The Degree of Non-parabolicity of the Surface Close to a Rotational Paraboloid

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Abstract

A measure of deviation from parabolicity of a convex smooth surface of rotation is introduced. The focus of the surface introdused is the one of the paraboloid of rotation, its axis and vertex coincide with the axis of the original surface. The relative area of the region filled with rays falling parallel to the axis of symmetry and reflecting from the surface is given and adopted as the measure of non-parabolicity. The measure of non-parabolicity of the spherical segment and the wave-like perturbed paraboloid of rotation was calculated.

Keywords

Convex rotational surface, reflector-type aerial, degree of parabolicity

1. Introduction

The interest in parabolic surfaces in engineering is primarily driven by their applications in antenna technology for satellite communication. The parabolic antenna was invented by the German physicist Heinrich Hertz in 1887. Hertz used cylindrical parabolic reflectors for sparking dipole antennas excitation during his experiments. Hertz successfully demonstrated the existence of electromagnetic waves, which had been predicted by Maxwell 22 years earlier.

Italian inventor Guglielmo Marconi used a parabolic reflector in the 1930s in his experiments to transmit signals to a boat in the Mediterranean Sea.

The first large parabolic antenna with a 9-meter reflector diameter was built in 1937 by radio astronomer Grote Reber. He used it to study the night sky.

In the 1960s, reflector-type aerials became widely used in terrestrial radio relaying communication networks. The first parabolic antenna used for satellite communication was constructed in 1962 in England for a communication satellite operation.

The basis of the operation of all parabolic antennas is the idea of transforming a plane electromagnetic wave into a spherical one or vice versa, transforming a spherical wave into a plane one. The larger the surface area of the antenna, the stronger the signal that can be obtained at its output. The efficiency of the antenna depends greatly on how close its surface approximates a paraboloid [1,2].

There are various approaches to evaluating the deviation of the antenna surface from a rotational paraboloid [3,4]. However, this issue still remains relevant.

2. Degree of Non-parabolicity of a Convex Rotational Surface.

In the article under discussion we will take into consideration some convex rotational surfaces Ω, which in the Cartesian coordinate system *Oxyz* are described by the equation $z = f(x, y), x^2 + y^2 < R^2$. The function $f(x, y)$ being twice differentiable in the circle D satisfies the following two conditions [5]:

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$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) < 0, \qquad \frac{\partial^2 f}{\partial x^2} > 0. \tag{1}$$

When $f(x,y) = \frac{1}{4c}(x^2 + y^2), c > 0$, then Ω is a paraboloid of revolution with a focus in the point F(0;0;c).

In this case, we assume that

$$f(x,y) = f(x^2 + y^2) \in C^2(x^2 + y^2 < R^2). \tag{2}$$

The function will satisfy the condition (1), when:

$$f''(t) > 0, f'(t)t > 0 \ (0 < t < R) \tag{3}$$

It is known that a rotational paraboloid possesses the property of focusing: rays parallel to its axis of symmetry (optical axis), after reflecting off its surface, pass through the focus of the paraboloid. When $f(x,y) = \frac{1}{4c}(x^2 + y^2)$, then all rays which are parallel to the axis of the applicate, after reflecting off the paraboloid surface will gather in point F(0;0;c).

We can say, that a rotational paraboloid is inscribed in a convex surface Ω(2) if their edges and vertices coincide. In this case, we will refer to the focus of the paraboloid as the conditional focus of the surface Ω.

Now, let Ω be a certain convex rotational surface with a hypothetical focus at the point F(0;0;c). Let D_f denote the region in the z=c plane where all the parallel axis-applied rays converge after reflecting off the surface Ω.

Definition. Let a convex rotational surface Ω is described by the equation $z = f(x^2 + y^2), (x^2 + y^2 \le R^2)$. We will refer to the degree on non-parabolicity of the surface Ω as $S(D_f)/(\pi R^2)$ where D_f is the area of the region.

The introduced concept here possesses such an interesting property.

We assume that

1) Ω_D^f *is* the surface described by the equations $z = f(x,y), (x,y) \in D$ i $f(x,y) \in C^2(D)$ is the function that satisfies the conditions (2).

2) We assume that $\rho(\Omega_D^f, \Omega_D^g) = |S(D_f) - S(D_g)|$.

In this case, M∋ Ω_D^f is the set of all surfaces Ω_D^f, *where* $c = const$ with metrics where $\rho(\Omega_D^f, \Omega_D^g)$ is a metric space.

The axioms of non-negativity and symmetry for the introduced metrics are obvious, and the triangle inequality is a consequence of this inequality $|S(D_f) - S(D_g)| + |S(D_g) - S(D_u)| \ge |S(D_f) - S(D_u)|$ for $\forall f, g, u \in M$.

2.1. Non-parabolicity of a Spherical Mirror.

To find a hypothetical focus and the degree of non-parabolicity of a spherical segment Ω: $z = R - \sqrt{R^2 - x^2 - y^2}, x^2 + y^2 \le r^2$ $(r \le R)$.

We must admit, that a spherical segment is a convex surface. To find its hypothetical focus we will write into Ω the paraboloid of revolution: $z = \frac{x^2+y^2}{R+\sqrt{R^2-r^2}}$. Thus,

$c = \frac{R+\sqrt{R^2-r^2}}{4}$.

The hypothetical focus of a spherical segment depends on its height $h = R - \sqrt{R^2 - r^2}$. The smaller the height, the larger the distance of a hypothetical focus from the surface vertex. When a spherical sector of the radius R is a hemisphere, it can reach the maximum possible height and $c = R/4$, but when its height $h \to 0$, then $c \to R/2$.

Area D represents a circle with the center in point (0;0;c). To find its radius, we will write the equation of a straight line in the plane *y=0*, making an angle that is equal to the angle between the a straight line x=x_0 (incident ray) and the radius of the arc of the circle $z = R - \sqrt{R^2 - x^2}$ in point K(x_0; $R - \sqrt{R^2 - x_0^2}$): $z = R - \sqrt{R^2 - x_0^2} - \frac{\sqrt{R^2 - x_0^2}}{x_0}(x - x_0)$. Reflected in point K the ray will obtain the equation $z = R - \sqrt{R^2 - x_0^2} + \frac{2x_0^2 - R^2}{2x_0\sqrt{R^2 - x_0^2}}(x - x_0)$.

The reflected ray is directed perpendicular to the axis of the sector, if the radius of the sector is $x_0 = R/\sqrt{2}$. In this case, if the radius of the sector is $r = R/\sqrt{2}$ the height is $h_0 = \frac{\sqrt{2}-1}{\sqrt{2}}R$ then, among reflected from the surface of the sector, some rays will be somehow close in their direction to the perpendicular ones to the axis of the sector, and the degree of its non-parabolicity will be infinitely large (fig. 1).

The ray reflected from the segment at point K intersects the plane at a hypothetical focus $z = c$ in the distance

$$d = \left| \frac{x_0(-3R\sqrt{R^2 - x_0^2} + 2R^2 + \sqrt{R^2 - r^2}\sqrt{R^2 - x_0^2}}{2(R^2 - 2x_0^2)} \right|$$

from the axis of the segment.

Radius p of the circle D is equal to the distance where the ray reflected from the edge of the hypothetical focus intersects the plane of the focus.

$$p = \left| \frac{r(3R\sqrt{R^2 - r^2} - 3R^2 + r^2)}{2(R^2 - 2r^2)} \right|, r < R/\sqrt{2}.$$

The degree of non-parabolicity of a spherical segment is equal to

$$S = \pi \left(\frac{r(3R\sqrt{R^2 - r^2} - 3R^2 + r^2)}{2(R^2 - 2r^2)} \right)^2.$$

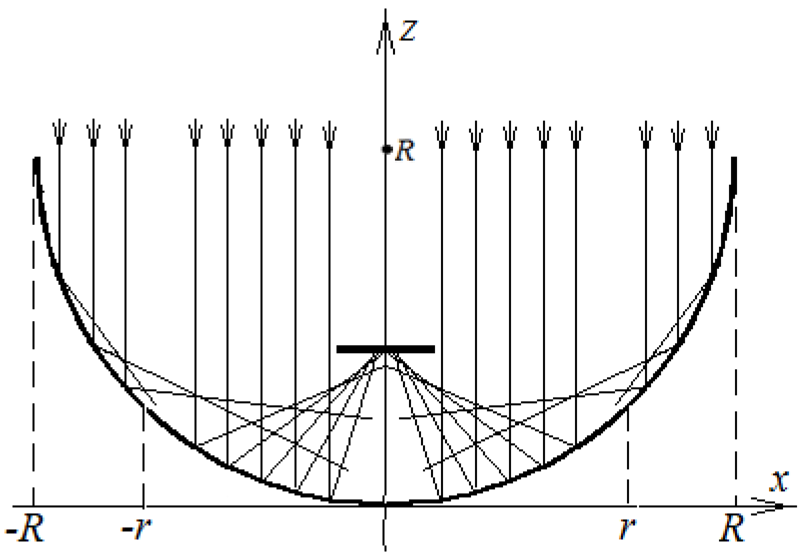

Figure 1. Spherical mirror of radius R. Applicate of the hypothetical focus $h = R - \sqrt{R^2 - r^2}$.

The spherical segment acts as a reflector-type aerial when the radius of its base is r $< R/\sqrt{2}$ (Fig. 2). Then the degree of its parabolicity is

$$\mathrm{S} = \left(\frac{\mathrm{r}(3\mathrm{R}\sqrt{\mathrm{R}^2 - \mathrm{r}^2} - 3\mathrm{R}^2 + \mathrm{r}^2)}{2R\sqrt{\mathrm{R}^2 - 2\mathrm{r}^2}} \right)^2$$

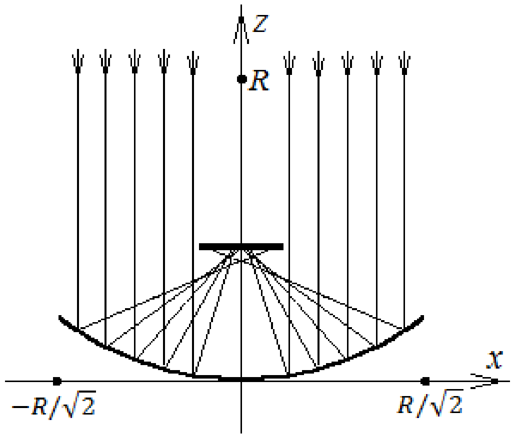

Figure 2. Track of the rays in a spherical segment that acts as an antenna mirror

2.2 Deviation from Parabolicity of a Wave-like Disturbed Paraboloid of Revolution.

Let the parabolic surface located in the cylindrical coordinate system $O\rho\varphi z$ described by the equation $z = \frac{1}{4c}\rho^2$ ($c = const, \rho \leq \rho_0$), disturbed by the deviation $\Delta z = A\sin\left(\frac{2\pi\rho}{\rho_0}\right)$, $0 \leq \rho \leq \rho_0$, remain convex (Fig. 3). We will find its degree of parabolicity as a function of the parameter A.

Let's study the surface

$$z = \frac{1}{4c}\rho^2 + A\rho\sin\left(\frac{2\pi\rho}{\rho_0}\right). \qquad (4)$$

$$z'' = \frac{1}{2c} - 4A\frac{\pi^2\rho}{\rho_0^2}\sin\left(\frac{2\pi\rho}{\rho_0}\right) + 2A\frac{\pi}{\rho_0}\cos\left(\frac{2\pi\rho}{\rho_0}\right).$$

The disturbed surface remains convex till

$$A < \frac{\rho_0}{4\pi cm},\ m = \max_{\rho=[0;\rho_0]}\left(\frac{2\pi\rho}{\rho_0}sin\frac{2\pi\rho}{\rho_0} - cos\frac{2\pi\rho}{\rho_0}\right).$$

Tangent of the angle between the incident ray and the normal and between the reflected ray is

$$k1 = \left(\frac{\rho}{2c} + 2A\frac{\pi}{\rho_0}sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)^{-1}.$$

Tangent of the angle between incident and reflected rays is

$$k2 = \frac{2\left(\frac{\rho}{2c} + 2A\frac{\pi}{\rho_0}sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)^{-1}}{1-\left(\frac{\rho}{2c} + 2A\frac{\pi}{\rho_0}sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)^{-2}}.$$

The angular coefficient of the reflected ray is

$$k = \frac{1}{2}\left(\left(\frac{\rho}{2c} + 2A\frac{\pi}{\rho_0}sin\left(\frac{4\pi\rho}{\rho_0}\right)\right) - \left(\frac{\rho}{2c} + 2A\frac{\pi}{\rho_0}sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)^{-1}\right).$$

Distance d of the point of cross-section of the reflected ray and the focus plane z=c from the axis of applicate

$$d = \left| 2\frac{c-\left(\frac{\rho^2}{4c}+A\sin\left(\frac{2\pi\rho}{\rho_0}\right)\right)}{\left(\frac{\rho}{2c}+A\frac{2\pi}{\rho_0}\sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)^2-1}\left(\frac{\rho}{2c}+A\frac{2\pi}{\rho_0}\sin\left(\frac{4\pi\rho}{\rho_0}\right)\right)+\rho \right|.$$

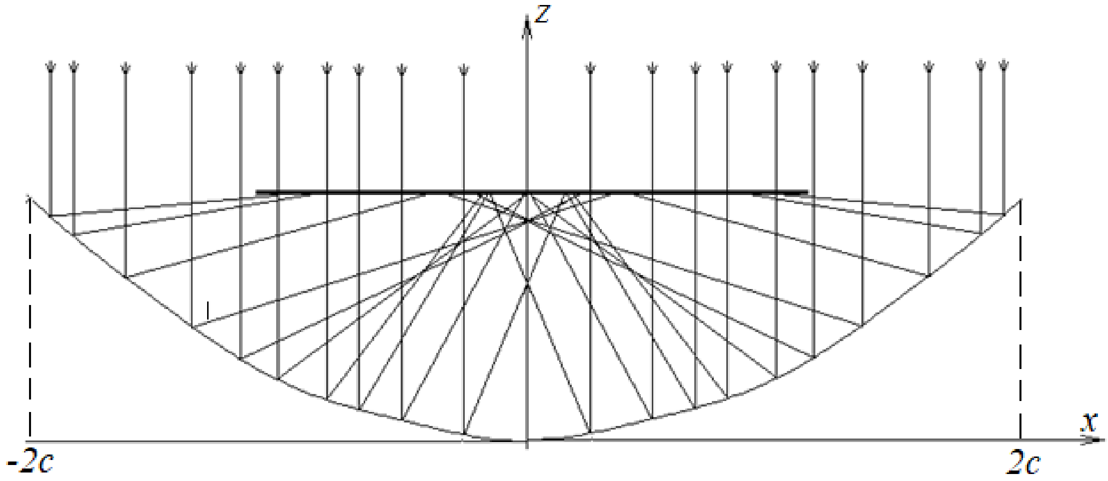

Figure 3. Wave-like perturbed parabolic mirror. Focus of the paraboloid is in point F(0;0;c). Amplitude of perturbance $A = 0.2c$

3. Conclusions

The proposed approach to assessing the deviation of a surface from a rotational paraboloid allows for the consideration of the efficiency of the antenna surface, taking into account energy losses during signal reception and transmission. This approach can be applied to analyze an antenna system subjected to wind loads and other natural disturbances that may induce random wave processes on the antenna surface. [6].

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