Reasoning in Non-normal Modal Description Logics

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Abstract

Non-normal modal logics, interpreted on neighbourhood models which generalise the usual relational semantics, have found application in several areas, such as epistemic, deontic, and coalitional reasoning. We present here preliminary results on reasoning in a family of modal description logics obtained by combining $\mathcal{ALC}$ with non-normal modal operators. First, we provide a framework of terminating, correct, and complete tableau algorithms to check satisfiability of formulas in such logics with the semantics based on varying domains. We then investigate the satisfiability problems in fragments of these languages obtained by restricting the application of modal operators to formulas only, and interpreted on models with constant domains, providing tight complexity results.

Keywords

Non-normal modal logics, Description logics, Tableau algorithms

1. Introduction

Contexts involving epistemic and doxastic [1, 2, 3], agency-based [4, 5] and coalitional [6, 7], as well as deontic [8, 9, 10], reasoning capabilities populate the wide spectrum of settings where modal logics have found natural applications. In such scenarios, modal operators can be used to represent and reason about what agents, or groups of agents, respectively know, believe, have the capability, or have the permission, to bring about.

The semantics of modal operators is usually given in terms of *relational models*, based on frames consisting of a set of possible worlds equipped with suitable accessibility relations. However, all the modal systems interpreted by means of this kind of semantics, known as *normal*, validate principles that have been considered problematic or debatable for the aforementioned applications, leading to counterintuitive or unacceptable conclusions. Among the unpleasant features discussed in the literature, one encounters for instance the problem of logical omniscience [3], as well as a number of so-called paradoxes in the representation of agents' abilities [5] and obligations [11, 12, 13].

To avoid the unwanted consequences of the relational semantics, several *non-normal* modal logics have been proposed and studied, tracing back to the seminal works by C.I. Lewis [14], Lemmon [15], Kripke [16], Scott [17], Montague [18], Segerberg [19], and Chellas [20]. The semantics of such systems can be given in terms of *neighbourhood models*, generalisations of

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the relational ones that were first introduced by Scott [17] and Montague [18]. In this setting, a frame consists of a set of worlds, each of which is associated with a set of subsets of worlds. Since a subset of worlds can be thought as a proposition (that is true in those worlds), this means that every world in a neighbourhood model is assigned to a set of propositions, those considered necessary with respect to that world. This semantics both generalises the relational one, and avoids the drawbacks of the latter, since the modal principles validated on relational frames that are deemed as problematic for epistemic, coalitional or deontic applications do not hold in general on neighbourhood models.

Non-normal modalities have been widely investigated as a way to extend propositional logic. A further line of research focuses on the behaviour of modal operators interpreted on neighbourhood frames in combination with first-order logic. In this direction, a few works have provided completeness results for first-order non-normal modal logics [21, 22]. In addition, non-normal modal extensions of *description logics*, seen as fragments of first-order logic with a good trade-off between expressive power and computational complexity, have been considered for knowledge representation applications [23, 24], also in multi-agent coalitional settings [25, 26].

In this paper, we investigate satisfiability of non-normal modal extensions of description logics. In particular, we study the logics characterised by the class of all neighbourhood frames ($\mathbf{E}$), supplemented neighbourhood frames ($\mathbf{M}$), neighbourhood frames closed under intersection ($\mathbf{C}$), and neighbourhood frames containing the unit ($\mathbf{N}$), and combine them with the prototypical $\mathcal{ALC}$ description logic. We provide a framework of terminating, correct, and complete tableau algorithms to check satisfiability in such logics interpreted in neighbourhood models with *varying domains* (in this kind of semantics, the domains of the interpretations at each world can differ; cf. Section 2 for details). We then investigate the satisfiability problems in fragments of these languages obtained by restricting the application of modal operators to formulas only, and provide complexity upper bounds with *constant domains* (in this case the domains of the interpretations at every world are the same). We leave satisfiability checking procedures for non-restricted languages interpreted on models with constant domain as open problems.

Full proof details are provided in an extended version of this paper [27].

2. Preliminaries

In this section, we provide preliminary definitions for non-normal modal description logics, first introducing their syntax, and then giving their semantics based on neighbourhood models.

Syntax Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and pairwise disjoint sets of *concept names* and *role names* respectively. An $ML^n_{\mathcal{ALC}}$ *concept* is an expression of the form

$$C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C \mid \Box_i C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $\Box_i$, with $i \in I = \{1, \ldots, n\}$, are modal operators called *boxes*. A *concept inclusion* (*CI*) is an expression of the form $C \sqsubseteq D$, where C, D are $ML^n_{\mathcal{ALC}}$ concepts. An $ML^n_{\mathcal{ALC}}$ *formula* takes the form

$$\varphi ::= C \sqsubseteq D \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi,$$

where $i \in I$. We will use the following standard definitions for concepts: $\bot := A \sqcap \neg A$, $\top := \neg\bot$; $\forall r.C := \neg\exists r.\neg C$; $(C \sqcup D) := \neg(\neg C \sqcap \neg D)$; $\Diamond_i C := \neg\Box_i\neg C$ (operators $\Diamond_i$ are called *diamonds*). Concepts of the form $\Box_i C$, $\Diamond_i C$ are called *modalised concepts*. Analogous conventions also hold for formulas, for which we set true $:= (\bot \sqsubseteq \top)$.

Semantics A *neighbourhood frame*, or simply *frame*, is a pair $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i\in I})$, where $\mathcal{W}$ is a non-empty set of *worlds* and, for each $i \in I = \{1, \dots, n\}$, $\mathcal{N}_i \colon \mathcal{W} \to 2^{2^{\mathcal{W}}}$ is called a *neighbourhood function*. A frame is: *supplemented* if, for all $i \in I$, $w \in \mathcal{W}$, $\alpha, \beta \subseteq \mathcal{W}$, $\alpha \in \mathcal{N}_i(w)$ and $\alpha \subseteq \beta$ implies $\beta \in \mathcal{N}_i(w)$; *closed under intersection* if, for all $i \in I$, $w \in \mathcal{W}$, $\alpha, \beta \subseteq \mathcal{W}$, $\alpha \in \mathcal{N}_i(w)$ and $\beta \in \mathcal{N}_i(w)$ implies $\alpha \cap \beta \in \mathcal{N}_i(w)$; and *contains the unit* if, for all $i \in I$, $w \in \mathcal{W}, \mathcal{W} \in \mathcal{N}_i(w)$. An $ML^n_{\mathcal{ALC}}$ *varying domain neighbourhood model*, or simply *model*, based on a neighbourhood frame $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, \mathcal{I})$, where $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i\in I})$ is a neighbourhood frame and $\mathcal{I}$ is a function associating with every $w \in \mathcal{W}$ an $\mathcal{ALC}$ *interpretation* $\mathcal{I}_w = (\Delta_w, \cdot^{\mathcal{I}_w})$, with non-empty *domain* Δ_w, and where $\cdot^{\mathcal{I}_w}$ is a function such that: for all $A \in \mathsf{N_C}$, $A^{\mathcal{I}_w} \subseteq \Delta_w$; for all $r \in \mathsf{N_R}$, $r^{\mathcal{I}_w} \subseteq \Delta_w \times \Delta_w$. An $ML^n_{\mathcal{ALC}}$ *constant domain neighbourhood model* is defined in the same way, except that, for all $w, w' \in \mathcal{W}$, we have that $\Delta_w = \Delta_{w'}$. Given a model $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ and a world $w \in \mathcal{W}$ of $\mathcal{F}$ (or simply w *in* $\mathcal{F}$), the *interpretation* $C^{\mathcal{I}_w}$ *of a concept* C *in* w is defined as:

$$\begin{aligned}
(\neg D)^{\mathcal{I}_w} &= \Delta_w \setminus D^{\mathcal{I}_w},\\
(D \sqcap E)^{\mathcal{I}_w} &= D^{\mathcal{I}_w} \cap E^{\mathcal{I}_w},\\
(\exists r.D)^{\mathcal{I}_w} &= \{d \in \Delta_w \mid \exists e \in D^{\mathcal{I}_w} : (d, e) \in r^{\mathcal{I}_w}\},\\
(\Box_i D)^{\mathcal{I}_w} &= \{d \in \Delta_w \mid [\![D]\!]^{\mathcal{M}}_d \in \mathcal{N}_i(w)\},
\end{aligned}$$

where, for all $d \in \bigcup_{w\in\mathcal{W}} \Delta_w$, the set $[\![D]\!]^{\mathcal{M}}_d = \{v \in \mathcal{W} \mid d \in D^{\mathcal{I}_v}\}$ is called the *truth set of* D *with respect to* d. We say that a concept C is *satisfied in* $\mathcal{M}$ if there is w in $\mathcal{F}$ such that $C^{\mathcal{I}_w} \neq \emptyset$, and that C is *satisfiable* (over varying or constant neighbourhood models, respectively) if there is a (varying or constant domain, respectively) neighbourhood model in which it is satisfied. The *satisfaction of an* $ML^n_{\mathcal{ALC}}$ *formula* φ *in* w *of* $\mathcal{M}$, written $\mathcal{M}, w \models \varphi$, is defined as follows:

$$\begin{aligned}
\mathcal{M}, w \models C \sqsubseteq D &\quad\text{iff}\quad C^{\mathcal{I}_w} \subseteq D^{\mathcal{I}_w},\\
\mathcal{M}, w \models \neg\psi &\quad\text{iff}\quad \mathcal{M}, w \not\models \psi,\\
\mathcal{M}, w \models \psi \land \chi &\quad\text{iff}\quad \mathcal{M}, w \models \psi \text{ and } \mathcal{M}, w \models \chi,\\
\mathcal{M}, w \models \Box_i\psi &\quad\text{iff}\quad [\![\psi]\!]^{\mathcal{M}} \in \mathcal{N}_i(w),
\end{aligned}$$

where $[\![\psi]\!]^{\mathcal{M}} = \{v \in \mathcal{W} \mid \mathcal{M}, v \models \psi\}$ is the *truth set of* ψ. As a consequence of the above definition, we obtain the following condition for diamond formulas: $\mathcal{M}, w \models \Diamond_i\psi$ iff $[\![\neg\psi]\!]^{\mathcal{M}} \notin \mathcal{N}_i(w)$. Given a neighbourhood frame $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i\in I})$ and a neighbourhood model $\mathcal{M} = (\mathcal{F}, \mathcal{I})$, we say that φ is *satisfied in* $\mathcal{M}$ if there is $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \varphi$, and that φ is *satisfiable* (over varying or constant domain neighbourhood models, respectively) if it is satisfied in some (varying or constant domain, respectively) neighbourhood model.

Given a class of frames $\mathcal{C}$, by the $ML^n_{\mathcal{ALC}}$ *formula satisfiability problem on* (*varying* or *constant domain*, respectively) *neighbourhood models based on a frame in* $\mathcal{C}$ we mean the problem of deciding whether an $ML^n_{\mathcal{ALC}}$ formula is satisfied in a (varying or constant domain, respectively) neighbourhood model based on a frame in $\mathcal{C}$. In the following, let $\mathsf{Log} = \{\mathbf{E}, \mathbf{M}, \mathbf{C}, \mathbf{N}\}$. Given $\mathbf{L} \in \mathsf{Log}$, the $\mathbf{L}^n_{\mathcal{ALC}}$ *formula satisfiability problem on* (*varying* or *constant domain*, respectively) *neighbourhood models* is the $ML^n_{\mathcal{ALC}}$ formula satisfiability problem on (varying or constant domain, respectively) neighbourhood models based on a frame in the class of:

- all neighbourhood frames, for $\mathbf{L} = \mathbf{E}$;
- supplemented neighbourhood frames, for $\mathbf{L} = \mathbf{M}$;
- neighbourhood frames closed under intersection, for $\mathbf{L} = \mathbf{C}$; and
- neighbourhood frames containing the unit, for $\mathbf{L} = \mathbf{N}$.

3. Tableaux for Non-normal Modal Description Logics

In this section, we provide terminating, sound and complete tableau algorithms to check satisfiability of formulas in varying domain neighbourhood models. The notation partly adheres to that of Gabbay et al. [28], while the model construction in the soundness proof is based on the strategy of Dalmonte et al. [29].

We require the following preliminary notions. For a concept or formula γ, we denote by $\dot{\neg}\gamma$ the negation of γ put in *negation normal form* (*NNF*), defined as usual. Given an $ML^n_{\mathcal{ALC}}$ formula φ, we assume without loss of generality that φ is in NNF, it contains CIs only of the form $\top \sqsubseteq C$, and every concept occurring in φ is also in NNF. We define the *weight* $|C|$ of a concept C in NNF as follows: $|A| = |\neg A| = 0$; $|\exists r.D| = |\forall r.D| = |\Diamond_i D| = |\Box_i D| = |D| + 1$; $|D \sqcap E| = |D \sqcup E| = |D| + |E| + 1$. The *weight* $|\varphi|$ of a formula φ in NNF is defined as: $|(C \sqsubseteq D)| = |\neg(C \sqsubseteq D)| = 0$; $\Box_i\psi = |\psi| + 1$; $|\psi \wedge \chi| = |\psi \vee \chi| = |\psi| + |\chi| + 1$. Observe that, for a concept or formula γ, we have that $|\gamma| = |\dot{\neg}\gamma|$. We denote by $\mathsf{con}(\varphi)$ and $\mathsf{for}(\varphi)$ the set of subconcepts and subformulas of φ, respectively, and then we set $\mathsf{con}_{\dot{\neg}}(\varphi) = \mathsf{con}(\varphi) \cup \{\dot{\neg}C \mid C \in \mathsf{con}(\varphi)\}$ and $\mathsf{for}_{\dot{\neg}}(\varphi) = \mathsf{for}(\varphi) \cup \{\dot{\neg}\psi \mid \psi \in \mathsf{for}(\varphi)\}$. The set $\mathsf{rol}(\varphi)$ is the set of role names occurring in φ. Let $\mathsf{Fg}(\varphi) = \mathsf{for}_{\dot{\neg}}(\varphi) \cup \mathsf{con}_{\dot{\neg}}(\varphi) \cup \mathsf{rol}(\varphi)$. Note that, by our assumption on the form of CIs in φ, we have $\top \in \mathsf{con}_{\dot{\neg}}(\varphi)$.

Moreover, let $\mathsf{N_V}$ be a countable set of *variables*, well-ordered by the relation $<$, and let $\mathsf{N_L}$ be a countable set of *labels*. Given an $ML^n_{\mathcal{ALC}}$ formula φ, an n*-labelled constraint for* φ takes the form $n : \psi$, or $n : C(x)$, or $n : r(x, y)$, where $n \in \mathsf{N_L}$, $\psi \in \mathsf{for}_{\dot{\neg}}(\varphi)$, $x, y \in \mathsf{N_V}$, $C \in \mathsf{con}_{\dot{\neg}}(\varphi)$, and $r \in \mathsf{rol}(\varphi)$. An n*-labelled constraint system for* φ is a set S_n of n-labelled constraints for φ. (A *labelled constraint for* φ is an n-labelled constraint for φ, for some $n \in \mathsf{N_L}$, and similarly for a *labelled constraint system for* φ). A *completion set* $\mathbf{T}$ is a non-empty union of labelled constraint systems for φ, and we set $\mathsf{L_T} = \{n \in \mathsf{N_L} \mid S_n \in \mathbf{T}\}$.

Concerning variables, we adopt the following terminology. A variable x *occurs in* S_n if S_n contains n-labelled constraints of the form $n : C(x)$ or $n : r(\tau, \tau')$, where $\tau = x$, or $\tau' = x$, and $n \in \mathsf{N_L}$. In addition, x is said to be *fresh for* S_n if x does not occur in S_n and $x > y$, for every y that occurs in S. (These notions can be used with respect to $\mathbf{T}$, whenever $S_n \subseteq \mathbf{T}$). Without loss of generality, we assume that, whenever x occurs in S_n, the n-labelled constraint $n : \top(x)$ is in S_n . Also, if $n : r(x, y) \in S_n$, we call y an r*-successor of* x with respect to S_n.

Finally, given variables x, y in an n-labelled constraint system S_n, we say that x is *blocked by y in S_n* if $x > y$ and $\{C \mid n : C(x) \in S_n\} \subseteq \{C \mid n : C(y) \in S_n\}$.

A completion set $\mathbf{T}$ contains a *clash* if $\{m : \psi, m : \neg\psi\} \subseteq \mathbf{T}$, or $\{m : C(x), m : \neg C(x)\} \subseteq \mathbf{T}$, for some $m \in \mathsf{N_L}$, and formula ψ or concept C. A completion set with no clash is *clash-free.* Given $\mathbf{L} \in \mathsf{Log}$, a completion set is $\mathbf{L}^n_{\mathcal{ALC}}$*-complete* if no $\mathbf{L}^n_{\mathcal{ALC}}$*-rule* from Figure 1 is applicable to $\mathbf{T}$, where γ_j is either $\psi_j \in \mathsf{for}_{\dot\neg}(\varphi)$ or $C_j(x_j)$, with $C_j \in \mathsf{con}_{\dot\neg}(\varphi)$, for $j = 1, \dots, k$, and δ is either $\chi \in \mathsf{for}_{\dot\neg}(\varphi)$ or $D(y)$, with $D \in \mathsf{con}_{\dot\neg}(\varphi)$, with respect to the following *application conditions* associated to each $\mathbf{L}^n_{\mathcal{ALC}}$-rule:

($\mathsf{R}_\wedge$) $\{n : \psi, n : \chi\} \not\subseteq \mathbf{T}$; ($\mathsf{R}_\sqcap$) $\{n : C(x), n : D(x)\} \not\subseteq \mathbf{T}$;

($\mathsf{R}_\vee$) $\{n : \psi, n : \chi\} \cap \mathbf{T} = \emptyset$; ($\mathsf{R}_\sqcup$) $\{n : C(x), n : D(x)\} \cap \mathbf{T} = \emptyset$;

($\mathsf{R}_\exists$) x is not blocked by any variable in S_n, there is no z such that $\{n : r(x, z), n : C(z)\} \subseteq \mathbf{T}$, and y is the $<$-minimal variable fresh for S_n;

($\mathsf{R}_\forall$) $n : C(y) \notin \mathbf{T}$;

($\mathsf{R}_=$) x occurs in an n-labelled constraint in $\mathbf{T}$ and $n : C(x) \notin \mathbf{T}$;

($\mathsf{R}_\neq$) x is the $<$-minimal variable fresh for S_n, and there is no y such that $n : \dot\neg C(y) \in \mathbf{T}$;

($\mathsf{R}_\mathbf{L}$) m is fresh for $\mathbf{T}$, and there is no $o \in \mathsf{N_L}$ such that $\{o : \gamma_1, \dots, o : \gamma_k, o : \delta\} \subseteq \mathbf{T}$, or $\{o : \dot\neg\gamma_j, o : \dot\neg\delta\} \subseteq \mathbf{T}$, for some $j \leq l$, where k and l are as in Figure 1.

The $\mathbf{L}^n_{\mathcal{ALC}}$-rules essentially state how to extend a completion set on the basis of the information contained in it. Branching rules entail a non-deterministic choice in the expansion of the completion set. For each $\mathbf{L} \in \mathsf{Log}$, we now define an algorithm based on $\mathbf{L}^n_{\mathcal{ALC}}$-rules for checking the $\mathbf{L}^n_{\mathcal{ALC}}$ formula satisfiability. We then prove that the algorithm terminates for every formula φ, and that it is sound and complete with respect to $\mathbf{L}^n_{\mathcal{ALC}}$ satisfiability.

Definition 1 ($\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm for φ)**.** *Given an $ML^n_{\mathcal{ALC}}$ formula φ, the* $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm for φ *runs as follows: set the initial completion set $\mathbf{T}_\varphi = \{0 : \varphi, 0 : \top(x)\}$, and expand it by means of the $\mathbf{L}^n_{\mathcal{ALC}}$-rules until a clash or an $\mathbf{L}^n_{\mathcal{ALC}}$-complete completion set is obtained.*

In the rest of this section, we prove termination, soundness and completeness of the tableau algorithms given above. We start by showing that the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm terminates.

Theorem 1 (Termination)**.** *Having started on the initial completion set $\mathbf{T}_\varphi = \{0 : \varphi, 0 : \top(x)\}$, the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm for φ terminates after at most $2^{p(|\mathsf{Fg}(\varphi)|)}$ steps, where p is a polynomial function.*

Proof. We first require the following claims.

Claim 1.1. *Let $\mathbf{T}$ be a completion set obtained by applying the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm for φ. For each $n \in \mathsf{L}_\mathbf{T}$, the number of n-labelled constraints for φ in $\mathbf{T}$ does not exceed $2^{q(|\mathsf{Fg}(\varphi)|)}$, where q is a polynomial function.*

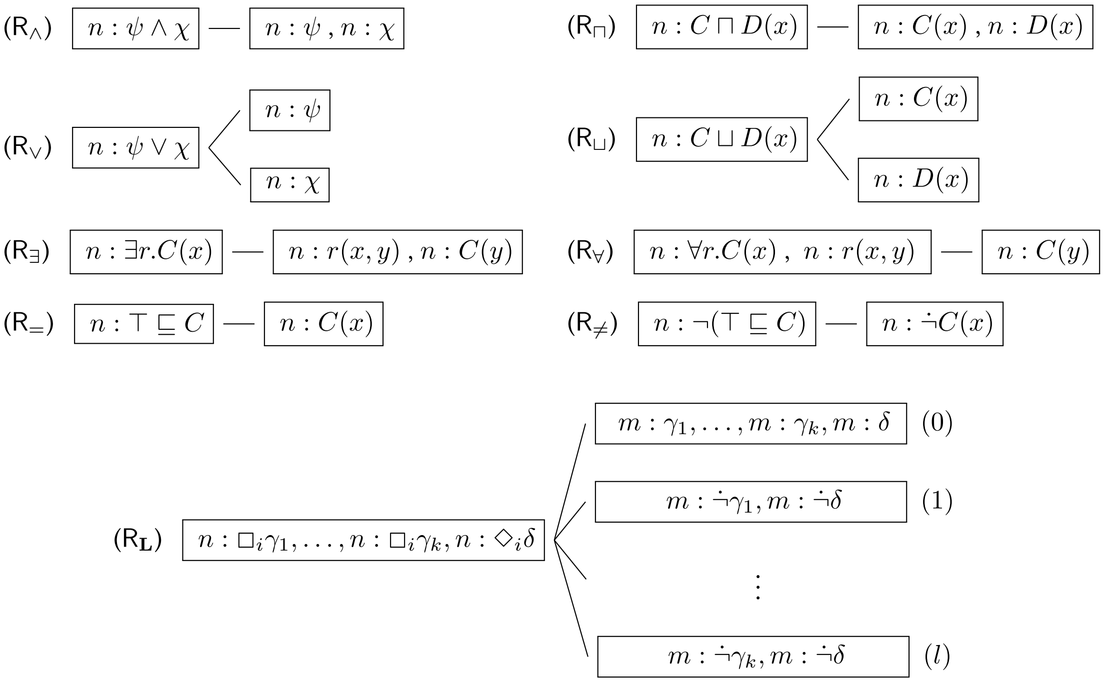

Figure 1: $\mathbf{L}^n_{\mathcal{ALC}}$-rules, where: for $\mathbf{L} = \mathbf{E}$, $k = l = 1$; for $\mathbf{L} = \mathbf{M}$, $k = 1$ and $l = 0$; for $\mathbf{L} = \mathbf{C}$, $k \geq 1$ and $l = k$; for $\mathbf{L} = \mathbf{N}$, $k = l = 1$ or $k = l = 0$;

Proof of Claim. We remark that, for each $S_n \subseteq \mathbf{T}$, the $\mathbf{L}^n_{\mathcal{ALC}}$ tableaux algorithm behaves exactly like a standard (non-modal) $\mathcal{ALC}$ tableaux algorithm (cf. e.g. [28, Theorem 15.4], noting also that in our case we do not have to deal with individual names). □

Claim 1.2. *Let* $\mathbf{T}$ *be a completion set obtained by applying the* $\mathbf{L}^n_{\mathcal{ALC}}$ *tableau algorithm for* φ*. For* $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$, $|\mathsf{L}_\mathbf{T}| \leq |\mathsf{Fg}(\varphi)|^2$*. For* $\mathbf{L} = \mathbf{C}$, $|\mathsf{L}_\mathbf{T}| \leq 2^{|\mathsf{Fg}(\varphi)|} \cdot |\mathsf{Fg}(\varphi)|$.

Proof of Claim. Labels n are generated in $\mathbf{T}$ by means of the application of the rule $\mathsf{R}_\mathbf{L}$. For $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$, this rule is applied to two n-labelled contraints $n : \Box_i\gamma, n : \Diamond_i\delta$ (for $\mathbf{L} = \mathbf{N}$ possibly also to a single constraint $n : \Diamond_i\delta$), while for $\mathbf{L} = \mathbf{C}$ it is applied to $k + 1$ n-labelled contraints $n : \Box_i\gamma_1, ...n : \Box_i\gamma_k, n : \Diamond_i\delta$. By the application condition of $\mathsf{R}_\mathbf{L}$, each such combination of constraints generates at most one label m. Therefore, the number of labels that can be generated in $\mathbf{T}$ is bounded by the number of possible such combinations, which is at most $|\mathsf{Fg}(\varphi)|^2$, for $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$, and at most $2^{|\mathsf{Fg}(\varphi)|} \cdot |\mathsf{Fg}(\varphi)|$, for $\mathbf{L} = \mathbf{C}$. □

The theorem is then a consequence of the following observations. Given a completion set $\mathbf{T}$ constructed by the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm, we have by Claim 1.2 that the number of applications of rule $\mathsf{R}_\mathbf{L}$ is bounded by $|\mathsf{L}_\mathbf{T}|$, which is at most $|\mathsf{Fg}(\varphi)|^2$, for $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$, and at most $2^{|\mathsf{Fg}(\varphi)|} \cdot |\mathsf{Fg}(\varphi)|$, for $\mathbf{L} = \mathbf{C}$. Moreover, since every application of the rules $\mathsf{R}_\wedge$ and $\mathsf{R}_\vee$ introduces a new formula to an n-labelled constraint, the total number of such rule applications is bounded by $|\mathsf{L}_\mathbf{T}| \cdot |\mathsf{Fg}(\varphi)|$. Finally, by Claim 1.1, the number of applications of rules $\mathsf{R}_\sqcap, \mathsf{R}_\sqcup, \mathsf{R}_\forall, \mathsf{R}_\exists, \mathsf{R}_=, \mathsf{R}_{\neq}$ per label n is bounded by $2^{q(|\mathsf{Fg}(\varphi)|)}$, where q is a polynomial function, since these rules add a new constraint to an n-labelled constraint system. Thus, the overall number of such rule applications is bounded by $|\mathsf{L}_{\mathbf{T}}| \cdot 2^{q(|\mathsf{Fg}(\varphi)|)}$. $\square$

We now proceed to prove that the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm is sound.

Theorem 2 (Soundness). *If, having started on the initial completion set* $\mathbf{T}_\varphi$*, the* $\mathbf{L}^n_{\mathcal{ALC}}$ *tableau algorithm constructs an* $\mathbf{L}^n_{\mathcal{ALC}}$*-complete and clash-free completion set for* φ*, then* φ *is* $\mathbf{L}^n_{\mathcal{ALC}}$ *satisfiable.*

Proof. Given an $\mathbf{L}^n_{\mathcal{ALC}}$-complete and clash-free completion set $\mathbf{T}$ for φ, define, for $n \in \mathsf{L}_{\mathbf{T}}$, $\psi \in \mathsf{for}_{\dot\neg}(\varphi)$, $C \in \mathsf{con}_{\dot\neg}(\varphi)$, and x occurring in $\mathbf{T}$,

$$\lfloor C \rfloor_x = \{n \in \mathsf{L}_{\mathbf{T}} \mid n : C(x) \in S_n\}, \qquad \lfloor \psi \rfloor = \{n \in \mathsf{L}_{\mathbf{T}} \mid n : \psi \in S_n\},$$

$$\lceil C \rceil_x = \mathsf{L}_{\mathbf{T}} \setminus \{n \in \mathsf{L}_{\mathbf{T}} \mid n : \dot\neg C(x) \in S_n\}, \qquad \lceil \psi \rceil = \mathsf{L}_{\mathbf{T}} \setminus \{n \in \mathsf{L}_{\mathbf{T}} \mid n : \dot\neg \psi \in S_n\}.$$

Moreover, define $\Gamma^x_n = \{\psi \mid n : \psi \in S_n\} \cup \{C \mid n : C(x) \in S_n\}$ and let γ, δ range over $ML^n_{\mathcal{ALC}}$ formulas or concepts, where: $\lfloor \gamma \rfloor_x = \lfloor \psi \rfloor$, if $\gamma = \psi$, and $\lfloor \gamma \rfloor_x = \lfloor C \rfloor_x$, if $\gamma = C$; and similarly for $\lceil \gamma \rceil_x$. We set $\mathcal{M} = (\mathcal{F}, \mathcal{I})$, with $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I})$ and $\mathcal{I}_n = (\Delta_n, \cdot^{\mathcal{I}_n})$, for $n \in \mathcal{W}$, defined as follows:

- $\mathcal{W} = \mathsf{L}_{\mathbf{T}}$;
- for every $i \in I = \{1, \ldots, n\}$, we set $\mathcal{N}_i \colon \mathcal{W} \to 2^{2^{\mathcal{W}}}$ such that:
  - for $\mathbf{L} = \mathbf{E}$: $\mathcal{N}_i(n) = \big\{\alpha \mid \text{for some } \Box_i\gamma \in \Gamma^x_n \colon \lfloor \gamma \rfloor_x \subseteq \alpha \subseteq \lceil \gamma \rceil_x\big\}$;
  - for $\mathbf{L} = \mathbf{M}$: $\mathcal{N}_i(n) = \big\{\alpha \mid \text{for some } \Box_i\gamma \in \Gamma^x_n \colon \lfloor \gamma \rfloor_x \subseteq \alpha\big\}$;
  - for $\mathbf{L} = \mathbf{C}$: $\mathcal{N}_i(n) = \big\{\alpha \mid \text{for some } \Box_i\gamma_1 \in \Gamma^{x_1}_n, \ldots, \Box_i\gamma_k \in \Gamma^{x_k}_n :$
    $\bigcap_{j=1}^k \lfloor \gamma_j \rfloor_{x_j} \subseteq \alpha \subseteq \bigcap_{j=1}^k \lceil \gamma_j \rceil_{x_j}\big\}$;
  - for $\mathbf{L} = \mathbf{N}$: $\mathcal{N}_i(n) = \big\{\alpha \mid \text{for some } \Box_i\gamma \in \Gamma^x_n \colon \lfloor \gamma \rfloor_x \subseteq \alpha \subseteq \lceil \gamma \rceil_x\big\} \cup \mathcal{W}$;
- $\Delta_n = \{x \in \mathsf{N_V} \mid x \text{ occurs in } S_n\}$;
- $A^{\mathcal{I}_n} = \{x \in \Delta_n \mid n : A(x) \in S_n\}$;
- $r^{\mathcal{I}_n} = \{(x, y) \in \Delta_n \times \Delta_n \mid n : r(x, y) \in S_n$ or $n : r(z, y) \in S_n$, for some z blocking x in S_n $\}$.

First, we observe the following.

- For $\mathbf{L} = \mathbf{M}$, we have that $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ is such that $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I})$ is supplemented. Indeed, for all $n \in \mathcal{W}$, $\alpha, \beta \subseteq \mathcal{W}$, suppose that $\alpha \in \mathcal{N}_i(n)$ and $\alpha \subseteq \beta$. By definition, this implies that: for some $\Box_i\gamma \in \Gamma^x_n$, $\lfloor \gamma \rfloor_x \subseteq \alpha \subseteq \beta$. Hence, $\beta \in \mathcal{N}_i(n)$.

- For $\mathbf{L} = \mathbf{C}$, we have that $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ is such that $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I})$ is closed under intersection. Indeed, for all $n \in \mathcal{W}$, $\alpha, \beta \subseteq \mathcal{W}$, suppose that $\alpha \in \mathcal{N}_i(n)$ and $\beta \in \mathcal{N}_i(n)$. Now suppose that, for some $\Box_i\gamma_1 \in \Gamma_n^{x_1}, \dots, \Box_i\gamma_k \in \Gamma_n^{x_k}$: $\bigcap_{j=1}^{k} \lfloor \gamma_j \rfloor_{x_j} \subseteq \alpha \subseteq \bigcap_{j=1}^{k} \lceil \gamma_j \rceil_{x_j}$ and, for some $\Box_i\delta_1 \in \Gamma_n^{y_1}, \dots, \Box_i\delta_h \in \Gamma_n^{y_h}$: $\bigcap_{j=1}^{h} \lfloor \delta_j \rfloor_{y_j} \subseteq \beta \subseteq \bigcap_{j=1}^{h} \lceil \delta_j \rceil_{y_j}$. Then for some $\Box_i\gamma_1 \in \Gamma_n^{x_1}, \dots, \Box_i\gamma_k \in \Gamma_n^{x_k}$ and some $\Box_i\delta_1 \in \Gamma_n^{y_1}, \dots, \Box_i\delta_h \in \Gamma_n^{y_h}$ the following holds, which in turn implies that $\alpha \cap \beta \in \mathcal{N}_i(n)$:
$$\bigcap_{j=1}^{k} \lfloor \gamma_j \rfloor_{x_j} \cap \bigcap_{j=1}^{h} \lfloor \delta_j \rfloor_{y_j} \subseteq \alpha \cap \beta \subseteq \bigcap_{j=1}^{k} \lceil \gamma_j \rceil_{x_j} \cap \bigcap_{j=1}^{h} \lceil \delta_j \rceil_{y_j}$$
- For $\mathbf{L} = \mathbf{N}$, we have that $\mathcal{M} = (\mathcal{F}, \mathcal{I})$, with $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I})$, is such that $\mathcal{F}$ contains the unit. Indeed, by construction, for all $n \in \mathcal{W}$, $\mathcal{W} \in \mathcal{N}_i(n)$.

We then require the following claims.

Claim 2.1. *For every $n \in \mathcal{W}$, $C \in \mathsf{con}_{\dot\neg}(\varphi)$, and $x \in \Delta_n$: if $n : C(x) \in S_n$, then $x \in C^{\mathcal{I}_n}$.*

Proof of Claim. We show the claim by induction on the weight of C (in NNF). The base case of $C = A$ comes immediately from the definitions. For the base case of $C = \neg A$, suppose that $n : \neg A(x) \in S_n$. Since $\mathbf{T}$ is clash-free, we have that $n : A(x) \notin S_n$, and thus $x \notin A^{\mathcal{I}_n}$ by definition of $A^{\mathcal{I}_n}$, meaning $x \in (\neg A)^{\mathcal{I}_n}$. The inductive cases of $C = D \sqcap E$ and $C = D \sqcup E$ come from the fact that S_n is closed under $\mathsf{R}_\sqcap$ and $\mathsf{R}_\sqcup$, respectively, and straightforward applications of the inductive hypothesis. We show the remaining cases (cf. also [28, Claim 15.2]).

$C = \exists r.D$. Let $n : \exists r.D(x) \in S_n$, meaning that $\exists r.D \in \Gamma_n^x$. We distinguish two cases.

- x is not blocked by any variable in S_n. Since S_n is closed under $\mathsf{R}_\exists$, there exists y occurring in S_n such that $n : r(x, y) \in S_n$ and $n : D(y) \in S_n$. Thus, by definition, $(x, y) \in r^{\mathcal{I}_n}$ and $n : D(y) \in S_n$. By inductive hypothesis, we obtain that $x \in (\exists r.D)^{\mathcal{I}_n}$.
- x is blocked by a variable in S_n, implying that there exists a $<$-minimal (since $<$ is a well-ordering) y occurring in S_n such that $y < x$ and $\{E \mid n : E(x) \in S_n\} \subseteq \{E \mid n : E(y) \in S_n\}$. In turn, this implies that y is not blocked by any other variable z in S_n (for otherwise z would block x, with $z < y$, against the fact that y is $<$-minimal). By reasoning as in the case above, since y is not blocked and S_n is closed under $\mathsf{R}_\exists$, we have a variable z occurring in S_n such that $n : r(y, z) \in S_n$ and $n : D(x) \in S_n$. Since y blocks x, by definition we have that $(x, z) \in r^{\mathcal{I}_n}$, and by inductive hypothesis we get from $n : D(z)$ that $z \in D^{\mathcal{I}_n}$. Thus, $x \in (\exists r.D)^{\mathcal{I}_n}$.

$C = \forall r.D$. Let $n : \forall r.D(x) \in S_n$, meaning that $\forall r.D \in \Gamma_n^x$, and suppose that $(x, y) \in r^{\mathcal{I}_n}$. By definition, either $n : r(x, y) \in S_n$ or $n : r(z, y) \in S_n$, for some z blocking x in S_n. In the former case, since S_n is closed under $\mathsf{R}_\forall$, we get that $n : D(y) \in S_n$. In the latter case, since z blocks x in S_n, we obtain $n : \forall r.D(z) \in S_n$; again, since S_n is closed under $\mathsf{R}_\forall$, this implies that $n : D(y) \in S_n$. Hence, in both cases, we have $n : D(y) \in S_n$. By inductive hypothesis, this means that $y \in D^{\mathcal{I}_n}$. Since y was arbitrary, we conclude that $x \in (\forall r.D)^{\mathcal{I}_n}$.

$C = \Box_i D$. Let $n : \Box_i D(x) \in S_n$, meaning that $\Box_i D \in \Gamma_n^x$. Consider $\mathbf{L} \in \mathsf{Log}$.

$\mathbf{L} = \mathbf{E}$. We have by inductive hypothesis that $\lfloor D \rfloor_x = \{n \in \mathcal{W} \mid n : D(x) \in S_n\} \subseteq \{n \in \mathcal{W} \mid x \in D^{\mathcal{I}_n}\} = [\![D]\!]_x^{\mathcal{M}}$. By inductive hypothesis (since $|\dot\neg D| = |D|$), we also have that

$\{n \in \mathcal{W} \mid n : \dot{\neg} D(x) \in S_n\} \subseteq \{n \in \mathcal{W} \mid x \in (\dot{\neg} D)^{\mathcal{I}_n}\} = \llbracket \dot{\neg} D \rrbracket_x^{\mathcal{M}} = \mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}}$. Hence, $\llbracket D \rrbracket_x^{\mathcal{M}} \subseteq \mathcal{W} \setminus \{w \in \mathcal{W} \mid n : \dot{\neg} D(x) \in S_n\} = \lceil D \rceil_x$. In conclusion, we have $\Box_i D \in \Gamma_n^x$ such that $\lfloor D \rfloor_x \subseteq \llbracket D \rrbracket_x^{\mathcal{M}} \subseteq \lceil D \rceil_x$. Thus, by definition, $\llbracket D \rrbracket_x^{\mathcal{M}} \in \mathcal{N}_i(n)$, as required.

$\mathbf{L} = \mathbf{M}$. We have by inductive hypothesis that $\lfloor D \rfloor_x = \{n \in \mathcal{W} \mid n : D(x) \in S_n\} \subseteq \{n \in \mathcal{W} \mid x \in D^{\mathcal{I}_n}\} = \llbracket D \rrbracket_x^{\mathcal{M}}$. Thus, we have $\Box_i D \in \Gamma_n^x$ such that $\lfloor D \rfloor_x \subseteq \llbracket D \rrbracket_x^{\mathcal{M}}$. By definition, this means $\llbracket D \rrbracket_x^{\mathcal{M}} \in \mathcal{N}_i(n)$, as required.

$\mathbf{L} \in \{\mathbf{C}, \mathbf{N}\}$. These cases are analogous to the case for $\mathbf{L} = \mathbf{E}$.

$C = \Diamond_i D$. Let $n : \Diamond_i D(x) \in S_n$. Consider $\mathbf{L} \in \mathsf{Log}$.

$\mathbf{L} \in \{\mathbf{E}, \mathbf{M}\}$. We distinguish two cases. (i) There exists no $\Box_i \gamma \in \Gamma_n^y$. This means that $\mathcal{N}_i(n) = \emptyset$. Thus, $\mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}} \notin \mathcal{N}_i(n)$, meaning that $x \in (\Diamond_i D)^{\mathcal{I}_n}$. (ii) There exists $\Box_i \gamma \in \Gamma_n^y$. We then reason similarly to the case for $\mathbf{L} = \mathbf{C}$.

$\mathbf{L} = \mathbf{C}$. We distinguish two cases. (i) There exist no $\Box_i \gamma_1 \in \Gamma_n^{y_1}, \ldots, \Box_i \gamma_k \in \Gamma_n^{y_k}$. As for $\mathbf{L} = \mathbf{E}$, we obtain $x \in (\Diamond_i D)^{\mathcal{I}_n}$. (ii) There exist $\Box_i \gamma_1 \in \Gamma_n^{y_1}, \ldots, \Box_i \gamma_k \in \Gamma_n^{y_k}$. Since $\mathbf{T}$ is $\mathbf{L}^n_{\mathcal{ALC}}$-complete, there exists $m \in \mathcal{W}$ such that: $\gamma_1 \in \Gamma_m^{y_1}, \ldots, \gamma_k \in \Gamma_m^{y_k}$ and $D \in \Gamma_m^x$; or $\dot{\neg} \gamma_j \in \Gamma_m^{y_j}$ and $\dot{\neg} D \in \Gamma_m^x$, for some $j \leq k$. By inductive hypothesis, the previous step implies that there exists $m \in \mathcal{W}$ such that: $\gamma_1 \in \Gamma_m^{y_1}, \ldots, \gamma_k \in \Gamma_m^{y_k}$ and $x \in D^{\mathcal{I}_m}$; or $\dot{\neg} \gamma_j \in \Gamma_m^{y_j}$ and $x \in \dot{\neg} D^{\mathcal{I}_m}$, for some $j \leq k$. Equivalently, it is not the case that, for every $v \in \mathcal{W}$: $\gamma_1 \in \Gamma_m^{y_1}, \ldots, \gamma_k \in \Gamma_m^{y_k}$ implies $x \notin D^{\mathcal{I}_m}$; and for all $j \leq k$, $x \in \dot{\neg} D^{\mathcal{I}_m}$ implies $\dot{\neg} \gamma_j \notin \Gamma_m^{y_j}$. In other words, it is not the case that: $\bigcap_{j=1}^{k} \lfloor \gamma_j \rfloor_{y_j} \subseteq \mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}}$; and $\mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}} \subseteq \bigcap_{j=1}^{k} \lceil \gamma_l \rceil_{y_l}$. Thus, $\mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}} \notin \mathcal{N}_i(n)$, i.e., $x \in (\Diamond_i D)^{\mathcal{I}_n}$, as required.

$\mathbf{L} = \mathbf{N}$. We distinguish two cases. (i) There exists no $\Box_i \gamma \in \Gamma_n^y$. This means that $\mathcal{N}_i(n) = \mathcal{W}$. Since $\mathbf{T}$ is $\mathbf{L}^n_{\mathcal{ALC}}$-complete, there exists $m \in \mathcal{W}$ such that $D \in \Gamma_m^x$, i.e., $m : D(x) \in S_m$. By inductive hypothesis, this implies $x \in D^{\mathcal{I}_m}$, that is, $\llbracket D \rrbracket_x^{\mathcal{M}} \neq \emptyset$. This holds iff $\mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}} \neq \mathcal{W}$, and thus $\mathcal{W} \setminus \llbracket D \rrbracket_x^{\mathcal{M}} \notin \mathcal{N}_i(n)$. Hence, $x \in (\Diamond_i D)^{\mathcal{I}_n}$. (ii) There exists $\Box_i \gamma \in \Gamma_n^y$. We then reason similarly to the case for $\mathbf{L} = \mathbf{C}$. ☐

Claim 2.2. *For every $w \in \mathcal{W}$ and $\psi \in \mathsf{con}_{\dot{\neg}}(\varphi)$: if $n : \psi \in S_n$, then $\mathcal{M}, w \models \psi$.*

Proof of Claim. We prove the claim by induction on the weight of φ (in NNF).

$\psi = (\top \sqsubseteq C)$. Let $n : \top \sqsubseteq C \in S_n$ and let $x \in \Delta_n$. Since S_n is closed under $(\mathsf{R}_=)$ and x occurs in S_n, we have that $n : C(x) \in S_n$. By Claim 2.1, we have that $x \in C^{\mathcal{I}_n}$. Given that x is arbitrary, we conclude that $\mathcal{M}, n \models \top \sqsubseteq C$.

$\psi = \neg(\top \sqsubseteq C)$. Let $n : \neg(\top \sqsubseteq C) \in S_n$. Since S_n is closed under $(\mathsf{R}_{\neq})$, there exists x occurring in S_n such that $n : \dot{\neg} C(x) \in S_n$. By Claim 2.1, we obtain that $x \in (\dot{\neg} C)^{\mathcal{I}_n}$, for some $x \in \Delta_w$. Hence, $\mathcal{M}, n \models \neg(\top \sqsubseteq C)$.

The inductive cases of $\psi = \chi \wedge \vartheta$ and $\psi = \chi \vee \vartheta$ follow from the definitions and straighforward applications of the inductive hypothesis. Moreover the inductive cases of $\psi = \Box_i \chi$ and $\psi = \Diamond_i \chi$ can be proved analogously to Claim 2.1. ☐

Since, by definition, we have $0 : \varphi \in S_0 \subseteq \mathbf{T}$, thanks to Claim 2.2 we obtain $\mathcal{M}, 0 \models \varphi$. ☐

We finally show completeness of the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm.

Theorem 3 (Completeness). *If φ is $\mathbf{L}^n_{\mathcal{ALC}}$ satisfiable, then, having started on the initial completion set $\mathbf{T}_\varphi$, the $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm constructs an $\mathbf{L}^n_{\mathcal{ALC}}$-complete and clash-free completion set for φ.*

Proof. Let $\mathcal{M} = (\mathcal{F}, \mathcal{I})$ be an $\mathbf{L}^n_{\mathcal{ALC}}$-model satisfying φ, with $\mathcal{F} = (\mathcal{W}, \{\mathcal{N}\}_{i\in I})$, i.e., $\mathcal{M}, w_\varphi \models \varphi$, for some $w_\varphi \in \mathcal{W}$. We require the following definitions and technical results. First, we let γ, δ (possibly indexed) range over $ML^n_{\mathcal{ALC}}$ concepts and formulas, with $[\![\gamma]\!]^{\mathcal{M}}_d = [\![\psi]\!]^{\mathcal{M}}$, if $\gamma = \psi$, and $[\![\gamma]\!]^{\mathcal{M}}_d = [\![C]\!]^{\mathcal{M}}_d$, if $\gamma = C$. Then, for $w \in \mathcal{W}$ and $d \in \bigcup_{v\in\mathcal{W}} \Delta_v$, define $\Phi^d_w = \{\psi \in \mathsf{for}_{\dot{\neg}}(\varphi) \mid \mathcal{M}, w \models \psi\} \cup \{C \in \mathsf{con}_{\dot{\neg}}(\varphi) \mid d \in C^{\mathcal{I}_w}\}$. Observe that, if $C \in \Phi^d_w$, then $d \in \Delta_w$. We now show that the following holds.

Claim 3.1. *For every $w \in \mathcal{W}$ and every $d_1, \dots, d_k, e \in \bigcup_{v\in\mathcal{W}} \Delta_v$: if $\Box_i\gamma_1 \in \Phi^{d_1}_w, \dots, \Box_i\gamma_k \in \Phi^{d_k}_w$ and $\Diamond_i\delta \in \Phi^e_w$, then there exists $v \in \mathcal{W}$ such that:*

(0) *$\gamma_1 \in \Phi^{d_1}_v, \dots, \gamma_k \in \Phi^{d_k}_v$ and $\delta \in \Phi^e_v$; or*

(1) *$\dot{\neg}\gamma_1 \in \Phi^{d_1}_v$ and $\dot{\neg}\delta \in \Phi^e_v$; or*

$\vdots$

(l) *$\dot{\neg}\gamma_l \in \Phi^{d_k}_v$ and $\dot{\neg}\delta \in \Phi^e_v$;*

where: for $\mathbf{L} = \mathbf{E}$, $k = l = 1$; for $\mathbf{L} = \mathbf{M}$, $k = 1$ and $l = 0$; for $\mathbf{L} = \mathbf{C}$, $k \geq 1$ and $l = k$; for $\mathbf{L} = \mathbf{N}$, $k = l = 1$ or $k = l = 0$.

Proof. We consider each $\mathbf{L} \in \mathsf{Log}$.

$\mathbf{L} = \mathbf{E}$. Assume $\Box_i\gamma \in \Phi^d_w, \Diamond_i\delta \in \Phi^e_w$, meaning that $[\![\gamma]\!]^{\mathcal{M}}_d \in \mathcal{N}_i(w)$ and $\mathcal{W} \setminus [\![\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$, i.e., $[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$. Towards a contradiction, suppose that, for every $v \in \mathcal{W}$, the following holds: ($\gamma \notin \Phi^d_v$ or $\delta \notin \Phi^e_v$) and ($\dot{\neg}\gamma \notin \Phi^d_v$ or $\dot{\neg}\delta \notin \Phi^e_v$). Equivalently, for every $v \in \mathcal{W}$: ($\gamma \in \Phi^d_v$ implies $\delta \notin \Phi^e_v$) and ($\dot{\neg}\delta \in \Phi^e_v$ implies $\dot{\neg}\gamma \notin \Phi^d_v$). By definition, we have that $\gamma \in \Phi^d_v$ iff $\dot{\neg}\gamma \notin \Phi^d_v$ and $\delta \notin \Phi^e_v$ iff $\dot{\neg}\delta \in \Phi^e_v$. Thus, the previous step means: ($[\![\gamma]\!]^{\mathcal{M}}_d \subseteq [\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e$) and ($[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \subseteq [\![\gamma]\!]^{\mathcal{M}}_d$), i.e., $[\![\gamma]\!]^{\mathcal{M}}_d = [\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e$, contradicting the assumption that $[\![\gamma]\!]^{\mathcal{M}}_d \in \mathcal{N}_i(w)$ and $[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$.

$\mathbf{L} = \mathbf{M}$. Assume $\Box_i\gamma \in \Phi^d_w, \Diamond_i\delta \in \Phi^e_w$, meaning that $[\![\gamma]\!]^{\mathcal{M}}_d \in \mathcal{N}_i(w)$ and $\mathcal{W} \setminus [\![\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$, i.e., $[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$. Towards a contradiction, suppose that, for every $v \in \mathcal{W}$, the following holds: $\gamma \notin \Phi^d_v$ or $\delta \notin \Phi^e_v$. Equivalently, for every $v \in \mathcal{W}$: $\gamma \in \Phi^d_v$ implies $\delta \notin \Phi^e_v$. By definition, the previous step means $[\![\gamma]\!]^{\mathcal{M}}_d \subseteq [\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e$. Since $\mathcal{M}$ is supplemented, we have that $[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \in \mathcal{N}_i(w)$, which is impossible.

$\mathbf{L} = \mathbf{C}$. Assume $\Box_i\gamma_1 \in \Phi^{d_1}_w, \dots, \Box_i\gamma_k \in \Phi^{d_k}_w, \Diamond_i\delta \in \Phi^e_w$, meaning that $[\![\gamma_j]\!]^{\mathcal{M}}_{d_j} \in \mathcal{N}_i(w)$, for $j = 1, \dots, k$, and $\mathcal{W} \setminus [\![\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$, i.e., $[\![\dot{\neg}\delta]\!]^{\mathcal{M}}_e \notin \mathcal{N}_i(w)$. Towards a contradiction, suppose that, for every $v \in \mathcal{W}$, none of the following holds: (0) $\gamma_1 \in \Phi^{d_1}_v, \dots, \gamma_k \in \Phi^{d_k}_v$ and $\delta \in \Phi^e_v$; (1) $\dot{\neg}\gamma_1 \in \Phi^{d_1}_v$ and $\dot{\neg}\delta \in \Phi^e_v$; ...; ($k$) $\dot{\neg}\gamma_k \in \Phi^{d_k}_v$ and $\dot{\neg}\delta \in \Phi^e_v$. Equivalently, for every $v \in \mathcal{W}$, it holds that (0) $\gamma_1 \in \Phi_v^{d_1}, \ldots, \gamma_k \in \Phi_v^{d_k}$ implies $\delta \notin \Phi_v^e$; and (1) $\dot{\neg}\delta \in \Phi_v^e$ implies $\dot{\neg}\gamma_1 \notin \Phi_v^{d_1}$; ... and (k) $\dot{\neg}\delta \in \Phi_v^e$ implies $\dot{\neg}\gamma_k \notin \Phi_v^{d_k}$. By definition, from the previous step we obtain (0) $\bigcap_{j=1}^{k} [\![\gamma_j]\!]_{d_j}^{\mathcal{M}} \subseteq [\![\dot{\neg}\delta]\!]_e^{\mathcal{M}}$; and (1) $[\![\dot{\neg}\delta]\!]_e^{\mathcal{M}} \subseteq [\![\gamma_1]\!]_{d_1}^{\mathcal{M}}$; ... and (k) $[\![\dot{\neg}\delta]\!]_e^{\mathcal{M}} \subseteq [\![\gamma_k]\!]_{d_k}^{\mathcal{M}}$. Hence $\bigcap_{j=1}^{k} [\![\gamma_j]\!]_{d_j}^{\mathcal{M}} = [\![\dot{\neg}\delta]\!]_e^{\mathcal{M}}$. Since $\mathcal{M}$ is closed under intersection, we obtain $[\![\dot{\neg}\delta]\!]_e^{\mathcal{M}} \in \mathcal{N}_i(w)$, a contradiction.

$\mathbf{L} = \mathbf{N}$. We distinguish two cases: (i) Let $k = l = 0$. That is, there exists no $\Box_i\gamma \in \Phi_w^d$, while $\Diamond_i\delta \in \Phi_w^e$, meaning that $\mathcal{W} \setminus [\![\delta]\!]_e^{\mathcal{M}} \notin \mathcal{N}_i(w)$. Towards a contradiction, suppose that, for every $v \in \mathcal{W}$, $\delta \notin \Phi_v^e$. Since, by definition, we have $\delta \notin \Phi_v^e$ iff $\dot{\neg}\delta \in \Phi_v^e$, the previous step means that $\mathcal{W} \subseteq [\![\dot{\neg}\delta]\!]_e^{\mathcal{M}}$, and hence $[\![\delta]\!]_e^{\mathcal{M}} = \emptyset$. Thus, $\mathcal{W} \notin \mathcal{N}_i(w)$, contradicting the fact that $\mathcal{M}$ contains the unit. (ii) Let $k = l = 1$. Hence, there exists $\Box_i\gamma \in \Phi_w^e$ and $\Diamond_i\delta \in \Phi_w^e$. We then reason similarly to the case for $\mathbf{L} = \mathbf{E}$. ☐

Given a completion set $\mathbf{T}$ for φ and $S_n \subseteq \mathbf{T}$, let $\Gamma_n^x = \{\psi \mid n : \psi \in S_n\} \cup \{C \mid n : C(x) \in S_n\}$. We say that a completion set $\mathbf{T}$ for φ is *$\mathcal{M}$-compatible* if there exists a function π from $\mathsf{L}_{\mathbf{T}}$ to $\mathcal{W}$, and, for every $n \in \mathsf{L}_{\mathbf{T}}$, there exists a function π_n from the set of variables occurring in S_n to $\Delta_{\pi(n)}$, such that $\gamma \in \Gamma_n^x$ implies $\gamma \in \Phi_{\pi(n)}^{\pi_n(x)}$. We then require the following claim.

Claim 3.2. *If a completion set $\mathbf{T}$ for φ is $\mathcal{M}$-compatible, then for every $\mathbf{L}_{\mathcal{ALC}}^n$-rule R applicable to $\mathbf{T}$ there exists a completion set $\mathbf{T}'$ obtained from $\mathbf{T}$ by an application of R such that $\mathbf{T}'$ is $\mathcal{M}$-compatible.*

Proof. Given an $\mathcal{M}$-compatible completion set $\mathbf{T}$ for φ and a label $n \in \mathsf{L}_{\mathbf{T}}$, let π and π_n be the functions provided by the definition of $\mathcal{M}$-compatibility. We need to consider each $\mathbf{L}_{\mathcal{ALC}}^n$-rule R. For $\mathsf{R} \in \{\mathsf{R}_\wedge, \mathsf{R}_\vee, \mathsf{R}_\sqcap, \mathsf{R}_\sqcup, \mathsf{R}_\forall, \mathsf{R}_\exists, \mathsf{R}_=, \mathsf{R}_\neq\}$, we proceed similarly to [28, Claim 15.14]. Here we consider the case of $\mathsf{R}_{\mathbf{L}}$: Suppose that $\mathsf{R}_{\mathbf{L}}$ is applicable to $\mathbf{T}$. Let $\Box_i\gamma_1 \in \Gamma_n^{x_1}, \ldots, \Box_i\gamma_k \in \Gamma_n^{x_k}, \Diamond_i\delta \in \Gamma_n^y$. Since $\mathbf{T}$ is $\mathcal{M}$-compatible, we have that $\Box_i\gamma_1 \in \Phi_{\pi(n)}^{\pi_n(x_1)}, \ldots, \Box_i\gamma_k \in \Phi_{\pi(n)}^{\pi_n(x_k)}$ and $\Diamond_i\delta \in \Phi_{\pi(n)}^{\pi_n(y)}$. Thus, by Claim 3.1, there exists $v \in \mathcal{W}$ such that: $\gamma_1 \in \Phi_v^{\pi_n(x_1)}, \ldots, \gamma_k \in \Phi_v^{\pi_n(x_k)}$ and $\delta \in \Phi_v^{\pi_n(y)}$; or $\dot{\neg}\gamma_j \in \Phi_v^{\pi_n(x_j)}$ and $\dot{\neg}\delta \in \Phi_v^{\pi_n(y)}$, for some $j \leq l$; where: for $\mathbf{L} = \mathbf{E}$, $k = l = 1$; for $\mathbf{L} = \mathbf{M}$, $k = 1$ and $l = 0$; for $\mathbf{L} = \mathbf{C}$, $k \geq 1$ and $l = k$; for $\mathbf{L} = \mathbf{N}$, $k = l = 1$ or $k = l = 0$. By applying the rule $\mathsf{R}_{\mathbf{L}}$ accordingly, one can obtain $\mathbf{T}'$ by adding $m : \gamma_1, \ldots, m : \gamma_k, m : \delta$, or $m : \dot{\neg}\gamma_j, m : \dot{\neg}\delta$, for some $j \leq l$, to $\mathbf{T}$ (recall that m is fresh for $\mathbf{T}$ and γ_j is either $\psi_j \in \mathsf{for}_{\dot{\neg}}(\varphi)$ or $C_j(x_j)$, with $C_j \in \mathsf{con}_{\dot{\neg}}(\varphi)$, for $j = 1, \ldots, k$, and δ is either $\chi \in \mathsf{for}_{\dot{\neg}}(\varphi)$ or $D(y)$, with $D \in \mathsf{con}_{\dot{\neg}}(\varphi)$). By extending π with $\pi(m) = v$, and π_m with $\pi_m(x_1) = \pi_n(x_1)$, ..., $\pi_m(x_k) = \pi_n(x_k)$, $\pi_m(y) = \pi_n(y)$, we obtain that $\mathbf{T}'$ is $\mathcal{M}$-compatible. ☐

To conclude, let $\mathbf{T}_\varphi = \{0 : \varphi, 0 : \top(x)\}$ be the initial completion set for φ. Define $\pi(0) = w_\varphi$ (where $\mathcal{M}, w_\varphi \models \varphi$) and $\pi_0(x) = d$, for an arbitrary $d \in \Delta_{w_\varphi}$. Clearly, these functions ensure that $\mathbf{T}_\varphi$ is $\mathcal{M}$-compatible. By Claim 3.2, we can apply the $\mathbf{L}_{\mathcal{ALC}}^n$-rules so that the obtained completion sets are $\mathcal{M}$-compatible as well. From Theorem 1, we have that the $\mathbf{L}_{\mathcal{ALC}}^n$ tableau algorithm eventually terminates, returning an $\mathbf{L}_{\mathcal{ALC}}^n$-complete completion set for φ that is clash-free by construction. ☐

By Theorem 1, we have that the non-deterministic $\mathbf{L}^n_{\mathcal{ALC}}$ tableau algorithm terminates after exponentially many steps in the size of the input formula. By Theorems 2 and 3, such algorithm is sound and complete with respect to satisfiability in varying domain neighbourhood models. Thus, we obtain the following result.

Theorem 4. *The $\mathbf{L}^n_{\mathcal{ALC}}$ formula satisfiability problem on varying domain neighbourhood models is decidable in NExpTime.*

To conclude this section, we observe that as an immediate consequence of the above results we also obtain a (constructive) proof of the following kind of *exponential model property.*

Corollary 5. *For $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$ (respectively, $\mathbf{L} = \mathbf{C}$), every $\mathbf{L}^n_{\mathcal{ALC}}$ satisfiable formula φ has a model with at most $p(|\mathsf{Fg}(\varphi)|)$ (respectively, at most $2^{p(|\mathsf{Fg}(\varphi)|)}$) worlds, each of them having a domain with at most $2^{q(|\mathsf{Fg}(\varphi)|)}$ elements, where p and q are polynomial functions.*

Proof. By Theorem 3, if φ is $\mathbf{L}^n_{\mathcal{ALC}}$ satisfiable, then there is a $\mathbf{L}^n_{\mathcal{ALC}}$-complete and clash-free completion set $\mathbf{T}$ for it. Then by Theorem 2, there exists a model $\mathcal{M} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I}, \mathcal{I})$ for φ where $\mathcal{W} = \mathsf{L}_{\mathbf{T}}$ and for each $n \in \mathcal{W}$, $\Delta_n = \{x \in \mathsf{N_V} \mid x \text{ occurs in } S_n\}$. By Theorem 1, Claim 1.2, it follows $|\mathcal{W}| \leq |\mathsf{Fg}(\varphi)|^2$ for $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{N}\}$, and $|\mathcal{W}| \leq 2^{|\mathsf{Fg}(\varphi)|} \cdot |\mathsf{Fg}(\varphi)|$ for $\mathbf{L} = \mathbf{C}$, finally by Theorem 1, Claim 1.1, for each $n \in \mathcal{W}$, $|\Delta_n|$ does not exceed $2^{q(|\mathsf{Fg}(\varphi)|)}$, where p and q are polynomial functions. ☐

4. Reasoning in Fragments without Modalised Concepts

An *$\mathcal{ALC}$-ML^n formula* is defined similarly to the $ML^n_{\mathcal{ALC}}$ case, by disallowing modalised concepts. Given $\mathbf{L} \in \mathsf{Log}$, the *$\mathcal{ALC}$-$\mathbf{L}^n$ formula satisfiability problem on constant domain neighbourhood models* is the $\mathcal{ALC}$-ML^n formula satisfiability problem on constant domain neighbourhood models based on neighbourhood frames in the respective class for $\mathbf{L}$ (cf. Section 2). An *ML^n formula*, instead, is defined analogously to $\mathcal{ALC}$-ML^n, except that we built it from the standard propositional (rather than $\mathcal{ALC}$) language over a countable set of *propositional letters* $\mathsf{N_P}$. The semantics of ML^n formulas is given in terms of *propositional neighbourhood models* (or simply *models*) $\mathcal{M}^{\mathsf{P}} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I}, \mathcal{V})$, where $(\mathcal{W}, \{\mathcal{N}_i\}_{i \in I})$ is a neighbourhood frame, with $I = \{1, \ldots, n\}$ in the following, and $\mathcal{V} : \mathsf{N_P} \to 2^{\mathcal{W}}$ is a function mapping propositional letters to sets of worlds (see [20, 30]). The *$\mathbf{L}^n$ formula satisfiability problem*, is the ML^n formula satisfiability problem on propositional neighbourhood models based on neighbourhood frames in the respective class for $\mathbf{L}$. A propositional neighbourhood model based on a neighbourhood frame in the respective class for $\mathbf{L}$ is called *$\mathbf{L}^n$ model.*

In Dalmonte et al. [24], it is shown that $\mathcal{ALC}$-$\mathbf{E}^n$ and $\mathcal{ALC}$-$\mathbf{M}^n$ formula satisfiability problems on constant domain neighbourhood models are ExpTime-complete. We now show tight complexity results for $\mathcal{ALC}$-$\mathbf{C}^n$ and $\mathcal{ALC}$-$\mathbf{N}^n$, using again the notion of a propositional abstraction of a formula (as in, e.g., [31]). Here, one can separate the satisfiability test into two parts, one for the description logic dimension and one for the modal dimension. The *propositional abstraction* φ_{prop} of an $\mathcal{ALC}$-ML^n formula φ is the result of replacing each $\mathcal{ALC}$ CI in φ by a propositional letter p, so that there is a $1 : 1$ relationship between the $\mathcal{ALC}$ CI π occurring in φ and the propositional letters p_π used for the abstraction. We set

$\mathsf{N_P}(\varphi) = \{p_\pi \in \mathsf{N_P} \mid \pi \text{ is an } \mathcal{ALC} \text{ CI in } \varphi\}$. Given an $\mathcal{ALC}$-ML^n formula φ, we say that a propositional neighbourhood model $\mathcal{M}^{\mathsf{P}} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I}, \mathcal{V})$ of φ_{prop} is *φ-consistent* if, for all $w \in \mathcal{W}$, the following $\mathcal{ALC}$ formula is satisfiable

$$\bigwedge_{p_\pi \in \mathsf{N_P}(w)} \pi \;\wedge\; \bigwedge_{p_\pi \in \overline{\mathsf{N_P}(w)}} \neg\pi,$$

where $\mathsf{N_P}(w) = \{p_\pi \in \mathsf{N_P}(\varphi) \mid w \in \mathcal{V}(p_\pi)\}$ and $\overline{\mathsf{N_P}(w)} = \mathsf{N_P}(\varphi) \setminus \mathsf{N_P}(w)$. We now formalise the connection between $\mathcal{ALC}$-ML^n formulas and their propositional abstractions with the following lemma, where $\mathbf{L} \in \{\mathbf{C}, \mathbf{N}\}$, obtained by adapting the proof of Dalmonte et al. [24, Lemma 1].

Lemma 6. *A formula φ is $\mathcal{ALC}$-$\mathbf{L}^n$ satisfiable on constant domain neighbourhood models iff φ_{prop} is satisfied in a φ-consistent $\mathbf{L}^n$ model.*

We assume that the primitive connectives used to build propositional formulas are $\neg$ and $\wedge$ ($\vee$ is expressed using $\neg$ and $\wedge$), and we denote by $\mathsf{sub}(\varphi_{\mathsf{prop}})$ the set of subformulas of φ_{prop} closed under single negation. A *valuation* for a propositional ML^n formula φ_{prop} is a function $\nu : \mathsf{sub}(\varphi_{\mathsf{prop}}) \to \{0, 1\}$ that satisfies the following conditions: (1) for all $\neg\psi \in \mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\psi) = 1$ iff $\nu(\neg\psi) = 0$; (2) for all $\psi_1 \wedge \psi_2 \in \mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\psi_1 \wedge \psi_2) = 1$ iff $\nu(\psi_1) = 1$ and $\nu(\psi_2) = 1$; and (3) $\nu(\varphi_{\mathsf{prop}}) = 1$. We say that a valuation for φ_{prop} is *φ-consistent* if any propositional neighbourhood model of the form $(\{w\}, \{\mathcal{N}_i\}_{i \in I}, \mathcal{V})$ satisfying $w \in \mathcal{V}(p_\pi)$ iff $\nu(p_\pi) = 1$, for all $p_\pi \in \mathsf{N_P}(\varphi)$, is φ-consistent. We now establish that satisfiability of φ_{prop} in a φ-consistent $\mathbf{C}^n$ (respectively, $\mathbf{N}^n$) model is characterized by the existence of a φ-consistent valuation satisfying the property described in Lemma 7 (respectively, Lemma 8).

Lemma 7. *A formula φ_{prop} is satisfied in a φ-consistent $\mathbf{C}^n$ model iff there is a φ-consistent valuation ν for φ_{prop} such that if $\Box_i\psi_1, \ldots, \Box_i\psi_k$ are in $\mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\Box_i\psi_j) = 1$ for all $1 \leq j < k$, and $\nu(\Box_i\psi_k) = 0$, then either $(\bigwedge_{j=1}^{k-1} \psi_j \wedge \neg\psi_k)$ or $(\neg\psi_j \wedge \psi_k)$ for some $1 \leq j < k$ is satisfied in a φ-consistent $\mathbf{C}^n$ model.*

Proof. ($\Rightarrow$) Suppose that φ_{prop} is satisfied in a world w of a φ-consistent $\mathbf{C}^n$ model $\mathcal{M}^{\mathsf{P}} = (\mathcal{W}, \{\mathcal{N}_i\}_{i \in I}, \mathcal{V})$. That is, $\mathcal{M}^{\mathsf{P}}, w \models \varphi_{\mathsf{prop}}$. We define a φ-consistent valuation for φ_{prop} by setting $\nu(\psi) = 1$ if $\mathcal{M}^{\mathsf{P}}, w \models \psi$ and $\nu(\psi) = 0$ if $\mathcal{M}^{\mathsf{P}}, w \not\models \psi$. It is easy to check that ν is indeed a φ-consistent valuation (given that $\mathcal{M}^{\mathsf{P}}$ is a φ-consistent $\mathbf{C}^n$ model). Assume $\Box_i\psi_1, \ldots, \Box_i\psi_k$ are in $\mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\Box_i\psi_j) = 1$ for all $1 \leq j < k$, and $\nu(\Box_i\psi_k) = 0$. Then $\mathcal{M}^{\mathsf{P}}, w \models \Box_i\psi_j$ for all $1 \leq j < k$, and $\mathcal{M}^{\mathsf{P}}, w \not\models \Box_i\psi_k$. By definition, $(\Box_i\psi_1 \wedge \ldots \wedge \Box_i\psi_{k-1}) \to \Box_i(\psi_1 \wedge \ldots \wedge \psi_{k-1})$ holds in $\mathbf{C}^n$ models. So $\mathcal{M}^{\mathsf{P}}, w \models \Box_i(\psi_1 \wedge \ldots \wedge \psi_{k-1})$ and $\mathcal{M}^{\mathsf{P}}, w \not\models \Box_i\psi_k$. This means that $\nu(\Box_i(\bigwedge_{j=1}^{k-1} \psi_j)) = 1$ while $\nu(\Box_i\psi_k) = 0$. By definition, $\mathcal{V}(\bigwedge_{j=1}^{k-1} \psi_j) \in \mathcal{N}_i(w)$ and $\mathcal{V}(\psi_k) \notin \mathcal{N}_i(w)$. So, $\mathcal{V}(\bigwedge_{j=1}^{k-1} \psi_j) \neq \mathcal{V}(\psi_k)$. Then, there is a world u in the symmetrical difference of these sets such that $\mathcal{M}^{\mathsf{P}}, u \models (\bigwedge_{j=1}^{k-1} \psi_j \wedge \neg\psi_k) \vee (\neg(\bigwedge_{j=1}^{k-1} \psi_j) \wedge \psi_k)$.

($\Leftarrow$) Suppose there is a φ-consistent valuation ν for φ_{prop} such that if $\Box_i\psi_1, \ldots, \Box_i\psi_k$ are in $\mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\Box_i\psi_j) = 1$ for all $1 \leq j < k$, and $\nu(\Box_i\psi_k) = 0$, then there is a φ-consistent $\mathbf{C}^n$ model

$$\mathcal{M}^{\mathsf{P}}_{\bigwedge_{j=1}^{k-1} \psi_j, \psi_k} = (\mathcal{W}_{\bigwedge_{j=1}^{k-1} \psi_j, \psi_k}, \{{\mathcal{N}_{\bigwedge_{j=1}^{k-1} \psi_j, \psi_k}}_i\}_{i \in I}, \mathcal{V}_{\bigwedge_{j=1}^{k-1} \psi_j, \psi_k})$$

and a world $w_{\bigwedge_{j=1}^{k-1}\psi_j,\psi_k} \in \mathcal{W}_{\bigwedge_{j=1}^{k-1}\psi_j,\psi_k}$ such that

$$\mathcal{M}^{\mathsf{P}}_{\bigwedge_{j=1}^{k-1}\psi_j,\psi_k}, w_{\bigwedge_{j=1}^{k-1}\psi_j,\psi_k} \models ((\bigwedge_{j=1}^{k-1}\psi_j)\land\neg\psi_k)\lor(\neg(\bigwedge_{j=1}^{k-1}\psi_j)\land\psi_k).$$

Let $\mathcal{M}^{\mathsf{P}}_1,\dots,\mathcal{M}^{\mathsf{P}}_m$ be an enumeration of the models $\mathcal{M}^{\mathsf{P}}_{\bigwedge_{j=1}^{k-1}\psi_j,\psi_k}$, as above. That is, we take one model $\mathcal{M}^{\mathsf{P}}_{\bigwedge_{l=1}^{k-1}\psi_l,\psi_k}$ for each pair $j=\bigwedge_{l=1}^{k-1}\psi_l,\psi_k$ where $\mathcal{M}^{\mathsf{P}}_j = (\mathcal{W}_j,\{\mathcal{N}_{j_i}\}_{i\in I},\mathcal{V}_j)$, and let $w_1,\dots,w_m$ be an enumeration of the worlds $w_{\bigwedge_{l=1}^{k-1}\psi_l,\psi_k}$, with $j=\bigwedge_{l=1}^{k-1}\psi_l,\psi_k$ and $w_j\in\mathcal{W}_j$. We assume without loss of generality that $\mathcal{W}_j\cap\mathcal{W}_k=\emptyset$ for $j\neq k$.

In the following, we define a φ-consistent $\mathbf{C}^n$ model $\mathcal{M}^{\mathsf{P}} = (\mathcal{W},\{\mathcal{N}_i\}_{i\in I},\mathcal{V})$ for φ_{prop}. Intuitively, we construct $\mathcal{M}^{\mathsf{P}}$ by taking the union of each $\mathcal{M}^{\mathsf{P}}_j$ with the addition of a new world w that will satisfy φ_{prop}. We define $\mathcal{W}$ as $\bigcup_{1\le j\le n}\mathcal{W}_j\cup\{w\}$, where w is fresh. Before defining $\mathcal{N}_i$ and $\mathcal{V}$, we define the function $J:\mathsf{sub}(\varphi_{\mathsf{prop}})\to 2^{\mathcal{W}}$ with $J(\psi)=\bigcup_{0\le j\le m}\mathcal{V}_j(\psi)$ for all $\psi\in\mathsf{sub}(\varphi_{\mathsf{prop}})$, where $\mathcal{V}_0:\mathsf{sub}(\varphi)\to 2^{\{w\}}$ is the function that assigns ψ to $\{w\}$, if $\nu(\psi)=1$, and to $\emptyset$, otherwise ($\mathcal{V}_j$, for $1\le j\le m$, is as above). By construction, we have that $J(\neg\psi)=\mathcal{W}\setminus J(\psi)$ and $J(\psi_1\land\psi_2)=J(\psi_1)\cap J(\psi_2)$. We define the assignment $\mathcal{V}$ as the function $\mathcal{V}:\mathsf{N}_{\mathsf{P}}(\varphi)\to 2^{\mathcal{W}}$ satisfying $\mathcal{V}(p_\pi)=J(p_\pi)$ for all $p_\pi\in\mathsf{N}_{\mathsf{P}}(\varphi)$.

It remains to define $\mathcal{N}_i$, for $i\in I$. For $u\in\mathcal{W}_j$ we first put $\alpha\subseteq W$ in $\mathcal{N}_i(u)$ precisely when $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi_\alpha$ and $\alpha=J(\psi_\alpha)$ for some $\Box_i\psi_\alpha\in\mathsf{sub}(\varphi)$. Then, we close $\mathcal{N}_i$ under intersection so that $\mathcal{M}^{\mathsf{P}}$ is a $\mathbf{C}^n$ model. The next two claims establish that $\mathcal{N}_i$ is as expected.

Claim 7.1. *If $\beta\in\mathcal{N}_i(u)$ and $\beta=J(\psi)$ for some $\Box_i\psi\in\mathsf{sub}(\varphi_{\mathsf{prop}})$, then $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi$.*

Proof of Claim. Indeed, since $\beta=J(\psi)\in\mathcal{N}_i(u)$, we must have that $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi_{1,\beta}$, …, $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi_{m,\beta}$ and $\beta=\bigcap_{l=1}^m J(\psi_{l,\beta})$ for some $\Box_i\psi_{1,\beta},\dots,\Box_i\psi_{m,\beta}\in\mathsf{sub}(\varphi_{\mathsf{prop}})$. Since $\mathcal{N}_i$ is closed under intersection, in fact, we have that $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i(\bigwedge_{l=1}^m\psi_{l,\beta})$. But since $J(\psi)=\bigcap_{i=1}^m J(\psi_{i,\beta})$, we also have $\mathcal{V}_j(\psi)=\bigcap_{l=1}^m\mathcal{V}_j(\psi_{l,\beta})$ (recall that $\mathcal{W}_j\cap\mathcal{W}_k=\emptyset$ for $k\neq j$), so $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi$ iff $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i(\bigwedge_{l=1}^m\psi_{l,\beta})$. It follows that $\mathcal{M}^{\mathsf{P}}_j,u\models\Box_i\psi$. ☐

Regarding the fresh world w introduced above in $\mathcal{W}$, we first put $\alpha\subseteq\mathcal{W}$ in $\mathcal{N}_i(w)$ precisely when $\nu(\Box_i\psi_\alpha)=1$ and $\alpha=J(\psi_\alpha)$ for some $\Box_i\psi_\alpha\in\mathsf{sub}(\varphi_{\mathsf{prop}})$. Then, we again close $\mathcal{N}_i$ under intersection so that $\mathcal{M}^{\mathsf{P}}$ is a $\mathbf{C}^n$ model.

Claim 7.2. *If $\beta\in\mathcal{N}_i(w)$ and $\beta=J(\bigwedge_{l=1}^{k-1}\psi_l)$ for some $\Box_i\psi_1,\dots,\Box_i\psi_{k-1}\in\mathsf{sub}(\varphi_{\mathsf{prop}})$ then $\nu(\Box_i\psi_l)=1$ for all $1\le l<k$.*

Proof of Claim. Indeed, since $\beta=J(\bigwedge_{l=1}^{k-1}\psi_l)\in\mathcal{N}_i(w)$ we must have that $\nu(\Box_i\psi_{1,\beta})=1,\dots,\nu(\Box_i\psi_{m,\beta})=1$ and $\beta=\bigcap_{i=1}^m J(\psi_{i,\beta})$ for some $\Box_i\psi_{1,\beta},\dots,\Box_i\psi_{m,\beta}\in\mathsf{sub}(\varphi_{\mathsf{prop}})$. Suppose now that $\nu(\bigwedge_{l=1}^{k-1}\psi_l)=0$. Then, by assumption, there exists a structure $\mathcal{M}^{\mathsf{P}}_j=(\mathcal{W}_j,\{\mathcal{N}_{j_i}\}_{i\in I},\mathcal{V}_j)$ and a world $w_j\in\mathcal{W}_j$ such that $\mathcal{M}^{\mathsf{P}}_j,w_j\models(\bigwedge_{l=1}^{k-1}\psi_{l,\beta}\land\neg(\bigwedge_{l=1}^{k-1}\psi_l))\lor(\neg(\bigwedge_{l=1}^{k-1}\psi_{l,\beta})\land(\bigwedge_{l=1}^{k-1}\psi_l))$. It follows that $\mathcal{V}_j(\bigwedge_{l=1}^{k-1}\psi_{l,\beta})\neq\mathcal{V}_j(\bigwedge_{l=1}^{k-1}\psi_l)$. Consequently $J(\bigwedge_{l=1}^{k-1}\psi_{l,\beta})\neq J(\bigwedge_{l=1}^{k-1}\psi_l)$, which is a contradiction. ☐

We now show by induction on the structure of formulas that $\mathcal{V}$ and J agree on $\mathsf{sub}(\varphi_{\mathsf{prop}})$. This holds by construction for atomic propositions. It is easy to deal with propositional connectives, since we know that $J(\neg\psi) = \mathcal{W} \setminus J(\neg\psi)$ and $J(\psi_1 \wedge \psi_2) = J(\psi_1) \cap J(\psi_2)$ and similarly for $\mathcal{V}$. Assume inductively that $\mathcal{V}(\psi) = J(\psi)$. Suppose first that $u \in J(\Box_i\psi)$. Then, either $u = w$ and $\nu(\Box_i\psi) = 1$ or $u \in \mathcal{W}_j$ and $\mathcal{M}^{\mathsf{P}}_j, u \models \Box_i\psi$. In either case we have that $J(\psi) \in \mathcal{N}_i(u)$. Since $\mathcal{V}(\psi) = J(\psi)$, it follows that $\mathcal{M}^{\mathsf{P}}, u \models \Box_i\psi$, that is, $u \in \mathcal{V}(\Box_i\psi)$. Suppose now that $u \in \mathcal{V}(\Box_i\psi)$, that is, $\mathcal{M}^{\mathsf{P}}, u \models \Box_i\psi$, or, equivalently, $\mathcal{V}(\psi) \in \mathcal{N}_i(u)$. Since $\mathcal{V}(\psi) = J(\psi)$ it follows that either $u = w$ and $\nu(\Box_i\psi) = 1$ or $u \in \mathcal{W}_j$ and $\mathcal{M}^{\mathsf{P}}_j, u \models \Box_i\psi$. In either case we have that $u \in J(\Box_i\psi)$.

Since $\nu(\varphi_{\mathsf{prop}}) = 1$, we have that $w \in J(\varphi_{\mathsf{prop}})$, and consequently $w \in \mathcal{V}(\varphi_{\mathsf{prop}})$. That is, $\mathcal{M}^{\mathsf{P}}, w \models \varphi_{\mathsf{prop}}$. The fact that $\mathcal{M}^{\mathsf{P}}$ is a $\mathbf{C}^n$ model follows from the definition of $\mathcal{N}_i$. The fact that $\mathcal{M}^{\mathsf{P}}$ is a φ-consistent model follows from the fact that ν, used to construct the assignment related to w, is φ-consistent and the models $\mathcal{M}^{\mathsf{P}}_1, \ldots, \mathcal{M}^{\mathsf{P}}_m$, used to define the remaining worlds in $\mathcal{W}$, are all φ-consistent models. ☐

The following result can be proved by adapting the proof of the previous lemma.

Lemma 8. *A formula φ_{prop} is satisfied in a φ-consistent $\mathbf{N}^n$ model iff there is a φ-consistent valuation ν for φ_{prop} such that*

1. *if $\Box_i\psi$ is in $\mathsf{sub}(\varphi_{\mathsf{prop}})$ and $\nu(\Box_i\psi) = 0$, then $\neg\psi$ is satisfied in a φ-consistent $\mathbf{N}^n$ model;*
2. *if $\Box_i\psi_1$ and $\Box_i\psi_2$ are in $\mathsf{sub}(\varphi_{\mathsf{prop}})$, $\nu(\Box_i\psi_1) = 1$, and $\nu(\Box_i\psi_2) = 0$, then $(\psi_1 \wedge \neg\psi_2) \vee (\neg\psi_1 \wedge \psi_2)$ is satisfied in a φ-consistent $\mathbf{N}^n$ model.*

To determine satisfiability of φ_{prop} in a φ-consistent model, we use Lemma 6 and the characterizations above. To establish complexity results, we use the fact that there are only quadratically many subformulas in φ_{prop}. Satisfiability in $\mathcal{ALC}$ is ExpTime-complete and so, one can determine in exponential time whether a valuation is φ-consistent. For an ExpTime upper bound, one can deterministically compute all possible φ-consistent valuations for $(\bigwedge_{j=1}^{k-1} \psi_j \wedge \neg\psi_k)$ (or $(\psi_1 \wedge \neg\psi_2)$) and decide satisfiability of φ_{prop} by a φ-consistent model using a bottom-up strategy (as in [31]). Since satisfiability in $\mathcal{ALC}$ is ExpTime-hard, our upper bound is tight.

Theorem 9. *The $\mathcal{ALC}$-$\mathbf{C}^n$ and $\mathcal{ALC}$-$\mathbf{N}^n$ formula satisfiability problems on constant domain neighbourhood models are ExpTime-complete.*

5. Discussion and Future Work

In this paper, we have presented first results on reasoning in non-normal modal description logics. After providing motivations and preliminaries for these logics, we have focused on the following two aspects. First, we have introduced terminating, sound and complete tableaux algorithms for checking satisfiability of multi-modal description logics formulas in varying domain neighbourhood models based on classes of frames that characterise different non-normal systems, that is, $\mathbf{E}^n$, $\mathbf{M}^n$, $\mathbf{C}^n$, and $\mathbf{N}^n$. We have then studied the complexity of the satisfiability problem restricted to fragments where modal operators can be applied to formulas only (thus without modalised concepts) and interpreted on neighbourhood models with constant domains. As future work, we plan to investigate along the following directions.

First, we are interested in adapting our tableau algorithms to check satisfiability of formulas on neighbourhood models with constant domains. This requires to address the introduction of fresh variables that do not occur in other previously expanded labelled constraints systems. For instance, by applying the $\mathbf{M}^n_{\mathcal{ALC}}$-rules to the n-labelled constraint system $S_n = \{n : \Diamond_i \exists r.A(x), \Box_i \neg A(x)\}$, we obtain the m-labelled constraint system $S_m = \{m : \exists r.A(x), m : \neg A(x), m : r(x,y), m : A(y)\}$. The fresh variable y in S_m does not allow us to directly extract a model with constant domain, since there would be no object in the domain of the world associated with S_n capable of representing y correctly.

A possible solution could be to define a suitable notion of *quasimodel* [28], to equivalently characterise satisfiability on constant domain neighbourhood models in terms of structures representing “abstractions” of the actual models of a formula. The representation of domain objects across worlds would be given in terms of suitably defined functions, called *runs*, to guarantee that they are well-behaved with respect to their modal properties, and that they do not violate the constant domain assumption. A similar approach is followed by Seylan and Erdur [23] and Seylan and Jamroga [25, 26], with suitable “copies” of worlds introduced to address the problem of the definition of runs. In these works, however, it is not made explicit how such a definition should be carried out in detail. We conjecture that an approach based on *marked variables*, as illustrated in Gabbay et al. [28], can be fruitfully adopted together with quasimodels to solve the issue of a constant domain model extraction from a complete and clash-free completion set for a formula.

In addition, we are interested in tight complexity results for $\mathbf{L}^n_{\mathcal{ALC}}$ formula satisfiability, with respect to varying and constant domain neighbourhood models. It is known that $\mathcal{ALC}$ formula satisfiability is ExpTime-complete. However, we do not know whether the upper bound for $\mathbf{L}^n_{\mathcal{ALC}}$ formula satisfiability problem on varying or constant domain neighbourhood models can be improved to ExpTime-membership, for any $\mathbf{L} \in \{\mathbf{E}, \mathbf{M}, \mathbf{C}, \mathbf{N}\}$. It has to be noted that, at the propositional level, the formula satisfiability problem for the systems $\mathbf{E}$, $\mathbf{M}$, and $\mathbf{N}$ is known to be NP-complete, with a rise to PSpace-completeness for systems containing $\mathbf{C}$ [30].

Finally, we plan to consider satisfiability in other combinations and extensions of non-normal modal description logics. This would naturally lead us to consider both the straightforward cases of $\mathbf{MC}$, $\mathbf{MN}$ and $\mathbf{CN}$ of the classical cube [32], as well as other logics tailored to applications in knowledge representation contexts. In particular, we intend to investigate non-normal modal description logics in epistemic, coalitional, and deontic settings.

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