Towards Detection of Partial Truth via Real Geometry

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Abstract

Computational algebraic geometry has already shown its great potential as the underlying tool for implementing automated reasoning algorithms in elementary geometry, in a complex numbers framework, currently available, and quite performant, at GeoGebra and GeoGebra Discovery. Although this complex-field approach is often fully operative for addressing geometry statements, not involving inequalities, over the real numbers, in our presentation we will describe our recent work concerning the specific treatement of geometry theorems over the reals, including those involving inequalities. On the one hand, we think that real quantifier elimination (RQE) is a robust tool for proving geometric statements, especially for proofs of inequalities. Thus, by combining RQE with a dynamic geometry system, a convenient user interface can be used to study non-trivial geometric inequalities in the plane. We present an updated implementation of the `ProveDetails` command in GeoGebra Discovery which exploits the free availability and competitive speed of the Tarski software system (based on QEPCAD B and Minisat). On the other hand, we will compare this approach with an alternate one, that tries to mimic, over the reals, the complex-geometry approach, focusing on the relevance of the truth over the real irreducible components of the hypotheses variety. This allows us to consider not just the proof of truth or falsity, but to address if a more delicate classification of truth can be obtained, similarly to the notion of "truth on parts" or "truth on components" in complex algebraic geometry.

Keywords

Automated Reasoning in Geometry, Real Quantifier Elimination, GeoGebra, Partial Truth

1. Introduction

The first complete study on proving geometric statements by using real quantifier elimination (RQE) is perhaps [1], being published more than 25 years ago. Even if several improvements were contributed since then, there are still hardly publicly available software tools with a simple user interface to study and prove geometry theorems via RQE. To fill the gap, we present some recent improvements of the tool *GeoGebra Discovery* which makes it possible to prove non-trivial geometric statements that way.

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Utilizing RQE can be unavoidable in some cases. Well-known techniques like Gröbner bases, may give an incomplete translation of the geometric setup into an algebraic system because they cannot handle inequalities.

In this research first we point to some former work [2, 3]. In addition, in this contribution we propose a protocol via RQE. Assume we are given a plane geometric construction in GeoGebra Discovery. The user types the input `ProveDetails(`S`)` where S is a statement to be proven. Now, the software mechanically sets up a hypothesis formula H and a thesis T (as sets of logical connectives of semi-algebraic formulas) that contain the free variables $u_1, u_2, \ldots, u_n$ and dependent variables $x_1, x_2, \ldots, x_m$. At this point the quantified formula $\forall x_1 \forall x_2 \ldots \forall x_m (H \Rightarrow T)$ is considered. It can be negated into the form $\exists x_1 \exists x_2 \ldots \exists x_m (H \wedge \neg T)$. Now a query will be formulated for the Tarski subsystem. It optimizes the input formula by using linear substitutions and other reductions, and applies cylindrical algebraic decomposition to find a quantifier-free formula F in the variables $u_1, u_2, \ldots, u_n$.

If F is false, then S is always true. If F is true, then S is false. But, in several cases, F forms connectives of semi-algebraic expressions in the variables $u_1, u_2, \ldots, u_n$ and the situation needs to be analyzed further. In some simple cases, F is a conjunction of not-equal relations: this means that the statement is true on a disjunction of equalities which has a smaller dimension than n, so we conclude that S is false in the overwhelming majority of cases (that is, “false”). In another simple case, $\neg F$ is a disjunction of some relations that contain at least one inequality: this means that the statement is true on an n-dimensional subset of $\mathbb{R}^n$ (if there are no other restrictions), so we conclude that S is true in a significant proportion of cases (that is, “true”).

A more detailed study of such questions is related to partial truth in complex algebraic geometry (see [4]), and is planned for future work.

2. Examples

In the following examples we discuss how our new protocol identifies the geometric problem, how it is translated to an algebraic setup, how dimension is read off, and how the RQE computations are performed. Also, a comparison with a former technique, based on Gröbner bases, is presented.

2.1. A segment is usually longer than zero

Consider the user query `ProveDetails(`$f > 0$`)` in GeoGebra Discovery, as shown in Fig. 1.

We consider a segment f with two endpoints, $A(v_1, v_2)$, $B(v_3, v_4)$, in the plane. The length of f will be denoted by v_5, that is, $h_1 : v_5^2 = (v_1 - v_3)^2 + (v_2 - v_4)^2$ and $h_2 : v_5 \geq 0$ hold. These are our hypotheses: $H = h_1 \wedge h_2$. Our thesis is $T : v_5 > 0$.

Without loss of generality, we can, however, assume that $A = (0, 0)$ and $B = (0, 1)$. The reason for this is that the left hand side of the thesis is a homogeneous polynomial of degree 1, therefore, if the statement holds for all v_1, v_2, v_3, v_4, except eventually for $(v_1, v_2) = (v_3, v_4)$, then a similarity transformation of $(v_1, v_2, v_3, v_4) \mapsto (v_1', v_2', v_3', v_4') = (0, 0, 0, 1)$ keeps all ratios of all the appearing lengths in the construction. (See also [5].) That is, a $v_5 \mapsto v_5'$ mapping keeps the same ratio, and since it is still a positive number, the case $v_5' > 0$ is equivalent to $v_5 > 0$.

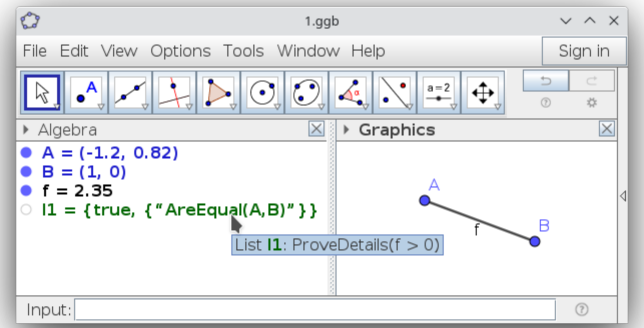

Figure 1: GeoGebra construction of two arbitrary points A and B and the segment f connecting them.

This approach, of course, skips the analysis of the case $(v_1, v_2) = (v_3, v_4)$, but this can be ignored as "skipped on purpose". In fact, it is implicitly assumed that points A and B are different, if the user creates them as two arbitrary points. That is, our new protocol just acknowledges the intuitive purpose of the user.

Now, after substituting the assumption $A = (0, 0)$ and $B = (0, 1)$ into the hypothesis, we get the statement S: $\forall v_5 (v_5^2 = 1 \wedge v_5 \geq 0 \Rightarrow v_5 > 0)$ whose negation has the form

$$\exists v_5 (v_5^2 = 1 \wedge v_5 \geq 0 \wedge v_5 \leq 0) \tag{1}$$

which is clearly false. That is, S is always true. We have to be, however, careful, because we implicitly assumed that $A \neq B$. That is, the correct communication of the protocol is the following statement: "If $A \neq B$, then $f > 0$."

Technically, after the user constructs A, B and f, and issues the command `ProveDetails(`$f > 0$`)`, GeoGebra Discovery sets up (1) and asks TARSKI to perform RQE by issuing a command like `(qepcad-api-call [ex v5 [(v5^2-1=0) /\ (v5>=0) /\ (~(v5>0))])`. This returns `false`, so GeoGebra Discovery outputs the result `{true, {"AreEqual(A,B)"}}`. This is to be interpreted: "If A and B differ, then the statement is true."

2.2. A segment is usually longer than another one

We continue our first example: Consider the user query `ProveDetails(`$f > g$`)` as shown if Fig. 2. We forget the substitutions for the coordinates v_1, v_2, v_3, v_4 at the first sight. Let us add a new point $C(v_6, v_7)$ and another segment $g = BC$. Also, we introduce v_8 to express the length g: $h_3 : v_8^2 = (v_3 - v_6)^2 + (v_4 - v_7)^2$, $h_4 : v_8 \geq 0$. To express that A, B and C form a non-degenerate triangle, we assume that

$$h_5 : v_9 \cdot \det \begin{pmatrix} v_1 & v_2 & 1 \\ v_3 & v_4 & 1 \\ v_6 & v_7 & 1 \end{pmatrix} - 1 = 0$$

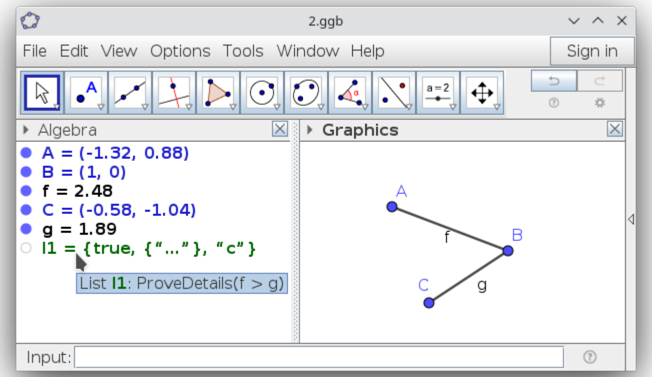

Figure 2: GeoGebra construction of three arbitrary points A, B, C and the segments f and g connecting them.

by using Rabinowitsch's trick[1] for a negation. (This could be avoided in real geometry, but at this point of our research, we chose to use this method which was borrowed from the old protocol.)

These are our hypotheses: $H = h_1 \wedge h_2 \wedge h_3 \wedge h_4$. Our thesis is $T : v_5 > v_8$, or equivalently, $v_5 - v_8 > 0$.

Again, we are allowed to set up some substitutions. $A = (0, 0)$ and $B = (0, 1)$ are considered again, for the same reason as above. (Here we confirm that the expression $v_5 - v_8$ from T is a degree 1 homogeneous polynomial.) On the other hand, C cannot be specialized. At the end of the day, our statement S has the form $\forall v_5, v_8, v_9 (v_5^2 = 1 \wedge v_5 \geq 0 \wedge v_8^2 = v_6^2 + (1 - v_7)^2 \wedge v_8 \geq 0 \wedge -v_6 v_9 - 1 = 0 \Rightarrow v_5 > v_8)$ – here we note that for the free point C we removed both of its variables v_6, v_7 from the list of quantified variables because we want to *learn a condition* for the truth. Now, the last formula has the negated form

$$\exists v_5, v_8, v_9 \quad (v_5^2 = 1 \wedge v_5 \geq 0 \wedge v_8^2 = v_6^2 + (1 - v_7)^2 \wedge v_8 \geq 0 \wedge -v_6 v_9 - 1 = 0 \wedge v_5 \leq v_8) \tag{2}$$

which is equivalent to

$$v_6 \neq 0 \wedge v_6^2 - 2v_7 + v_7^2 \geq 0 \tag{3}$$

after performing RQE.

Geometrically, this problem is a two-dimensional one, in the variables v_6, v_7 of point C. Therefore, to learn the truth about statement S we should analyze (3): how much part of $\mathbb{R}^2$ is covered by the non-solutions? Is it possible that (3) is a zero-dimensional set? If yes, we can say, that S is false on a "small" subset of $\mathbb{R}^2$. But, we should keep in mind, that such analysis should

[1] Let $p \in \mathbb{Q}[v_1, v_2, \ldots, v_n]$. To express $p \neq 0$, we can use the equivalent formula $v_{n+1} \cdot p - 1 = 0$. See [6].

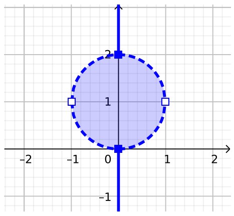

Figure 3: Where to put $C = (v_6, v_7) = (x, y)$ to ensure $f > g$?

be performed automatically and quickly enough. So, without answering this question at the moment, we also consider the positive variant of S and compute the negation of (3), by hoping that the other formulation will be simpler to study. We get

$$v_6 = 0 \lor v_7^2 - 2v_7 + v_6^2 < 0. \tag{4}$$

Now we hope for the best about the RQE implementation and assume that all parts of the (4) is in its simplest form, that is, for example, the quadratic inequality part is a two-dimensional set[2]. (See Fig. 3 for a visualization of (4): $(v_6, v_7) \mapsto (x, y)$.) Here, by using a very simple algorithm, we can conclude that there is a two-dimensional subset of $\mathbb{R}^2$ on which S holds, because (4) is a *disjunction* of sets, and one of them is an *inequality* of type $<$, $\leq$, $\geq$ or $>$.

Note that it is inconvenient to see $v_6 = 0$ in (4) because we excluded it in hypothesis h_5. But we note that h_5 is a one-dimensional restriction for the possible set of points C in statement S, so we are still on the safe side when ignoring that one-dimensional "noise" in the output.

Actually, the obtained circle meets our expectations because it exactly describes those points C such that $f > g$ is implied.

Technically, after the user constructs C and g, and issues the command `ProveDetails(`$f > g$`)`, GeoGebra Discovery sets up (2) and asks Tarski to perform RQE by issuing a command like `(qepcad-api-call [ex v5,v8,v9 [v5^2=1 /\ v5>=0 /\ v8^2=v6^2+(1-v7)^2 /\ v8>=0 /\ -v6 v9-1=0 /\ v5<=v8]])`. This returns $v_6 \neq 0 \land v_7^2 - 2v_7 + v_6^2 \geq 0$ which requires further investigation, so Tarski's `(t-neg %)` command is used to get $v6 = 0 \lor v_7^2 - 2v_7 + v_6^2 < 0$. Finally, GeoGebra Discovery outputs the result `{true, {"..."}, {"c"}}` which means that the statement is true on a remarkable subset of all possible values of C, but it is too difficult for the program to give a more precise explanation. (Here `c` recalls the abbreviation of "component" from the old protocol.)

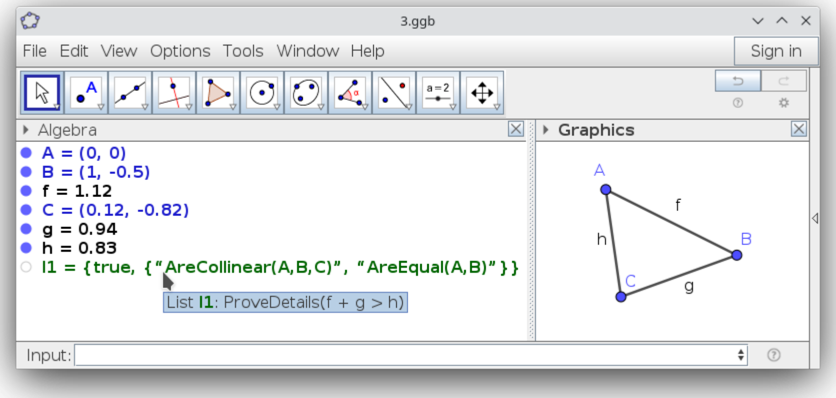

Figure 4: GeoGebra construction of three arbitrary points A, B, C and the segments f, g and h connecting them.

2.3. The Triangle Inequality and a generalization of the Pythagorean Theorem

Let us continue our example by adding segment $h = BC$ (see Fig. 4). This requires the addition of two further hypotheses: $h_6 : v_{10}^2 = (v_1 - v_6)^2 + (v_2 - v_7)^2$, $h_7 : v_{10} \geq 0$. W.l.o.g. we assume again that $A = (0, 0)$ and $B = (0, 1)$. This turns h_6 into $v_{10}^2 = v_6^2 + v_7^2$.

When performing our protocol given above, the following Tarski command studies if the Triangle Inequality $f + g > h$ holds: `(qepcad-api-call [ex v5,v8,v9,v10 [v5^2=1 /\ v5>=0 /\ v8^2=v6^2+(1-v7)^2 /\ v8>=0 /\ -v6 v9-1=0 /\ v10^2 = v6^2+v7^2 /\ v10 >=0 /\ ~(v5+v8>v10)]])`, and it returns `false`, this means that the statement always holds (under the non-degeneracy conditions $A \neq B$ and that $\triangle ABC$ is non-degenerate, this is communicated by GeoGebra Discovery with the output `{true, {"AreCollinear(A,B,C)", "AreEqual(A,B)"}}`).

Similarly, we can study some different statements. Alternatively, when checking if $f + g < h$ holds, we get $v_6 \neq 0$ for the negated condition and therefore $v_6 = 0$ for the positive condition. That is, this statement can be true at most on a zero-dimensional subset of $\mathbb{R}^2$, thus it is considered false.

Let us study some variants of the Pythagorean Theorem. As well-known, in a right triangle $f^2 + g^2 = h^2$ holds. But here we do not assume a right triangle, so we are interested, in general, in what prerequisites are required for $f^2 + g^2 = h^2$. By following our protocol, we only have to change the negated thesis to `~(v5^2+v8^2=v10^2)` and we obtain $v_6 \neq 0 \wedge v_7 - 1 \neq 0$ which can be studied better in its positive form: $v_6 = 0 \vee v_7 - 1 = 0$. This clearly means that $\triangle ABC$ must be degenerate (which is disallowed) or the second coordinate of C must be 1 (which is

[2]For example, a form like $v_6^2 + v_7^2 \leq 0$ would be deceptive. It is an inequality and seems to describe a two-dimensional subset of $\mathbb{R}^2$, but in fact it is equivalent to $(v_6, v_7) = (0, 0)$ which defines a zero-dimensional set. That is, we assume that the RQE implementation communicates such a subset in the "simplest" form $v_6 = 0 \wedge v_7 = 0$ and not like $v_6^2 + v_7^2 \leq 0$.

exactly the expected prerequisite that at vertex B the angle is right). Since both parts of the disjunction are one-dimensional, they cannot be considered as a "large enough" subset of $\mathbb{R}^2$. Therefore, the Pythagorean Theorem without its well-known assumption has to be considered false.

Here we remark, that the above mentioned negative form $v_6 \neq 0 \wedge v_7 - 1 \neq 0$ is a conjunction of $\neq$ relations. In all such cases the statement is clearly false. From an algorithmic perspective means that the computation of the positive form can be skipped.

One might also be interested in whether $f^2 + g^2 > h^2$ holds in general. By modifying the input accordingly, we obtain the negative form $v_6 \neq 0 \wedge v_7 - 1 \geq 0$ which has the positive form $v_6 = 0 \vee v_7 - 1 < 0$. This is clearly a two-dimensional subset of $\mathbb{R}^2$ because of the inequality part. By contrast, $f^2 + g^2 \leq h^2$ also holds in general, because its positive form, $v_6 = 0 \vee v_7 - 1 >= 0$ is also a two-dimensional subset of $\mathbb{R}^2$. This leads to the interesting fact that, in some sense, a thesis T and also its negation $\neg T$ can be true *at the same time.*

3. Comparison of the old and the new protocol

We give an example to compare the old and new protocols. The old approach computes an elimination ideal to learn the required prerequisities for a statement to be true.

Let us consider a simplified, degenerate version of the Triangle Inequality. Let us assume that $A = (0, 0)$, $B = (0, 1)$, and $C = (0, v_7)$, moreover let $h_1 : v_8^2 = (1 - v_7)^2$ and $h_2 : v_{10}^2 = v_7^2$. We would like to prove that $T : v_8 + v_{10} = 1$.

The old protocol [4] is based on elimination over the field of coefficients (usually $\mathbb{Q}$) and gets geometric conclusions over the complex numbers. Since it deals with polynomials and not equations, we consider the hypotheses and the thesis as polynomials, that is, $h_1 = v_8^2 - (1 - v_7)^2$, $h_2 = v_{10}^2 - v_7^2$, $T = v_8 + v_{10} - 1$. The old protocol performs the following steps:

1. We identify a maximum-size set of independent variables. Here it is $\{v_7\}$.
2. The elimination ideal $I_1 = \langle h_1, h_2, v_{11} \cdot T - 1\rangle \cap \mathbb{Q}[v_7]$ is computed (here v_{11} is a new dummy variable, to support Rabinowitsch's trick). If I_1 differs from the zero ideal, then the statement is *generally true*, possibly under certain conditions (that are contained in I_1). In our case, $I_1 = \langle 0\rangle$, so we continue.
3. Otherwise, the elimination ideal $I_2 = \langle h_1, h_2, T\rangle \cap \mathbb{Q}[v_7]$ is computed. If I_2 differs from the zero ideal, then the statement is *generally false.* In our case, $I_2 = \langle 0\rangle$, so we continue.
4. Otherwise, the statement is *true on components.* To identify the components, for example, Maple's command `PrimaryDecomposition` can be helpful, by using the hypotheses as input. Here, the primary decomposition of h_1, h_2 gives the output $\langle v_{10} \pm v_7, v_7 \pm v_8 - 1\rangle$, that is, it consists of 4 components, and it can be interpreted that on some components $|v_8 - v_{10}| = 1$ holds.

In fact, from the geometrical point of view, these components belong over the reals to the cases $v_7 < 0$, $0 \leq v_7 \leq 1$ and $1 < v_7$. Algebraically, however, there is a 4th, "invisible" case. This can be checked when T is changed to the thesis $T' : (v_8 + v_{10} - 1) \cdot (v_8 + v_{10} + 1) \cdot (v_8 - v_{10} - 1) \cdot (v_8 - v_{10} + 1) = 0$, and here the second factor does not have a geometrical meaning (but all other three factors do, see also [7]).

By contrast, the new protocol performs the following steps:

1. We determine the "naive dimension" d of the independent variables. Here the only independent variable is $\{v_7\}$, so $d = 1$.
2. We collect all dependent variables. Here they are v_8 and v_{10}.
3. Perform RQE on $H \wedge \neg T$ where the dependent variables are existentially quantified.
4. If the result is "false", then the statement is *always true* under the hypotheses. If it is "true", then the statement is *false* under the hypotheses. In our case, the RQE is computed with the Tarski command `(qepcad-api-call [ex v8,v10 [v8^2=(1-v7)^2 /\ v10^2=v7^2 /\ v8>=0 /\ v10>=0 /\ ~(v8+v10-1=0)]])` which yields

$$v_7 \neq 0 \wedge v_7 - 1 \neq 0 \wedge (v_7 < 0 \vee v_7 - 1 > 0). \tag{5}$$

   So we continue.
5. We determine the dimension d_n of the result of the RQE. If $d_n < d$, then the statement is *true* under the negation (that is, the positive form) of the result of the RQE. In our case, $d_n = 1$ (see below), so we continue. If d_n cannot be determined for some reason, continue.
6. Otherwise, that is, if $d_n = d$, we compute the dimension d_p of the positive form of the result of the RQE. If $d_p = d$, then the statement is *true* under the positive form of the result of the RQE. In our case, this holds.
7. Otherwise, that is, if $d_p < d$, the statement is *false*.
8. Otherwise, that is, if d_p cannot be determined for some reason, the statement is undecided.

In our case, it is not completely straightforward how to determine d_n. For some reason, Tarski's output in (5) is not the simplest possible formulation, but it is equivalent to $v_7 \notin [0, 1]$, and this can be confirmed by using some further basic operations in Tarski. That is, $d_n = 1$, because (5) is a union of two intervals.

Finally, $d_p = 1$ can be identified by negating (5) which is clearly $0 \leq v_7 \leq 1$, so we learn that the statement is true under the condition $v_7 \in [0, 1]$. Geometrically, this means that the statement is true if point C is an element of the segment AB.

In fact, the dimension of a the result of the RQE could already be obtained with some minor programming work in the Tarski subsystem. It is an on-going work to read off the dimension in such a way, and conclude truth or falsity simply and reliably.

4. Conclusion

We introduced a new protocol on proving geometric inequalities by using RQE in the educational software package GeoGebra Discovery. Our method uses the Tarski computer algebra system.

The provided examples are not completely trivial, however, using more free or dependent variables may quickly change the computational complexity infeasible. Therefore, we continue our research to automatically reduce the number of variables for a large set of possible inputs. Also, exact detection of the dimension of the RQE output is subject of future research.

Some further examples, including Viviani's theorem and Clough's conjecture [8] (they are, in some sense, further generalizations of the Triangle Inequality), can be found on the release announcement of GeoGebra Discovery version 2023Apr07, see https://github.com/

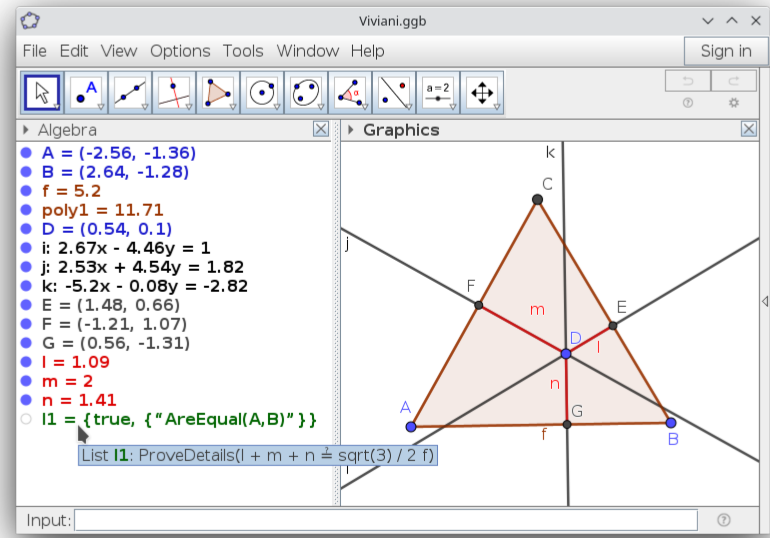

Figure 5: GeoGebra Discovery proves Viviani's theorem on a Lenovo ThinkPad i7 (2018), in 6 seconds, when adding command line option `--prover=timeout:10`. Point D is attached to regular triangle ABC. Lines j, k and l are perpendicular to the appropriate sides of $\triangle ABC$.

kovzol/geogebra/releases/tag/v5.0.641.0-2023Apr07. (See also Fig. 5 that shows a successful proof of Viviani's theorem after computing `(epc [ex sqrt3, v10, v11, v12, v13, v14, v15, v16, v17, v18, v19, v20, v21, v22, v23, v24, v25, v26, v5, v6, v7, v8, v9 [(v23>=0) /\ (v24>=0) /\ (v25>=0) /\ (v26>=0) /\ (sqrt3>=0) /\ (v5=-1/2) /\ (v7=1/2) /\ (v22=0) /\ (v23=1) /\ (v6=v8) /\ (v16=v10+1) /\ (v15=v9) /\ (v21=v9) /\ (v11=-v8+v9) /\ (v13=v8+v9) /\ (((v10>0) /\ ((-v9 v8+v10/2-v10+v8)>0) /\ ((v9 v8-v10/2)>0)) \/ ((0>v10) /\ (0>(-v9 v8+v10/2-v10+v8)) /\ (0>(v9 v8-v10/2)))) /\ (v8>0) /\ (sqrt3^2=3) /\ (~((-sqrt3+2 v24+2 v25+2 v26)/2=0)) /\ (4 v8^2-3=0) /\ (2 v10-2 v12-1=0) /\ (2 v10-2 v14-1=0) /\ (-v10 v17-v10 v8+v10 v9+v17 v12+v8 v18-v9 v12=0) /\ (-2 v17 v8+2 v8-v18=0) /\ (-v10 v19+v10 v8+v10 v9+v19 v14-v8 v20-v9 v14=0) /\ (2 v19 v8-v20=0) /\ (-v10^2+2 v10 v18-v18^2-v17^2+2 v17 v9-v9^2+v24^2=0) /\ (-v10^2+2 v10 v20-v20^2-v19^2+2 v19 v9-v9^2+v25^2=0) /\ (-v10^2+v26^2=0)]])` via Tarski, here the epc script is an efficient "black-box" method to compute RQE for non-trivial inputs. The epc script is maintained at https://github.com/chriswestbrown/epcx.) They contain some references to border cases when the old protocol is triggered by the program. We highlight here that in some cases the old protocol is still preferred because of substantially better speed. On the other hand, some problem settings require using real geometry, therefore the new protocol fits much better.

Our research is made even more topical by the fact that the SC^2 project has a particular focus on automated reasoning in the classroom (see [9]). Indeed, all our examples can be of educational interest, since GeoGebra Discovery offers a familiar look and intuitive use for young learners in many languages for free of charge.

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