A Simple Heuristic for the Graph Tukey Depth Problem with Potential Applications to Graph Mining

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Abstract

We study a recently introduced adaptation of Tukey depth to graphs and discuss its algorithmic properties and potential applications to mining and learning with graphs. In particular, since it is NP-hard to compute the Tukey depth of a node, as a first contribution we provide a simple heuristic based on maximal closed set separation in graphs and show empirically on different graph datasets that its approximation error is small. Our second contribution is concerned with geodesic core-periphery decompositions of graphs. We show empirically that the geodesic core of a graph consists of those nodes that have a high Tukey depth. This information allows for a parameterized deterministic definition of the geodesic core of a graph.

Keywords

graph Tukey depth, geodesic closure, closed set separations, geodesic core-periphery decomposition

1. Introduction

Centrality measures are of high importance in data analysis, as they typically capture the elements' "importance" quantitatively. Of course, the meaning of *importance* depends on the choice of the particular centrality measure. Different types of centrality measures have been introduced for networks (see, e.g., [1]), including *degree centrality*, *eigenvector centrality*, *Katz centrality*, *closeness centrality*, *betweenness centrality*, *page rank*, and *hubs and authorities*. In Fig. 1 we present a graphical illustration of some of these centrality measures for some small graphs for a visual comparison.

Graph Tukey depth is a relatively new centrality measure, introduced in [3]. It is based on the original notion of Tukey depth that is defined over finite subsets of $\mathbb{R}^d$ [4, 5]. Informally speaking, the semantics of Tukey depth is as follows: An element e has *high* Tukey depth if it is "hard" to separate it from the rest of the finite ground set using separating hyper-planes only. Conversely, e has *low* Tukey depth if it is "easy" to separate it from the rest of the set. "Hard" resp. "easy" in this context means that the half-space bounded by the separating hyper-plane that contains e contains many resp. a few other elements of the finite ground set. Thus, the elements' "importance" defined by ordinary Tukey depth in $\mathbb{R}^d$ relies on the possibility of separating them from other elements. This property directly connects the Tukey depth to all

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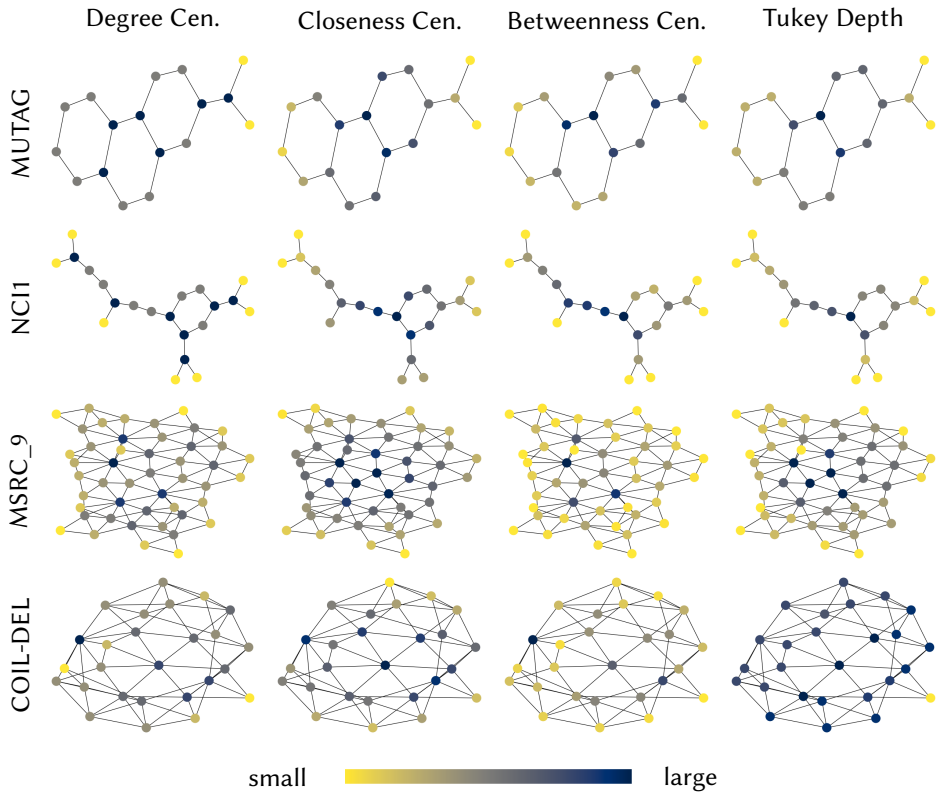


Figure 1: The *Degree Centrality*, *Closeness Centrality*, *Betweenness Centrality* and *Tukey Depth* of nodes in graphs selected from different graph datasets [2]. The centrality (resp. depth) values are *normalized* (i.e, mapped to the interval $[0, 1]$) by their maximum values in the graph. In particular, nodes of the smallest (resp. highest) centrality values are denoted by yellow (resp. blue).

machine learning approaches that are based on hyper-plane separations [6, 7]. For $\mathbb{R}^d$ and in other more general metric spaces [8], Tukey depth has been studied in the context of *machine learning*, in particular, for object classification [9]. In case of learning linear classifiers, it is shown in [10] that the *Tukey median*, i.e., the points with the highest Tukey depth, are related to the Bayes point.

An important advantage of Tukey depth over other centrality measures is that the *exact* geometric position of the points in $\mathbb{R}^d$ is *not* relevant for their depth. Thus, the Tukey depth is relatively stable regarding outliers and is therefore used for outlier detection [11, 12]. Moreover, as Tukey depth relies on separations, it is possible to adapt it to other domains enabling (abstract) hyper-plane or half-space separation. In particular, this property of the original Tukey depth is utilized in its adaptation to *geodesic closure systems* over *graphs* [3].

Similarly to the fact that the computation of the Tukey depth in $\mathbb{R}^d$ is NP-hard [13], it is also NP-hard to compute the graph Tukey depth of a node [3]. Motivated by this negative result, one of our main contributions is a *heuristic* algorithm for *approximating* graph Tukey depth. It runs in time *polynomial* in the size of the input graph and approximates the Tukey depth of a node with one-sided error by *overestimating* it. Our experimental results with small graphs

clearly demonstrate that the approximation is close to the exact Tukey depth, by noting that for larger graphs we were not able to evaluate the approximation performance of our algorithm, as it was *not* possible to calculate the *exact* Tukey depth in practically feasible time.

Our heuristic is based on our greedy algorithm designed in [14] for solving the more general *maximal closed set separation* (*MCSS*) problem. In the particular case of graphs, the problem is as follows: Given two subsets A, B of the node set, find, if they exist, two (inclusion) maximal disjoint geodesically closed node sets that separate A and B from each other. As maximal disjoint closed sets provide a separation of sets, it is, therefore, natural to ask the following question: *Is there a connection between graph Tukey depth and node separation with geodesically closed node sets?* We give an affirmative answer to this question by showing experimentally on small graph datasets that the solutions of the *MCSS* problem provide a good approximation quality of the exact Tukey depth. This mainly relies on the following fact: For any closed set containing at least one node of high Tukey depth, there exists *no* large disjoint closed set.

It follows from the definition of graph Tukey depth that it is related to other concepts based on geodesic convexity. One of these notions is the recent *probabilistic* definition of *geodesic core-periphery decomposition* of graphs. It was introduced in [15] and studied in [15, 16, 17, 18]. Hence, our second question is concerned with the following problem: *Is there a connection between graph Tukey depth and geodesic core-periphery decompositions?* The geodesic core-periphery decomposition breaks up some types of graphs (including interaction graphs, such as social networks) into a *dense* core and a *sparse* "surrounding" *periphery* [15] (see Fig. 3 for a visual example). While some graphs (e.g., Erdős-Rényi, Barabási-Albert, and Watts-Strogatz random graphs) seem to have no periphery, others (e.g., trees and fully connected graphs) seem to have no core. Since this behavior is not well-understood up to now, we try to understand it by studying the nodes' Tukey depth. It seems that the geodesic core of a graph consists of those nodes that are of high Tukey depth (see Fig. 4 for some examples). This observation allows for a *parameterized deterministic* definition of the cores. That is, the core of a graph can be defined by the set of those nodes that have a Tukey depth greater than a user-specified threshold. Our empirical results clearly demonstrate that using the right threshold, the probabilistic definition of cores in [15] *coincides* with our deterministic one.

The rest of the paper is organized as follows. In Sec. 2 we first collect the necessary notions and notation. Sec. 3 contains some examples showing that graph Tukey depth is strongly related to such mining and learning algorithms on graphs that rely on graph geodesic convexity. In Sec. 4 we present our heuristic for approximating graph Tukey depth and evaluate it empirically on small graph datasets. Finally, in Sec. 5 we mention some open questions for future research.

2. Preliminaries

In this section, we collect the necessary notions and fix the notation. For a graph $G = (V, E)$, $V(G)$ and $E(G)$ denote the set V of nodes and the set E of edges. Furthermore n and m denotes $|V(G)|$ and $|E(G)|$, respectively. By graphs, we always mean undirected graphs without loops and multi-edges. For any $u, v \in V(G)$, the (geodesic) interval $[u, v]$ is the set of *all* nodes on *all* shortest paths between u and v (see Fig. 2a for an example). A set of nodes $X \subseteq V(G)$ is called (geodesically) closed iff for all $u, v \in V(G)$, $u, v \in X$ implies $[u, v] \subseteq X$. The closure $\rho(X)$ of a

(a) (b)

Figure 2: This figure shows (a) the geodesic interval $[u, v]$, i.e., the nodes on all shortest paths between u, v (blue) and (b) the geodesic closure $\rho(\{u, v\})$, i.e., the smallest set of nodes that contains u, v and all nodes on all shortest paths between arbitrary node pairs of the set (red).

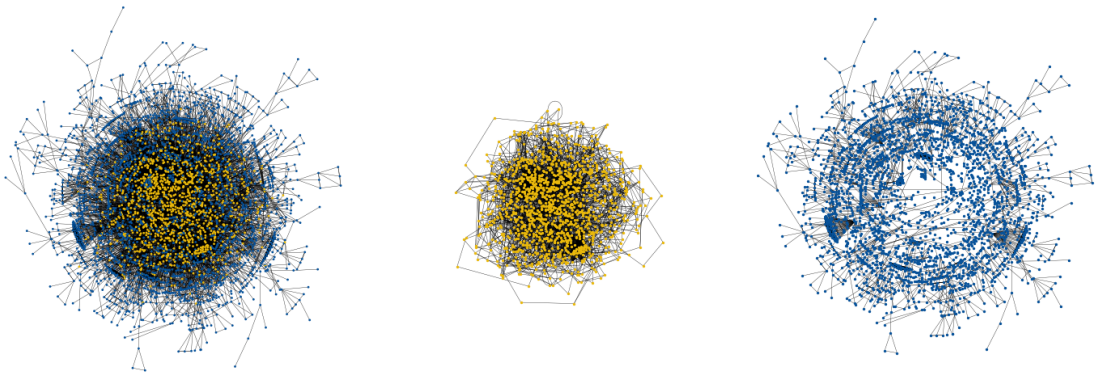
set $X \subseteq V(G)$ is the smallest closed set containing X (see Fig. 2b for an example of $\rho(\{u, v\})$).

The Graph Tukey Depth Problem For a graph G, the *Tukey depth* of a node $v \in V(G)$ is defined as follows [3]: Let $C \subseteq V(G)$ be a closed set of *maximum* cardinality such that $v \notin C$. Then the *Tukey depth* of v is defined by $\mathrm{td}(v) := |V(G)| - |C|$. That is, v has a high Tukey depth if all closed sets *not* containing v are of small cardinality. We are interested in the *Graph Tukey depths* problem defined as follows: Given a graph G compute $\mathrm{td}(v)$ for all $v \in V(G)$.

Geodesic Cores *Geodesic cores* have been defined in [15] in a *probabilistic* way. Informally, the geodesic core of a graph consists of those nodes which are contained in most geodesic closed sets that are generated by a small number of random nodes. Of course, the core defined in this way can be empty. However, it turns out that this is not the case for interaction graphs (e.g. social networks) [15]. We define the geodesic core of a graph G, denoted by $\mathcal{C}$ as follows. Let $X_1, X_2, \ldots$ be a sequence of sets, where each set consists of $k > 0$ nodes selected independently and uniformly at random from $V(G)$. Then $\mathcal{C} = \bigcap_{j=1}^{i} \rho(X_j)$ is the core of G, where i is the *smallest* integer satisfying $\bigcap_{j=1}^{i} \rho(X_j) = \bigcap_{j=1}^{i+1} \rho(X_j)$. Obviously, this definition is *not* deterministic since different choices of X_j and k can lead to different cores. Nevertheless, the experiments in [16] with large real-world networks show that for $k \approx 10$, the core (if it exists) does *not* depend on the particular choice of the generator elements. The *core-periphery decomposition* of a graph is composed of the subgraph induced by the core nodes and that by the remaining nodes, called *periphery*. In Fig. 3 we give a visual example of the core-periphery decomposition of the CA-GrQc network [19][1].

MCSS problem for Graphs We will use the *maximal closed set separation* (*MCSS*) problem restricted to geodesic closures over graphs (see [14] for the generic definition): *Given* a graph $G = (V, E)$ and $A, B \subseteq V(G)$ with disjoint geodesic closures, *find* two geodesic closed sets $A', B' \subseteq V(G)$ with $A' \cap B' = \emptyset$ and $A \subseteq A', B \subseteq B'$ that are maximal, i.e., there exist no proper supersets of A' resp. B' that fulfill the above properties.

[1]This network is built by the co-authorships in the general relativity and quantum cosmology community.



(a) Entire Network (b) (Geodesic) Core (c) Periphery

Figure 3: Example of a geodesic core-periphery decomposition (c.f. [16]) (a) the CA-GrQc network [19] with core in orange and periphery in blue, (b) its (geodesic) core, (c) its periphery.

3. Potential Applications to Mining and Learning with Graphs

Before we turn to the algorithmic aspects of the *graph Tukey depth* problem, in this section we first present some of its *potential* applications to mining and learning with graphs. In particular, this section deals with some connections of graph *Tukey depth* to *node separations* and to *geodesic core-periphery decompositions*. We state three important properties of Tukey depth. In particular, Proposition 1 clarifies the role of Tukey depth in the context of geodesic closed sets.

Proposition 1. *Let G be a graph, $v \in V(G)$ with $\mathrm{td}(v) = n - c$, and $C \subseteq V(G)$ a geodesically closed node set with $|C| > c$. Then $v \in C$.*

Proposition 2 ([3]). *Let G be a graph, $X \subseteq V(G)$, and C be the closure of X. Then the graph Tukey depth is a quasi-concave function, i.e., for all $c \in C$ we have $td(c) \geq \min\{td(x) : x \in X\}$.*

Proposition 3 ([3]). *Let G be a graph, $k \in \mathbb{N}$, and $X = \{v \in V(G) : \mathrm{td}(v) \geq k\}$. Then X is geodesically closed.*

To underline the importance of these three statements, we give two examples that show how they (can) influence machine learning and data mining methods based on geodesic closures.

Example 1: Node Classification and Active Learning In [14, 20, 21, 22], disjoint half-spaces and closed sets are used for binary classification in closure systems, for node classification, and active learning in graphs using geodesic convexity. Given the Tukey depth $\mathrm{td}(v)$ of a node v, Proposition 1 immediately implies that a separating half-space or closed set not containing v cannot have a cardinality greater than $n - \mathrm{td}(v)$. Thus, for nodes of *high* Tukey depth, there is *no* large geodesic closed set *not* containing them. Hence, Proposition 1 implies a nice theoretical connection between graph Tukey depth and the maximum size of separating half-spaces and closed sets. Using approximate graph Tukey depths, the predictive performance of all the above methods can possibly be improved.

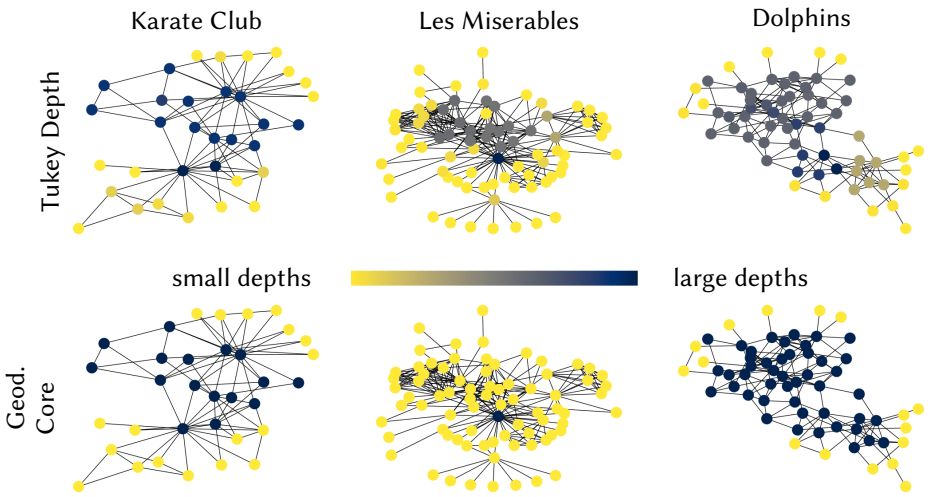


Figure 4: Tukey depth (top) vs. geodesic core-periphery decomposition (bottom) for the Karate Club [23], Les Miserables character [24], and Dolphins social networks [25]. For the different Tukey depths we use sequential colors. Core and periphery nodes are denoted by blue and yellow, respectively.

Example 2: Geodesic core-periphery decomposition The geodesic core-periphery decomposition of graphs was analyzed in [15, 16, 17]. In particular, it was found in [15, 17] that many social networks consist of a dense geodesic core "surrounded" by a sparse periphery (see Fig. 3 for an example). While some graphs, especially tree-like graphs, seem to have no core, others, such as graphs sampled from random graph models (e.g., Erdős-Rényi, Barabási-Albert and Watts-Strogatz) seem to have no periphery. Moreover, the closure of a small number of randomly chosen graph nodes (≈ 10) always contains the geodesic core (if it exists). Furthermore, if the nodes are sampled from the geodesic core only, then the closure of the nodes is the geodesic core itself. If we compute the closure of, say, 10 randomly chosen nodes from the entire network (Fig. 3a), then the closure always contains the core (orange nodes in Fig. 3b). If all random nodes belong to the core (orange nodes in Fig. 3a), then their closure is the core itself. The above statements explain this behavior. Using that the core is always contained in the closure of a small number of randomly chosen nodes, from Proposition 2 it follows that the nodes in the core are those with the highest Tukey depths. Moreover, the quasi-concave property of graph Tukey depth implies that if the core is generated by a few nodes from the core, then the core nodes must have a very close Tukey depth. Finally, using Proposition 3, we have that the set of nodes in a graph with a Tukey depth above some threshold is always geodesically closed; cores arise as a special case of this property. These three properties motivate the following *deterministic* definition of geodesic cores:

Definition 1. *The k-geodesic core of a graph G is defined by*

$$C_k := \{v \in V(G) : \mathrm{td}(v) \geq k\} \ .$$

To empirically confirm our claim that the core contains the nodes with the highest Tukey depths, consider the three graphs in Fig. 4. We computed the exact Tukey depths (top) and

their geodesic cores (bottom). For the Karate Club network (left) considered also in [3], the core exactly matches the set of nodes of the highest Tukey depth. Furthermore, there is not much fluctuation in the depths of the core nodes. In fact, all nodes of Tukey depth of at most 3 belong to the periphery and all nodes of Tukey depth 19 or 21 to the core, by noting that there are *no* nodes of Tukey depth between 4 and 18. In case of the Les Miserables character network (middle), there is only a single node with a very high Tukey depth of 57, surrounded by nodes of depth less than 35. In this case, the core algorithm returns only the node with the highest Tukey depth, showing that the graph Tukey depth can possibly be used to improve core-periphery decompositions. This is also the case for the Dolphin community graph (right), where the core consists of nodes with Tukey depth greater than 2, while all nodes in the periphery have a Tukey depth of at most 2.

4. Approximating the Tukey Depth

Motivated by the negative complexity result concerning the calculation of Tukey depth, in Sec. 4.1 below we first propose a *heuristic* based on the algorithm in [14] that solves the *MCSS* problem. It approximates the nodes' Tukey depths with one-sided error. We show experimentally on different types of *small* graphs that the results obtained by our heuristic are *fairly close* to the exact ones. Furthermore, even on small graphs, our algorithm is up to 200 times faster than the exact one (Sec. 4.2). It is important to emphasize that we had to resort to such graphs, as it was not possible to calculate the exact Tukey depths for larger graphs in a practically feasible time.

4.1. The Heuristic

Recall that the *exact* Tukey depth of a node v is defined by $\mathrm{td}(v) := |V(G)| - |C|$, where $|C|$ is the *maximum* cardinality of a closed set C *not* containing v. It can be computed *exactly* using an *integer linear program* (see [3] for the details). The computationally *hard* part of the problem is to find a closed set of *maximum* size. Our heuristic addresses this problem by considering an inclusion *maximal* closed set only, instead of a maximum cardinality closed set. This relaxation, which distorts of course the exact value of Tukey depth, allows us to apply the efficient greedy algorithm proposed in [14] for solving the *MCSS* problem. In what follows, for any $v \in V(G)$, $\widehat{\mathrm{td}}(v)$ denotes the approximation of $\mathrm{td}(v)$ obtained with our heuristic.

Given a graph G, the rough idea to approximate the Tukey depth of a node $v \in V(G)$ is to find an inclusion maximal geodesically closed set $C \subseteq V(G)$ with $v \notin C$. Such a set C can be found by applying the *MCSS* algorithm [14] with input $\{v\}$ and $\{v'\}$. Then the output of the algorithm is a solution of the *MCSS* problem, i.e., it consists of two closed node sets $H_v, H_{v'} \subseteq V(G)$ with $v \in H_v$ and $v' \in H_{v'}$ that are disjoint and inclusion maximal. That is, there exist no proper closed supersets of $H_v, H_{v'}$ with the same properties. The Tukey depth can then be approximated using the cardinalities of H_v resp. $H_{v'}$. For a fixed node v, the result depends on the particular choice of v'. To improve the approximation quality, we therefore call the *MCSS* algorithm for each node v several times, with *different* nodes $v' \neq v$.

The pseudo-code of the above heuristic is given in Alg. 1. In Line 1 we initialize the Tukey depth of all nodes in G by setting them to the maximum possible value, i.e., to $|V(G)|$. We repeat the procedure described above for all nodes $v \in V(G)$ and all their neighbors $v' \in \Gamma(v)$ (see the

Algorithm 1: Approximation of Graph Tukey Depth

Input : graph G
Output: approximation $\widehat{\mathrm{td}}(v)$ of $\mathrm{td}(v)$ for all $v \in V(G)$

1 $\widehat{\mathrm{td}}(v) \longleftarrow |V(G)|$ for all $v \in V(G)$;
2 **forall** $v \in V(G)$ **do**
3 **forall** $v' \in \Gamma(v)$ **do**
4 $H_{v'}, H_v = MCSS(\{v'\}, \{v\})$;
5 **forall** $x \in V(G)$ **do**
6 **if** $x \notin H_{v'}$ **then**
7 $\widehat{\mathrm{td}}(x) = \min\{\widehat{\mathrm{td}}(x), |V(G)| - |H_{v'}|\}$;
8 **if** $x \notin H_v$ **then**
9 $\widehat{\mathrm{td}}(x) = \min\{\widehat{\mathrm{td}}(x), |V(G)| - |H_v|\}$;
10 **return** $\widehat{\mathrm{td}}(v)$ for all $v \in V(G)$

for-loops in Line 2 and 3). In this way we solve the *MCSS* problem for all input sets $\{v\}, \{v'\}$, i.e., separate v from all of its neighbors v' by maximal disjoint closed sets $H_v, H_{v'}$ (see Line 4). Note that the Tukey depth of a node x is based on a closed set of *maximum* cardinality not containing x. Thus, if x does not lie in the set H_v (resp. $H_{v'}$), then the cardinality of H_v (resp. $H_{v'}$) is smaller than or equal to a closed set of maximum cardinality not containing x. Hence, we can update the current Tukey depth approximation of *all* nodes $x \in V(G)$ as follows: Take the minimum over the old and the new approximation which is the cardinality of $V(G) \setminus H_v$ if $x \notin H_v$ or that of $V(G) \setminus H_{v'}$ if $x \notin H_{v'}$ (see Line 7 and Line 9).

By construction, Alg. 1 finds only *maximal* and *not* maximum closed sets, resulting in a one-sided error in the estimate of Tukey depths. This is formulated in the proposition below.

Proposition 4. *Alg. 1 overestimates the Tukey depth, i.e., for the output* $\widehat{\mathrm{td}}(v)$ *returned by Alg. 1 we have* $\widehat{\mathrm{td}}(v) \geq \mathrm{td}(v)$*, for all* $v \in V(G)$.

Regarding the runtime of Alg. 1, note that the inner part of the loop in Lines 3-9 is executed $O(m)$ times because we iterate over all neighbors (i.e., all edges are considered twice). The runtime of the inner loop (Lines 3–9) is dominated by the *MCSS* algorithm called in Line 4, which calls the closure operator at most $O(n)$ times [14]. Since the geodesic closure can be computed in time $\mathcal{O}(mn)$ [26], we have the following result for the total runtime of Alg. 1:

Proposition 5. *Alg. 1 returns an upper bound of the Tukey depth for all nodes of* G *in* $\mathcal{O}(m^2n^2)$ *time.*

The runtime of the approximation algorithm can be improved by considering for each node v a *fixed* number of distinct nodes v', or by considering a fixed subset $W \subseteq V(G)$, instead of the whole node set $V(G)$ in the outer loop (see Lines 2–9). It is left to further research to analyze how these changes affect the quality of the approximation performance.

4.2. Experimental Evaluation

In this section we empirically evaluate the approximation quality and runtime of Alg. 1 on datasets containing *small* graphs[2]. Regarding the approximation quality, we compare the results obtained by our heuristic algorithm to the exact Tukey depths computed with the algorithm in [3]. For the evaluation, we consider 19 graph datasets of different types (small molecules, small graphs from bioinformatics and computer vision, and small social networks) from [2] (see columns 2–4 of Tab. 1 for the number of graphs and their average number of nodes and edges). The *average size* of the graphs ranges from 14 (*PTC_MM*) up to 82 (*OHSU*); their *average edge numbers* from 14 to 200. The reason for considering small graphs only is that the exact algorithm [3] was unable to calculate the Tukey depth for larger graphs within one day (see the last two columns of Tab. 1). For practical reasons, we removed all disconnected graphs from the original datasets, by noting that our heuristic works for disconnected graphs as well.

The results are presented in Tab. 1. It contains the approximation qualities measured in different ways (columns 5–10) and the runtime of the exact (column 11) and our heuristic algorithm (column 12). The datasets are sorted according to their *absolute approximation error* (column 5 of Tab. 1), i.e., the sum of all differences between the approximation and the exact Tukey depth over all nodes and all graphs in the dataset.

Regarding the *absolute error* (column 5), our approximation results are equal to the exact Tukey depths for 5 out of the 19 datasets, while their computation was faster by a factor of up to 100 (see *PTC_MM*). Our algorithm has the largest absolute error of 4155 on the *COIL-DEL* graphs, by noting that this dataset consists of 3900 graphs. Hence, the average error per graph is only slightly above one. Additionally, we look at the *relative errors* (column 6), i.e., the absolute error divided by the sum of all depths. We use this measure to validate that our algorithm performs very well, by noting that the relative errors are below $4 \cdot 10^{-3}$ for all graph datasets. The *per node error* (column 7) is the average error our algorithm makes per node, while the *per graph error* (column 8) is the error it has on average per graph. Regarding the *per node error*, the worst-case is for the *COIL-DEL* dataset (last row) with an average error of 0.05. For the *per graph error*, the worst result was obtained for the *OHSU* dataset, where we overestimate the sum of all node depths by 1.65 per graph on average. This shows that our approximation algorithm performs very well, especially, if considering the average results over the datasets.

Finally, we studied also the worst-case approximations for nodes and graphs. In particular, the columns *Max. Node Error* resp. *Max. Graph Error* denote the *maximum error* of the algorithm on single nodes resp. on single graphs. The results show that there is a very low error of at most 3 per node for 13 out of the 19 datasets. For three graph datasets, the *maximum error per node* is at most 7 and we have a maximum error between 11 and 19 in three cases. Regarding the *maximum error per graph*, a similar behavior can be observed by noting that except for *OHSU* and *Peking_1*, the *maximum node errors* and *maximum graph errors* are close to each other. This implies that there are only a *few* nodes with a *high* approximation error. It is an interesting problem to find the structural properties of such nodes and graphs that are responsible for the high approximation errors. The last two columns show the *runtimes* of the two algorithms. Our algorithm (last column) is *faster* than the exact one by at least *one order* of magnitude. In summary, the evaluation of Alg. 1 clearly demonstrates that our heuristic performs well in

[2]See https://github.com/fseiffarth/AppOfTukeyDepth for the code.

Data	Number of Graphs	Avg. Nodes	Avg. Edges	Error (absolute)	Error (relative)	Error per Node	Error per Graph	Max. Node Error	Max. Graph Error	Exact Runtime (s)	Approx. Runtime (s)
BZR	405	35.75	38.36	0	0	0	0	0	0	56.60	1.04
PTC_MM	336	13.97	14.32	0	0	0	0	0	0	21.09	0.21
COX2	467	41.22	43.45	0	0	0	0	0	0	76.10	1.27
Cuneiform	267	21.27	44.80	0	0	0	0	0	0	2.00	0.61
DHFR	756	42.43	44.54	0	0	0	0	0	0	266.33	3.19
PTC_FR	351	14.56	15.00	1	4.50e-05	1.96e-04	2.85e-03	1	1	23.81	0.25
PTC_FM	349	14.11	14.48	1	4.80e-05	2.03e-04	2.86e-03	1	1	20.64	0.22
MUTAG	188	17.93	19.79	1	6.50e-05	2.97e-04	5.32e-03	1	1	3.01	0.09
PTC_MR	344	14.29	14.69	2	9.20e-05	4.07e-04	5.81e-03	1	1	23.29	0.23
KKI	83	26.96	48.42	12	1.01e-03	5.36e-03	1.45e-01	2	4	40.45	2.43
IMDB-BINARY	1000	19.77	96.53	19	4.37e-04	9.61e-04	1.90e-02	1	3	723.07	113.14
NCI1	3530	29.27	31.88	34	5.20e-05	3.29e-04	9.63e-03	6	8	1194.63	7.76
Peking_1	85	39.31	77.35	40	1.30e-03	1.20e-02	4.71e-01	3	10	4761.33	48.08
MSRC_21C	209	40.28	96.60	86	1.28e-03	1.02e-02	4.11e-01	12	12	51.78	3.06
MSRC_9	221	40.58	97.94	89	1.21e-03	9.92e-03	4.03e-01	7	8	49.33	2.92
OHSU	79	82.01	199.66	130	8.54e-04	2.01e-02	1.65e+00	2	13	42887.32	235.40
ENZYMES	569	31.68	61.44	307	1.68e-03	1.70e-02	5.40e-01	7	10	933.79	8.25
MSRC_21	563	77.52	198.32	877	1.39e-03	2.01e-02	1.56e+00	19	25	3679.24	58.98
COIL-DEL	3900	21.54	54.24	4155	4.05e-03	4.95e-02	1.07e+00	11	17	2242.05	43.00

Table 1
Graph data of different sizes selected from [2]. Disconnected graphs are removed from the original datasets. The columns regarding the approximation quality denote the following. *Error (absolute)* denotes the overall error on the dataset, *Error (relative)* denotes the relative error regarding the depths, *Error per Node* denotes the average error per node, *Error per Graph* denotes the average error per graph, *Max. Node Error* denotes the maximum error for a node and *Max. Graph Error* denotes the maximum error on a graph. The last two columns show the runtimes of the exact and approximation algorithm in seconds.

approximating the graph Tukey depth. It is faster (sometimes more than 100 times) than the exact algorithm, even on small graph datasets. Regarding larger graphs, this gap in runtime will increase because of the exponential runtime of the exact algorithm. Additionally, the very small relative errors (at most $4 \cdot 10^{-3}$), the average errors (at most 1.65 per graph), and also the worst case errors show that the algorithm can be used effectively for further applications based on the Tukey depth (see Sec. 3).

5. Concluding Remarks

Graph Tukey depth appears an interesting and promising new concept for mining and learning with graphs. The study of the relationship of graph Tukey depth to other node centrality measures is an interesting question for further research (cf. Fig. 1). For example, while the centroid(s) in trees [27, 28] are exactly the nodes with the highest Tukey depth, this is not necessarily the case for more general graphs beyond trees.

Another important issue is to understand the semantics of graph Tukey depth better. For example, what are the properties of the nodes with the highest depth (cf. the def. of Tukey-median in $\mathbb{R}^d$)? We have empirically demonstrated that graph Tukey depth can closely be approximated in small graphs. It is a question of whether this result holds for (very) large graphs as well. To answer this question, the scalability of our approximation algorithm should be improved on the one hand. On the other hand, one needs (possibly tight) theoretical upper bounds on graph Tukey depths. Another interesting question is to identify graph classes for which our heuristic always results in the *exact* graph Tukey depth. While this is the case for trees, it is unclear whether it holds for outerplanar graphs as well. We believe that this question can be answered affirmatively by using techniques from [16]. We show experimentally that there is a connection between geodesic core-periphery decompositions and the deterministic definition of k-geodesic cores. This suggests that using our core approximation algorithm [16], we can closely approximate the set of nodes with the highest Tukey depth.

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