end{cases}$ we obtain for $\alpha|\,\gamma$ (when $\mathcal{C}$ is co-regular) $inc' = -\gamma + \alpha \cdot \frac{\gamma}{\alpha} = 0$. If $\alpha|\,\gamma$ does not hold we obtain $inc' = -\gamma - \alpha \cdot (-\lfloor \frac{\gamma}{\alpha} \rfloor - 1) = -(\gamma - \alpha \cdot \lfloor \frac{\gamma}{\alpha} \rfloor) + \alpha = \alpha - \gamma \; mod \; \alpha$. □

4. Cycloid Isomorphisms and Reduction Equivalence

A net isomorphism between two cycloids (Definition 1) does not necessarily preserve forward or backward places. To preserve these properties we define the notion of a cycloid isomorphism.

Definition 12. *Given two cycloids* $\mathcal{C}_1 = \mathcal{C}_1(\alpha_1, \beta_1, \gamma_1, \delta_1) = (S_1, T_1, F_1)$ *and* $\mathcal{C}_2 = \mathcal{C}_2(\alpha_2, \beta_2, \gamma_2, \delta_2) = (S_2, T_2, F_2)$ *a mapping* $f : X_1 \to X_2$ *(*$X_i = S_i \cup T_i, S_i = S_i^{\to} \cup S_i^{\leftarrow}$*) is a* cycloid morphism *if*
$f(F_1 \cap (S_1^{\to} \times T_1)) \subseteq (F_2 \cap (S_2^{\to} \times T_2)) \cup id$ *and*
$f(F_1 \cap (S_1^{\leftarrow} \times T_1)) \subseteq (F_2 \cap (S_2^{\leftarrow} \times T_2)) \cup id$ *and*
$f(F_1 \cap (T_1 \times S_1^{\to})) \subseteq (F_2 \cap (T_2 \times S_2^{\to})) \cup id$ *and*
$f(F_1 \cap (T_1 \times S_1^{\to})) \subseteq (F_2 \cap (T_2 \times S_2^{\to})) \cup id.$
f *is an isomorphism if* f *is a bijection and the inverse* f^{-1} *is a also a morphism Then,* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are called cycloid isomorphic denoted* $\mathcal{C}_1 \simeq_{cyc} \mathcal{C}_2$. *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are cycloid systems with initial markings* M_0^1 *and* M_0^2*, respectively, then the definition of a cycloid isomorphism is extended by* $f(S_1^{\to} \cap M_0^1) = S_2^{\to} \cap M_0^2$ *and* $f(S_1^{\leftarrow} \cap M_0^1) = S_2^{\leftarrow} \cap M_0^2$.

Lemma 3. *The cycloid* $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ *of Theorem 2.2 is cycloid isomorphic to the transformed cycloids.*

Proof. In the proof of Theorem 2.2 only properties of the Petri space are used. Therefore it proves that these transformations are invariant with respect to cycloid isomorphism. □

Theorem 4.1. *Two cycloid systems* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are cycloid isomorphic (Definition 12) if and only if they are* $\gamma\delta$*-reduction equivalent (Definition 6):*
$\mathcal{C}_1 \simeq_{cyc} \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \simeq_{\gamma\delta} \mathcal{C}_2$.

Proof. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are cycloid isomorphic then they have the same values of α and β by Lemma 2: $|f(S_1^{\to} \cap M_0^1)| = |S_2^{\to} \cap M_0^2| = \beta$ and $|f(S_1^{\leftarrow} \cap M_0^1)| = |S_2^{\leftarrow} \cap M_0^2| = \alpha$.
Case a) $\alpha \neq \beta$: $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same values of A, cyc and we compute γ, δ by Theorem 2.5. As proved in [5], all solutions are $\gamma\delta$-equivalent. A unique value is obtained by the γ-reduced equivalent in the case $\alpha \leq \beta$ and the δ-reduced equivalent in the case $\alpha > \beta$.
Case b) $\alpha = \beta$: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are cycloid isomorphic then inclinations inc are equal and their areas A are identical. Using Theorem 3.2 δ, γ are computed $modulo \;\; \alpha$ and by α-reduction we obtain a unique result.
Conversely, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are reduction equivalent, by Lemma 3 they are cycloid isomorphic. □

To illustrate the case $\alpha \neq \beta$ in the proof of the theorem consider the cycloid net system $\mathcal{C}(5, 3, 2, 6, M_0)$ of Figure 8. We find $\beta = 3$ since $S^{\to} = \{$**s1f, s24f, s36f**$\}$ has three elements and $\alpha = 5$ since there are 5 marked backward places. Using the Synthesis Theorem 2.5 we calculate $\gamma \ mod \ 5 = \frac{1}{5-3} \cdot (5 \cdot 8 - 36) = 2$ with a solution $\gamma = 2$ and $\delta = \frac{1}{5} \cdot (36 - 3 \cdot 2) = 6$.

For the case $\alpha = \beta$ consider the cycloid systems $\mathcal{C}_1 = \mathcal{C}(3, 3, 1, 8, M_0^1)$ and $\mathcal{C}_2 = \mathcal{C}(3, 3, 7, 2, M_0^2)$ in Figure 6 both with $inc = 2$. From $\delta \ mod \ 3 = 2$ we obtain $\delta = 2, 5, 8$ and with $A = \alpha\delta + \beta\gamma$ also $\gamma = 7, 4, 1$. All further such steps result in not positive values.

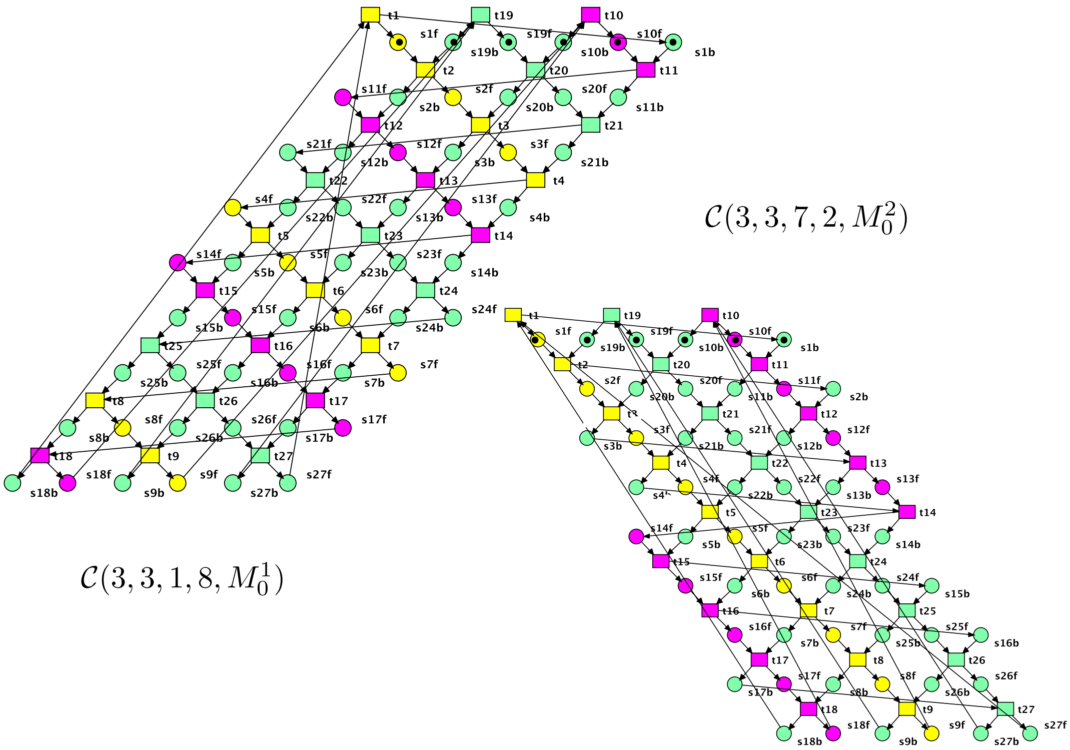

Figure 6: Cycloid systems with $\alpha = \beta$ for illustrating Theorem 4.1.

5. Reduction of Cycloids without initial marking

In this section we investigate cycloid nets without initial marking in order to find suitable parameters $\alpha, \beta, \gamma, \delta$. Therefore we consider a transformation of the first two parameters.

Theorem 5.1. *The following cycloids are cycloid isomorphic (Definition 12) to* $\mathcal{C}(\alpha, \beta, \gamma, \delta)$*:*
a) $\mathcal{C}(\alpha + \gamma, \beta - \delta, \gamma, \delta)$ *if* $\beta > \delta$,
b) $\mathcal{C}(\alpha - \gamma, \beta + \delta, \gamma, \delta)$ *if* $\alpha > \gamma$.

Proof. Let be $\mathcal{C} = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ with matrix $\mathbf{A}$ (Definition 3), $\mathcal{C}_1 = \mathcal{C}_1(\alpha \pm \gamma, \beta \mp \delta, \gamma, \delta)$ with matrix $\mathbf{A}_1 = \begin{pmatrix} \alpha \pm \gamma & \gamma \\ -(\beta \mp \delta) & \delta \end{pmatrix}$ and the vector $\overrightarrow{mn} := (m, n) \in \mathbb{Z}^2$. By Theorem 2.1 with respect to $\mathcal{C}_1$ we obtain: $\vec{x}_1 \equiv \vec{x}_2 \Leftrightarrow \exists\, \overrightarrow{mn} \in \mathbb{Z}^2 : \vec{x}_2 - \vec{x}_1 = \mathbf{A}_1 \cdot \overrightarrow{mn} = \begin{pmatrix} m \cdot \alpha + (n \pm m) \cdot \gamma \\ -m \cdot \beta + (n \pm m) \cdot \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix} \begin{pmatrix} m \\ n \pm m \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} m \\ n \pm m \end{pmatrix}$. Hence, the equivalence relations of $\mathcal{C}$ and $\mathcal{C}_1$ are the same. □

Similar to Theorem 2.2 the transformations of Theorem 5.1 correspond to a shearing. While the invariant edge of the fundamental parallelogram is the edge between O and Q it is called O-Q-shearing to distinguish it from the shearing of Figure 4, which is a O-P-shearing by the use of this terminology. An example of such a O-Q-shearing is given in Figure 7. To give

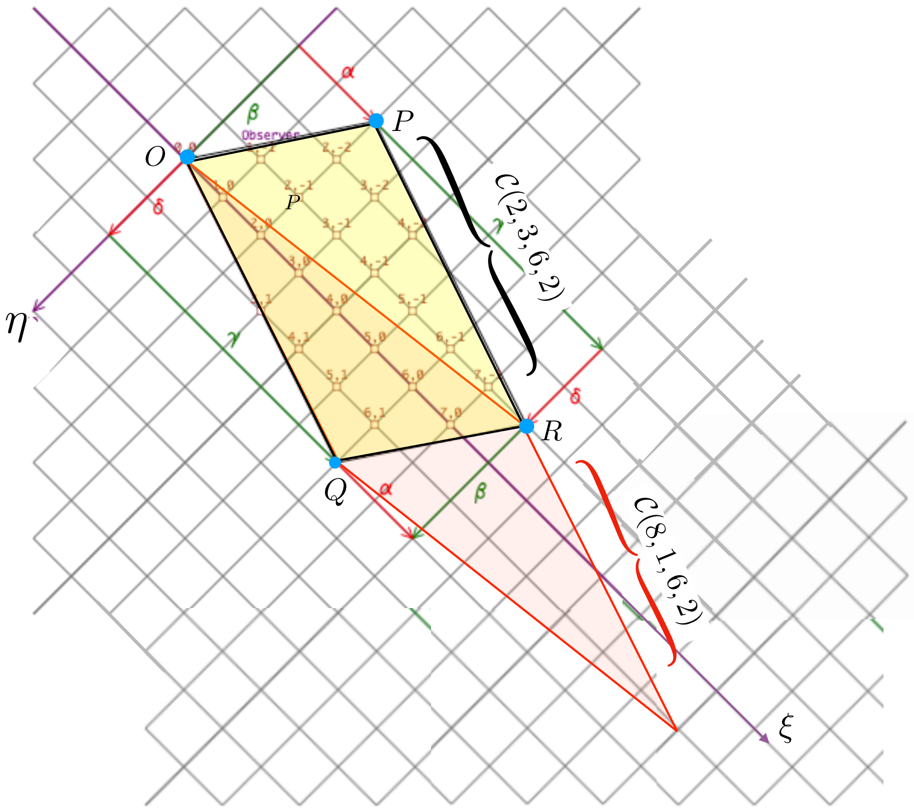

Figure 7: A O-Q-shearing with $\mathcal{C}(2,3,6,2)$ to $\mathcal{C}(8,1,6,2)$.

an example for a transformation which does not preserve isomorphism, consider the cycloids $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ and $\mathcal{C}(\gamma,\delta,\alpha,\beta)$. Obviously they have the same area, but are not isomorphic in general. For instance the cycloids $\mathcal{C}(3,1,1,1)$ and $\mathcal{C}(1,1,3,1)$ have the same area $A = 4$, but different minimal cycle length 2 and 4, respectively. To prepare an algorithm for computing the parameters $\alpha,\beta,\gamma,\delta$ of a cycloid net as in Figure 8, now ignoring the initial marking, we give a formula for the interval of the ξ-axis belonging to the fundamental parallelogram.

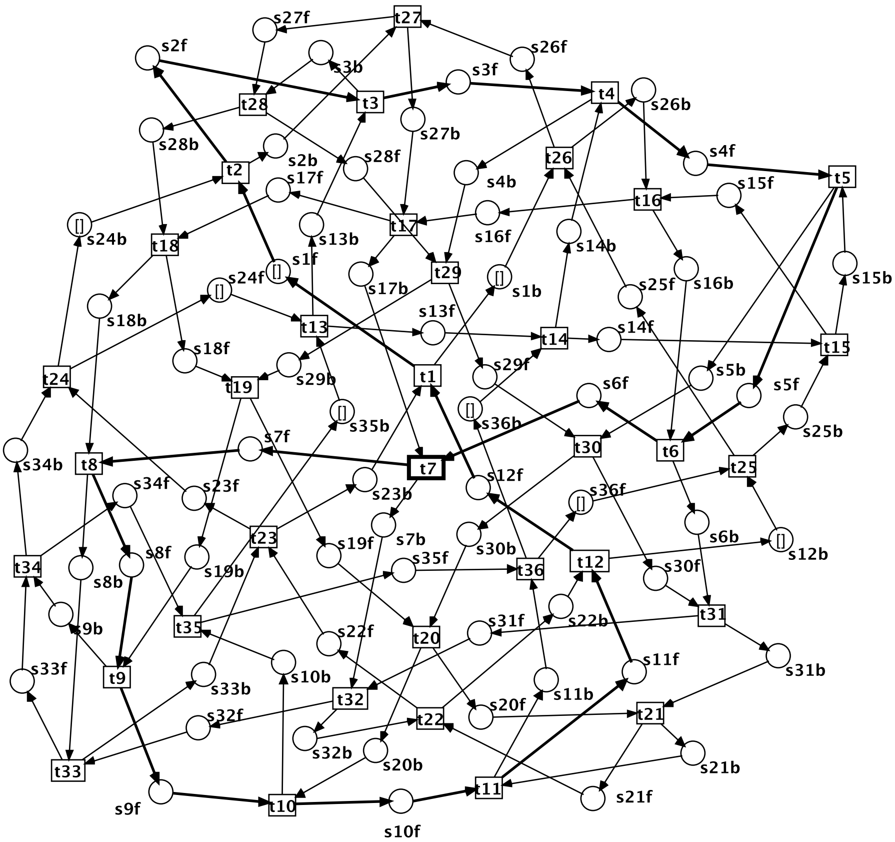

Figure 8: A cycloid net of the cycloid system $\mathcal{C}(5, 3, 2, 6, M_0)$.

Lemma 4. *For a cycloid* $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ *the interval of the* ξ*-axis within the fundamental parallelogram extends from the origin* $t_{0,0}$ *up to* $t_{\xi_{max},0}$ *where* $\xi_{max} = \lceil \frac{A}{max(\beta,\delta)} \rceil - 1$.

Proof. The condition for a transition $t_{\xi,0}$ to lie within the fundamental parallelogram is by Theorem 2.6: $\begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} - \mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix}$ or $\mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with $m = \lfloor \frac{1}{A}(u\delta - v\gamma) \rfloor = \lfloor \frac{1}{A}(\xi \cdot \delta - 0 \cdot \gamma) \rfloor = \lfloor \frac{\xi \cdot \delta}{A} \rfloor$ and $n = \lfloor \frac{1}{A}(v\alpha + u\beta) \rfloor = \lfloor \frac{1}{A}(0 \cdot \alpha + \xi \cdot \beta) \rfloor = \lfloor \frac{\xi \cdot \beta}{A} \rfloor$. This gives $\mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \alpha \cdot \lfloor \frac{\xi \cdot \delta}{A} \rfloor + \gamma \cdot \lfloor \frac{\xi \cdot \beta}{A} \rfloor \\ -\beta \cdot \lfloor \frac{\xi \cdot \delta}{A} \rfloor + \delta \cdot \lfloor \frac{\xi \cdot \beta}{A} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. From the first row we obtain $\lfloor \frac{\xi \cdot \delta}{A} \rfloor = 0$ and $\lfloor \frac{\xi \cdot \beta}{A} \rfloor = 0$ which satisfies also the second row. The overall condition is therefore $\xi < \frac{A}{\delta} \wedge \xi < \frac{A}{\beta}$ or $\xi < \frac{A}{max(\beta,\delta)}$. The largest integer satisfying this condition is $\xi_{max} = \lceil \frac{A}{max(\beta,\delta)} - 1 \rceil = \lceil \frac{A}{max(\beta,\delta)} \rceil - 1$. □

A more geometric way to obtain this result starts with the observation that ξ_{max} is the largest integer value on the ξ-axis before the intersection of the ξ-axis with the lines containing Q, R

or P, R of the fundamental parallelogram. The line containing Q and R is given by the equation $\eta = -\frac{\beta}{\alpha}(\xi - \gamma) + \delta$ (see [5]). Setting $\eta = 0$ gives $\xi = \frac{A}{\beta}$. The line containing P and R is given by the equation $\eta = \frac{\delta}{\gamma}(\xi - \alpha) - \beta$. Again, setting $\eta = 0$ gives $\xi = \frac{A}{\delta}$. Therefore we obtain the overall condition $\xi < \frac{A}{\delta} \ \wedge \ \xi < \frac{A}{\beta}$ and proceed as in the proof before. For the cycloid $\mathcal{C}(4, 2, 2, 3)$ of Figure 3 b) we obtain $A = 16$ and $\xi_{max} = \lceil \frac{16}{max(2,3)} - 1 \rceil = \lceil \frac{16}{3} - 1 \rceil = \lceil \frac{13}{3} \rceil = 5$. As can be seen in the figure, the ξ-axis overlaps with the fundamental parallelogram in the transitions from $t_{0,0}$ to $t_{5,0}$. The values of ξ_{max} for the cycloids of Figure 7 are 7 and 10.

Lemma 5. *For a cycloid* $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ *the output transition of* $t_{0,0}^{\overset{\bullet}{\leftarrow}}$ *is* $t_{\alpha,1-\beta}$.

Proof. For any cycloid the output transition $t_{0,1}$ of $t_{0,0}^{\overset{\bullet}{\leftarrow}}$ is not contained in the fundamental parallelogram. Again, we calculate the equivalent $\vec{x}$ of $t_{0,1}$ within the fundamental parallelogram using Theorem 2.6: $\vec{x} = \vec{u} - \mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix}$ where $\vec{u} = (u, v) = (0, 1)$ and $m = \lfloor \frac{1}{A}(u\delta - v\gamma) \rfloor = \lfloor \frac{1}{A}(0 \cdot \delta - 1 \cdot \gamma) \rfloor = \lfloor \frac{-\gamma}{A} \rfloor = -1$ and $n = \lfloor \frac{1}{A}(v\alpha + u\beta) \rfloor = \lfloor \frac{1}{A}(1 \cdot \alpha + 0 \cdot \beta) \rfloor = \lfloor \frac{\alpha}{A} \rfloor = 0$. Hence we obtain $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 - \beta \end{pmatrix}$. □

Also, this result is obtained in a more geometric way as follows. The position of the output transition of $t_{0,0}^{\overset{\bullet}{\leftarrow}}$ in the fundamental parallelogram is one step from P in direction of the η-axis: $P + (0, 1) = (\alpha, -\beta) + (0, 1) = (\alpha, 1 - \beta)$. Similar to Definition 10 a reduction rule is defined using Theorem 5.1.

Definition 13. *For cycloids* $\mathcal{C}_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ *and* $\mathcal{C}_2(\alpha_2, \beta_2, \gamma_2, \delta_2)$ *the following conditional reduction rule is defined:*

R3: $\alpha_2 = \alpha_1 + \gamma_1, \beta_2 = \beta_1 - \delta_1, \gamma_2 = \gamma_1$ *and* $\delta_2 = \delta_1$ *if* $\beta_1 > \delta_1$.

If this rule cannot be applied the cycloid is said to be β-reduced. *If a cycloid* $\mathcal{C}_1$ *can be obtained from a cycloid* $\mathcal{C}_2$ *by iterated applications of the transformations in Theorem 5.1 they are called* $\alpha\beta$-reduction equivalent, *denoted* $\mathcal{C}_1 \simeq_{\alpha\beta} \mathcal{C}_2$. *They are called* reduction equivalent, *denoted* $\mathcal{C}_1 \simeq_{red} \mathcal{C}_2$, *if the transformations of both, Theorem 5.1 and Theorem 2.2, are used.*

Theorem 5.2. *For a cycloid-net* $\mathcal{C}_1$ *(where the parameters* $\alpha, \beta, \gamma, \delta$ *are not known) a* β*-reduced cycloid* $\mathcal{C}_2 = \mathcal{C}(\alpha_2, \beta_2, \gamma_2, \delta_2)$ *can be computed which is cycloid isomorphic to* $\mathcal{C}_1$. *The parameters* α_3, β_3 *of each cycloid, which is* $\alpha\beta$*-equivalent to* $\mathcal{C}_2$*, is represented by cut points of forward and backward cycles of in the graph of* $\mathcal{C}_1$.

Proof. Let be $\mathcal{C}_1 = \mathcal{C}(\alpha, \beta, \gamma, \delta)$ a cycloid and $t_{\xi_0,\eta_0}, t_{\xi_1,\eta_1}, t_{\xi_2,\eta_2}, \cdots$ a backward cycle (Definition 7) starting in $t_{\xi_0,\eta_0} = t_{0,0}$. By Lemma 5 the second element is $t_{\xi_1,\eta_1} = t_{\alpha,1-\beta}$. Since $1 - \beta \leq 0$ then follow elements $t_{\alpha,2-\beta}, t_{\alpha,3-\beta}, \cdots$ until the ξ-axis in the Petri space is reached in a transition t_{α,η_r} with $\eta_r = 0$ and $r = \beta$. The ξ-axis in the Petri space, however, is folded to the forward cycle in the fundamental parallelogram and there may be different meeting points of the backward an forward cycle, both started in $t_{0,0}$ (for instance $t_{10,-1}$ and $t_{10,-2}$ in the cycloid $\mathcal{C}(10, 3, 2, 2)$ of Figure 9). We make the choice to select the transition t_{ξ_r,η_r} of the backward cycle meeting the forward cycle with minimal r ($t_{10,-2}$ in the cycloid $\mathcal{C}(10, 3, 2, 2)$ of Figure 9). Thus we obtain $\beta_2 = r$ and $\alpha_2 = q$ ($\beta_2 = 1$ in $\mathcal{C}(10, 3, 2, 2)$ and $\alpha_2 = 12$ since the

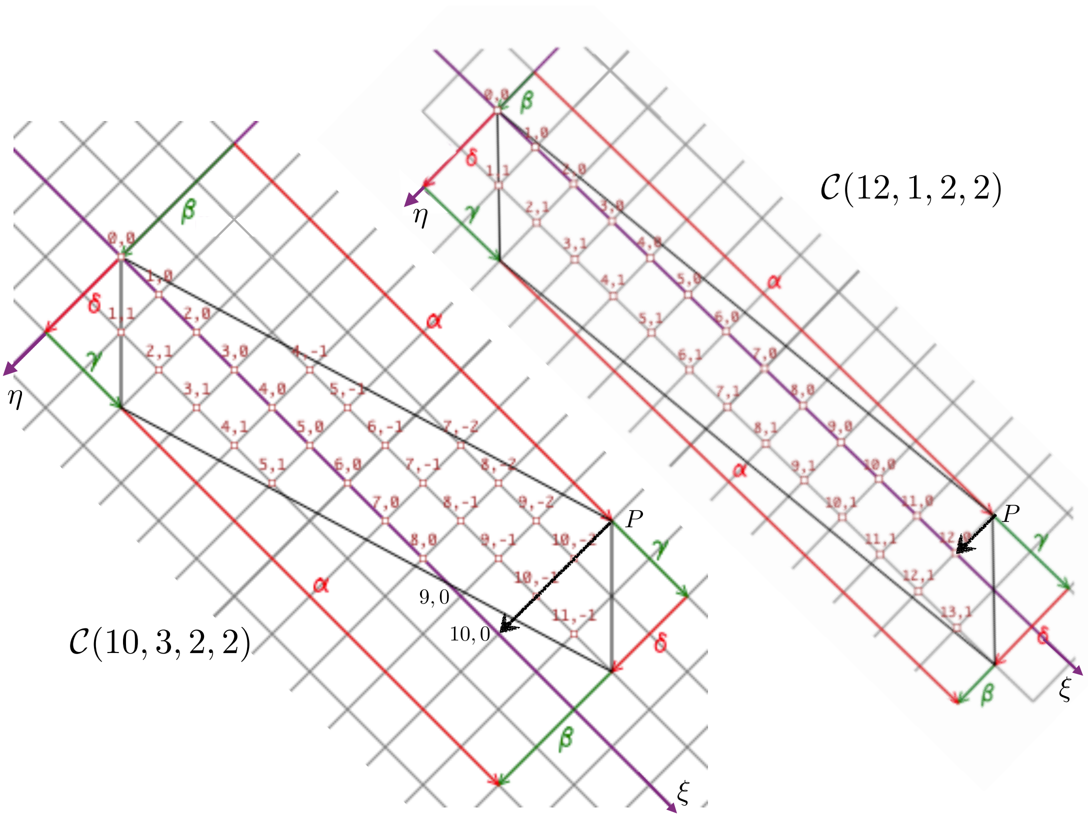

Figure 9: Fundamental parallelograms of $\mathcal{C}(10, 3, 2, 2)$ and $\mathcal{C}(12, 1, 2, 2)$.

initial section of the forward cycle is $t_{0,0}, t_{1,0}, \cdots, t_{8,0}, t_{7,-2}, t_{8,-2}, t_{9,-2}, t_{10,-2}$ having length 12 without counting $t_{0,0}$ and the β-reduced cycloid $\mathcal{C}(12, 1, 2, 2)$ is obtained.).

γ_2 and δ_2 are computed in a similar way, since the input transition of the backward input place of $t_{0,0}$ is t_{γ_2,δ_2-1}, which is proved by $\begin{pmatrix} \gamma_2 \\ \delta_2 - 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ by Theorem 2.1 again: $\frac{1}{A} \cdot \begin{pmatrix} \delta_2 & -\gamma_2 \\ \beta_2 & \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \gamma_2 \\ \delta_2 \end{pmatrix} = \frac{1}{A} \cdot \begin{pmatrix} \delta_2 \cdot \gamma_2 - \gamma_2 \cdot \delta_2 \\ \beta_2 \cdot \gamma_2 + \alpha_2 \cdot \delta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^2$.
In $\mathcal{C}_1$ we use the forward cycle from $t_{0,0}$ again. Then we look for the first transition $t_{\xi,0}$ on this forward cycle, when going from $t_{0,0}$ backwards on the backward cycle of length r (not counting $t_{0,0}$). Then $\gamma_2 = \xi$ and $r + 1 = \delta_2$.

If the cycloid $\mathcal{C}_2$ is β-reduced ($\beta \leq \delta$) then in the construction of α_2 and β_2 above, the meeting point t_{ξ_r,η_r} is on the ξ-axis **within** the fundamental parallelogram. This holds if $\xi_{max} \geq \alpha$ which is proved as follows: using $\frac{\beta \cdot \gamma}{\delta} > 0 \Rightarrow \lceil \alpha + \frac{\beta \cdot \gamma}{\delta} \rceil \geq \alpha + 1$ and Lemma 4 we deduce $\xi_{max} = \lceil \frac{A}{max(\beta,\delta)} \rceil - 1 = \lceil \frac{\alpha \cdot \delta + \beta \cdot \gamma}{\delta} \rceil - 1 = \lceil \alpha + \frac{\beta \cdot \gamma}{\delta} \rceil - 1 \geq \alpha + 1 - 1$. (See the β-reduced equivalent $\mathcal{C}(12, 1, 2, 2)$ of $\mathcal{C}(10, 3, 2, 2)$ in Figure 9.)

Having found all β-reduced cycloid $\mathcal{C}_2$ we now show how to find the $\alpha\beta$-equivalent cycloids from the graph of the cycloid net of $\mathcal{C}_1$. To this end we prove that within the fundamental parallelogram for any $k \in \mathbb{N}$ by going $k \cdot \gamma_2$ steps backwards on the forward cycle from $t_{\alpha_2,0}$ the transition $t_{\alpha_2-k\cdot\gamma_2,0}$ on the ξ-axis is equivalent to the transition $t_{\alpha_2,k\cdot\delta_2}$ going $k\cdot\delta_2$ steps forward from $t_{\alpha_2,0}$ on the backward cycle. This is proved by the equivalence: $\begin{pmatrix}\alpha_2\\ k\cdot\delta_2\end{pmatrix} \equiv \begin{pmatrix}\alpha_2-k\cdot\gamma_2\\ 0\end{pmatrix}$. By Theorem 2.1 we obtain $\frac{1}{A}\cdot\mathbf{B}\cdot(\begin{pmatrix}\alpha_2\\ k\cdot\delta_2\end{pmatrix} - \begin{pmatrix}\alpha_2-k\cdot\gamma_2\\ 0\end{pmatrix}) = \frac{1}{A}\cdot\begin{pmatrix}\delta_2 & -\gamma_2\\ \beta_2 & \alpha_2\end{pmatrix}\cdot\begin{pmatrix}k\cdot\gamma_2\\ k\cdot\delta_2\end{pmatrix} = \frac{1}{A}\cdot\begin{pmatrix}\delta_2\cdot k\cdot\gamma_2-\gamma_2\cdot k\cdot\delta_2\\ \beta_2\cdot k\cdot\gamma_2+\alpha_2\cdot k\cdot\delta_2\end{pmatrix} = \begin{pmatrix}0\\ k\end{pmatrix} \in \mathbb{Z}^2$. The subset of these intersection points with $\alpha_2-k\cdot\gamma_2 > 0$ correspond to the first two parameters in the cycloids $\mathcal{C}(\alpha_2-k\cdot\gamma_2, \beta_2+k\cdot\delta_2, \gamma_2, \delta_2)$ with $\alpha_2 - k\cdot\gamma_2 > 0$ which are β-equivalent to the β-reduced form $\mathcal{C}_2(\alpha_2, \alpha_2, \gamma_2, \delta_2)$. Observe that starting from the β-reduced cycloid $\mathcal{C}_2$we went to the $\alpha\beta$-equivalent versions of the cycloid.

In this proof we started with the origin $t_{0,0}$ of the fundamental parallelogram. If this transition is not known in the cycloid net, by the symmetry of the cycloid, a randomly selected transition can be chosen instead. ☐

As example for the procedure, as described in the proof of Theorem 5.2, consider the randomly selected transition t_{ξ_0,η_0} = **t7** in the cycloid net of Figure 8 (ignoring the initial marking). Then by following the forward places we construct the forward cycle of length $p = 12$ starting in this transition: **t7 t8 t9 t10 t11 t12 t1 t2 t3 t4 t5 t6**. The backward cycle **t7 t32 t22 t12 t25 ... t17** of length $p' = 36$ also starting in **t7** is meeting the forward cycle for the first time in **t12**. The length of the section from **t7** to **t12** is $\alpha_2 = 5$ in the forward cycle and $\beta_2 = 3$ in the backward cycle (by not counting the initial element **t7** in both cases). To compute γ_2 and δ_2, starting again in t_7 we are going backwards on the backward cycle **t7 t17 t27 t2 t24 t34 t9** of length $\delta_2 = 6$ where the forward cycle is met. The length of the latter from **t7** to **t9** is $\gamma = 2$ (without counting **t7** in both cases). From the intersection points we select one where the section of the forward cycle is minimal, i.e. not t_2 in the example. In summary, have calculated the β-reduced cycloid $\mathcal{C}_2 = \mathcal{C}(5, 3, 2, 6)$.

Next we attach the count labels of the path sections to the reached transitions. Above, for the transition **t12** the label is $[5, 3]$ corresponding to $\mathcal{C}(5, 3, 2, 6)$. Going from **t12** a number of $\gamma = 2$ steps backwards on the forward cycle and $\delta = 6$ steps forwards on the backwards cycle we reach transition **t10** with label $[3, 9]$, corresponding to $\mathcal{C}(3, 9, 2, 6)$. Doing the same again we come to **t8** with label $[1, 15]$, corresponding to $\mathcal{C}(1, 15, 2, 6)$. A further such step leads to negative values. In this way, from the net we have deduced all $\alpha\beta$-equivalent cycloids of the β-reduced cycloid $\mathcal{C}_2 = \mathcal{C}(5, 3, 2, 6)$.

Corollary 1. *Two cycloids* $\mathcal{C}_i = \mathcal{C}_i(\alpha_i, \beta_i, \gamma_i, \delta_i),\ i \in \{1, 2\}$ *are cycloid isomorphic (Definition 12) if and only if they are reduction equivalent (Definition 13):* $\mathcal{C}_1 \simeq_{cyc} \mathcal{C}_2 \Leftrightarrow \mathcal{C}_1 \simeq_{red} \mathcal{C}_2$.

Proof. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are cycloid isomorphic by Theorem 5.2 the cycloids $\mathcal{C}'_1$ and $\mathcal{C}'_2$ are constructed which are β-reduced and cycloid isomorphic to both cycloids. They have the same values of α and β. If $\alpha \neq \beta$ we compute the γ or δ-reduced equivalent to obtain the same values for γ and δ by Theorem 3.1. If $\alpha = \beta$ Theorem 11 is used, instead. Conversely, if $\mathcal{C}_1 \simeq_{red} \mathcal{C}_2$ then they are cycloid-isomorphic by Lemma 3 and Theorem 5.1. ☐

6. Conclusion

The theory of cycloids is extended by the technique of reduction. It allows for easier computation of properties like the minimal length of cycles and the cycloid parameters α, β, γ and δ. Reductions can be used to prove cycloid isomorphism which considerably improves the complexity of the problem of testing for cycloid isomorphism. As a byproduct new insights in structural properties of cycloids are gained.

References

[1] C. A. Petri, Nets, Time and Space, Theoretical Computer Science (153) (1996) 3–48. doi:10.1016/0304-3975(95)00116-6.

[2] R. Valk, On the Two Worlds of Carl Adam Petri's Nets, in: W. Reisig, G. Rozenberg (Eds.), Carl Adam Petri: Ideas, Personality, Impact, Springer, Cham, 2019, pp. 37–44. doi:10.1007/978-3-319-96154-5.

[3] O. Kummer, M.-O. Stehr, Petri's Axioms of Concurrency - a Selection of Recent Results, in: Application and Theory of Petri Nets 1997, volume 1248 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, 1997, pp. 195 – 214. doi:10.1007/3-540-63139-9_37.

[4] U. Fenske, Petris Zykloide und Überlegungen zur Verallgemeinerung, Diploma Thesis, Dep. of Informatics, Univ. Hamburg, 2008.

[5] R. Valk, Formal Properties of Petri's Cycloid Systems, Fundamenta Informaticae 169 (2019) 85–121. doi:10.3233/FI-2019-1840.

[6] B. Jessen, D. Moldt, Some Simple Extensions of Petri's Cycloids, in: M. Köhler-Bussmeier, E. Kindler, H. Rölke (Eds.), PNSE 2020 Petri Nets and Software Engineering, CEUR Workshop Proceedings, http://ceur-ws.org/Vol-2651/paper13.pdf, 2020, pp. 194–212.

[7] R. Valk, Circular Traffic Queues and Petri's Cycloids, in: Application and Theory of Petri Nets and Concurrency, volume 12152 of *Lecture Notes in Computer Science*, Springer-Verlag, Cham, 2020, pp. 176 – 195. doi:10.1007/978-3-030-51831-8.

[8] E. Smith, W. Reisig, The semantics of a net is a net – an exercise in general net theory, in: K. Voss, J. Genrich, G. Rozenberg (Eds.), Concurrency and Nets, Springer-Verlag, Berlin, 1987, pp. 461–479. doi:10.1007/978-3-642-72822-8_29.

[9] C. A. Petri, R. Valk, On the Physical Basics of Information Flow - Results obtained in cooperation Konrad Zuse, 2008. URL: https://www2.informatik.uni-hamburg.de/TGI/mitarbeiter/profs/petri/Xian_Petri_Valk.pdf.

[10] R. Valk, Deciphering the Co-car Anomaly of Circular Traffic Queues using Petri Nets, in: Application and Theory of Petri Nets and Concurrency, volume 12734 of *Lecture Notes in Computer Science*, Springer-Verlag, Cham, 2021, pp. 443 – 462. doi:10.1007/978-3-030-76983-3.

[11] R. Valk, On the Structure of Cycloids Indroduced by Carl Adam Petri, in: Application and Theory of Petri Nets and Concurrency, volume 10877 of *Lecture Notes in Computer Science*, Springer-Verlag, Cham, 2018, pp. 294 – 314. doi:10.1007/978-3-319-91268-4_15.