Simulation the Impact of Time-Delay in Richardson Arms Race Models

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Abstract

Based on statistical data for the years 2014 - 2018 from open sources, a specific model of the Richardson-type arms race is recorded in terms of system of ordinary differential equations (ODE), as traditional. A numerical analysis of its behavior was carried out. For this purpose was used well-known Maple program tools. In real life an answer of any actions can't to be immediate reaction. With the aim to improve adequacy, traditional model is modified by taking into account the time delay in the adversary's response. And new model is rewrites in term of system of functional-differential equations (FDE). There are well-known results from common FDE theory, that fact of involving time lag into study of a dynamic process give the significantly influences on the qualitative characteristics of behavior (stability, type of equilibrium points, bifurcation, etc.). To demonstrate this dependence, phase portraits were constructed for both models, and their comparative analysis was carried out.

Keywords [1]

Differential equation systems, stability, phase portrait, deviated argument.

1. Introduction

Lewis Richardson was born in 1881 in Newcastle and studied mathematical psychology. Richardson believed that each state steadily increases its equipment, as if obliged, forced to do so, which may be related to primitive instincts, or to the lack of a spiritual and moral basis for establishing borders. Based on this hypothesis, he built a mathematical model of the arms race [1,2]. There are many different definitions of arms races, but for the purposes of this paper they can be seen as long-lasting rivalries between pairs of hostile states that encourage the competitive acquisition of military power. One option is a two-person game, specifically a prisoner's dilemma, where the choice is to arm or not to arm, and the dominant strategy for both is not Pareto optimal [3,4]. The other is Richardson's model, as an action-reaction process represented by a pair of differential equations [5-7]. There is a sufficient number of scientific and popular science works devoted to both, the support and development of Richardson's ideas in this direction, and their substantial criticism, which sometimes reaches the point of complete denial [6,8,9]. The answer to this question is beyond the scope of this work.

2. Formulation of the problem

The basic idea of Richardson's model is that in a situation where there are two parties who are potential enemies, each of them responds to the actions of the other, increasing or decreasing its level of aggressiveness, or, as he puts it, "readiness for war." Let's call ***X*** and ***Y*** the two sides, respectively. The dynamics of aggressiveness between two parties ***X*** and ***Y*** can be described by the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= \alpha D(R_x(y), x) \\ \frac{dy}{dt} &= \beta D\big(R_y(x), x\big) \end{aligned} \qquad (1)$$

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Here, $x = R_x(y)$ is a function that, for any given level $\boldsymbol{y}$ of the action performed by party $\boldsymbol{Y}$, gives the value $\boldsymbol{x}$ of the action given in response by party $\boldsymbol{X}$, $y = R_y(x)$ – similarly for the other party; $D(z,w)$ is the distance (some metrics) between the arguments $\boldsymbol{z}$ and $\boldsymbol{w}$, according to some criteria, and α and β are positive constants. The idea is that the value of the change in aggressiveness is proportional to the distance between the optimal level, that is, that given by the response function, and the actual level. The response functions chosen by Richardson for his model are linear. Therefore, the differential equations describing conflict escalation in Richardson's model have the form

$$\begin{aligned} \frac{dx}{dt} &= ky(t) - \alpha x(t) + p \\ \frac{dy}{dt} &= lx(t) - \beta y(t) + q \end{aligned} \tag{2}$$

Here, k and l are protection coefficients, α and β are fatigue and expense coefficients, and p and q are indicators of past complaints (claims) between the two parties.

Solving the system of algebraic equations

$$\begin{aligned} ky + \alpha x + p = 0 \\ lx + \beta y + q = 0 \end{aligned}$$

We will obtain the equilibrium point

$$(x_{0,}y_0)\colon x_0 = R_X(y_0) \text{ i } y_0 = R_x(x_0)$$

It will be stable if

$$R'_X(y_0)R'_Y(x_0) < 1.$$

In the linear case (2), which was considered by Richardson, this condition turns into

$$\alpha\beta > kl$$

This can be interpreted as follows - for guaranteed stability, it is necessary that considerations regarding the cost of weapons have more weight than defense issues.

2.1. Some modern applications of the Richardson model

In work [10], the Richardson model was used to investigate the existence of an arms race between Pakistan and India for the period 1972–2010. The use of the generalized method of moments in the work showed that Pakistan's claims in the model are positive, and India's - negative. The defense expenditures of both countries in the past period are negative, because they are related to the change in their own defense expenditures due to the economic or administrative influence of the arms race. In addition, the coefficients of defense or reaction in the specified model confirm the existence of an arms race between the two countries. The values of these coefficients are positively consistent with the classic Richardson model, suggesting that an arms race does exist between Pakistan and India.

In work [11], the mode of the high power military game was considered, which contains two military means, namely, conventional and nuclear weapons. A country needs to break down its gross national product into daily consumption, conventional weapons, and nuclear weapons. On the one hand, nuclear weapons can be used as a military deterrent more effectively and improve national defense and security, and the budget savings can be spent on consumption.

On the other hand, the nuclear arms race will bring the world to the threat of a nuclear disaster. Unlike conventional weapons, nuclear weapons, as durable goods, can accumulate over time. Based on the above framework, this thesis established the endogeneity of Richardson's one-dimensional nuclear arms race model with a symmetric game.

Richardson-type models were used not only for the study of hostile conflicts. For example, the purpose of the work [3] is to apply some of the most widely known mathematical abstractions of war to the study of competition in commercial firms (in particular, the mobile phone industry of Greece).

Also interesting is the fact that the proposed systems of differential equations (1) (actually (2)) are quite universal. Thus, by applying a different interpretation of them, Lanchester-type combat models are successfully studied [12-14]. Even more useful information on the issue discussed in this subsection can be found in works [9,15], as well as the literature referenced in them.

2.2. Russia-Ukraine military confrontation

We will use the data obtained in work [16] for the model of the arms race between Russia and Ukraine (the original data source is the websites of the State Statistics Service of Ukraine and the Federal Statistics Service of the Russian Federation for Ukraine and Russia, respectively), Table 1,2,3.

Table 1
Expenditures of Ukraine and Russia in 2014-2018 (billion dollars)

Years	Ukraine	Absolute deviation	Russia	Absolute deviation
2014	5.5	-	84.5	-
2015	3.06	-2.44	67.0	-17.5
2016	2.329	-0.731	48.4	-18.6
2017	2.451	0.122	66.3	17.9
2018	2.937	0.486	45.69	-20.61

Table 2
Gross domestic product of Ukraine 2012-2017

Years	GDP in actual prices (billion UAH)	Absolute deviation	Dollar exchange rate	GDP in actual prices (billion USD)	Absolute deviation
2012	1404.669	-	7.9898	175.81	-
2013	1465.198	60.529	7.993	183.31	7.5
2014	1586.914	121.717	8.2714	191.86	8.55
2015	1988.544	401.629	16.2836	122.12	-69.74
2016	2385.367	396.823	25.5089	93.51	-28.61
2017	2982.920	597.553	28.1473	105.98	12.47

Table 3
Gross domestic product of the Russian Federation 2012-2017

Years	GDP in actual prices (billion UAH)	Absolute deviation	Dollar exchange rate	GDP in actual prices (billion USD)	Absolute deviation
2012	68163.9	-	31.879	2138.62	-
2013	73133.9	4970	30.4215	2404.02	265.4
2014	79199.7	6065.8	32.6587	2425.07	21.05
2015	83387.2	4187.5	56.2376	1482.77	-942.3
2016	86148.6	2761.4	72.9299	1181.25	-301.52
2017	92037.2	5888.6	59.8961	1536.61	355.36

Let us denote Ukraine by country X, and Russia, in turn, by Y. Accordingly, the defense and army expenditures of these countries will be denoted as $x = x(t)$, and $y = y(t)$. Based on the data in Tables 1-3, constant coefficients for the model were already calculated in [16], so we will take them in the form of Table 4 and substitute them in equations system (2).

We will get the finished model:

$$\begin{aligned} \frac{dx}{dt} &= 0.014y(t) - 0.44x(t) + 42.37 \\ \frac{dy}{dt} &= 3.3x(t) - 0.42y(t) - 121 \end{aligned} \quad (3)$$

First, let's find the equilibrium points of our system. The equations are linear, so let's equate the right-hand side of the system to zero and solve the system of equations. Let's get a point:

$$(x = 116.1717172, y = 624.6825397) \quad (4)$$

Table 4
Coefficients of the arms race model

	Ukraine	Russia	
X	2.937	45.69	***Y***
α	0.014	3.3	Β
γ	0.44	0.42	Δ
a	42.38	4.00	B
c	0	125	D

Now let's define the type of this point. For this we will find eigen values.
Values are real, different, of the same sign, negative

$$(\lambda 1 = -0.6452, \lambda 2 = -0.2148)$$

Therefore, the equilibrium point is a "stable node".

2.3. Consideration of the response delay factor

If one takes into account the decision-making time in the development and implementation of types of weapons, as well as technical delays in their implementation, then the system of ordinary differential equations with delay is more adequate. Unfortunately, if the solution of a linear stationary system without delay (2) in an analytical form can be obtained using the usual exponent, then for systems with even one constant delay, finding the general solution becomes more complicated. Richardson's model with lagging of general vector-matrix form can be written in the form of a system of differential-difference equations with the time-delay argument [7]

$$\frac{dx}{dt} = Ax(t) + By(t-\tau) + f(t), \quad \tau = const > 0 \tag{5}$$

Method for analytical finding of the solution for *special* cases of models of a system with time-delay (5) was considered in [7]. But for the general cases similar results have not yet been obtained. Therefore, in this work, we propose to carry out a numerical simulation of the effect of delay on the behavior of the Richardson arms race model, and to specify the results, we will use the system that describes the relations of the confronting parties from the previous subsection.

Taking as a basis the initial original model of Richardson (2), we modify its elements, without limiting the generality, we will introduce time-delay in the response of one of the parties,

for ***Y***:

$$\begin{aligned} \frac{dx}{dt} &= ky(t-\tau) - \alpha x(t) + p \\ \frac{dy}{dt} &= lx(t) - \beta y(t-\tau) + q \end{aligned} \tag{6}$$

for ***X***:

$$\begin{aligned} \frac{dx}{dt} &= ky(t) - \alpha x(t-\tau) + p \\ \frac{dy}{dt} &= lx(t-\tau) - \beta y(t) + q \end{aligned} \tag{7}$$

3. Numerical simulations

For the purpose of conducting numerical experiments, it is convenient to use one of the widely used software packages for solving similar problems (Mathematica, Maple, MatLab Simulink, Sage, etc.) [17-20]. We will use Maple in this research.

Let's build phase portraits of the arms race systems being studied - first we will do it for model (3), which corresponds to the confrontation between Ukraine and Russia according to the results of 2014-2018. Fig.1 clearly shows an equilibrium asymptotically stable point - a "node" with coordinates (4) .

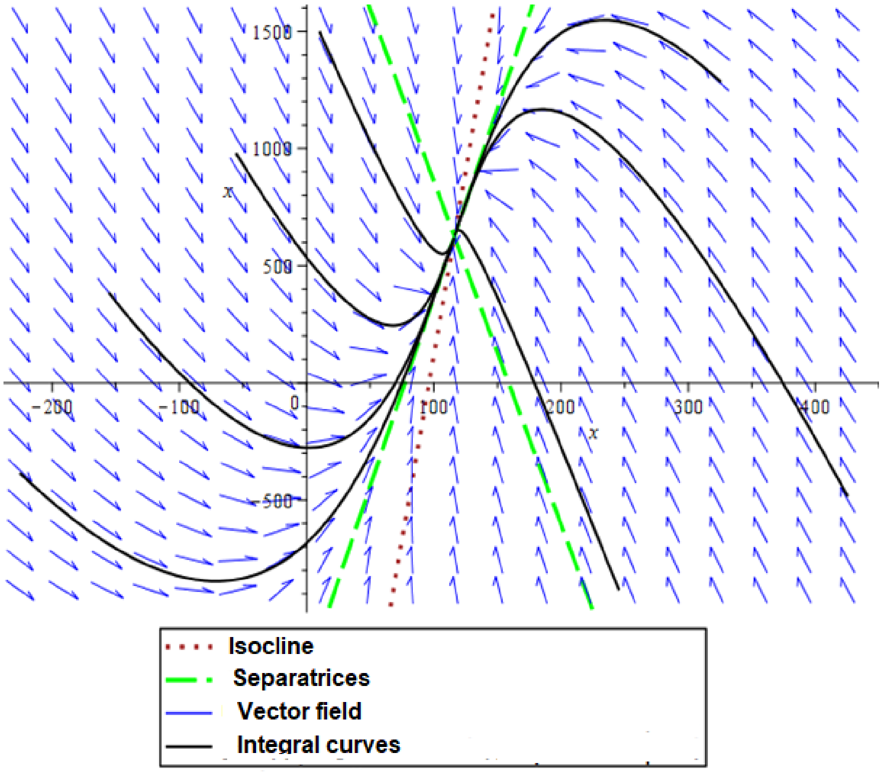

Figure 1. Asymptotically stable confrontation point "node" for model (3)

Next, we will proceed to the study of the impact of a delay in the response of one of the parties to the conflict. For the sake of clarity, let's focus on the variant of system (6).

We will successively consider certain values of the time delay τ=0.2; 1; 3; 5; 7. For each of the values of the selected time delay, we will construct the corresponding trajectories of the behavior of the system (6).

Graphs of these trajectories, for the purpose of comparison, will be superimposed on the phase portrait of the original system without deviation of the argument (Fig. 2 - Fig. 6).

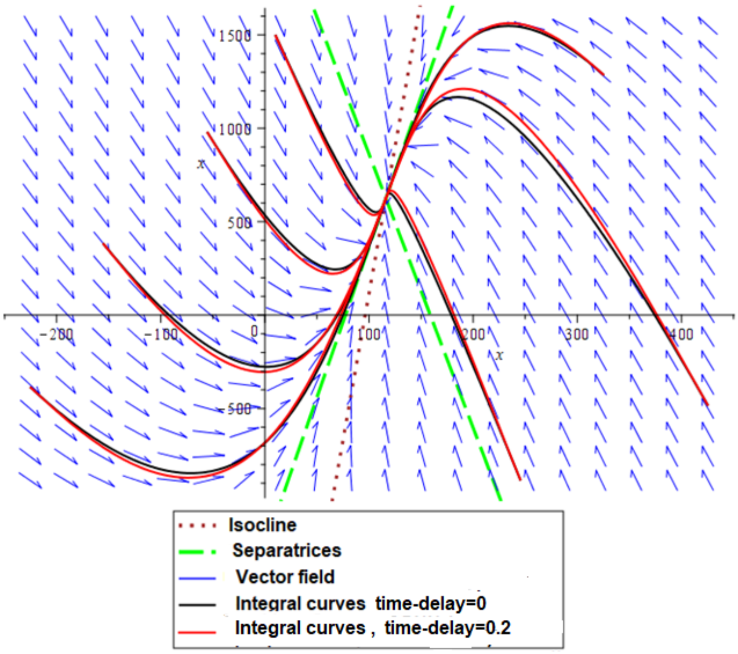

Figure 2. The behavior of the trajectories for model (6) with a delay value of τ=0.2 against the background of a normal phase portrait of model (3)

The given value of the argument deviation of 2 units is too small for the system to undergo any significant changes. We see small deviations, but, in general, the behavior of the system does not differ from the original one.

The next step is to increase the deviation. Analyzing the behavior of the corresponding trajectories of the system depicted in Fig. 3, we see that with a time-delay value equals to 1, the deviation of the integral curves from the initial ones (model (3)) becomes more noticeable, but still there are no qualitative fundamental changes. Curves also go to a equilibrium point "node".

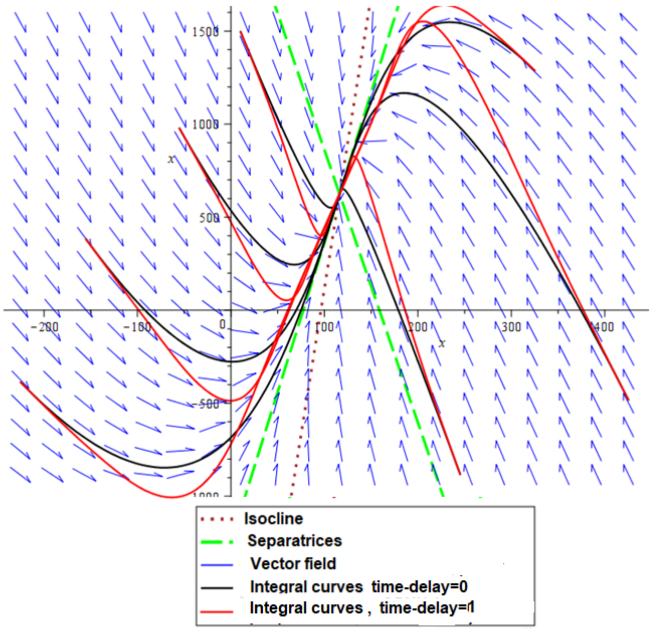

Figure 3. The behavior of the trajectories for model (6) with a delay value of *τ=1* against the background of a normal phase portrait of model (3)

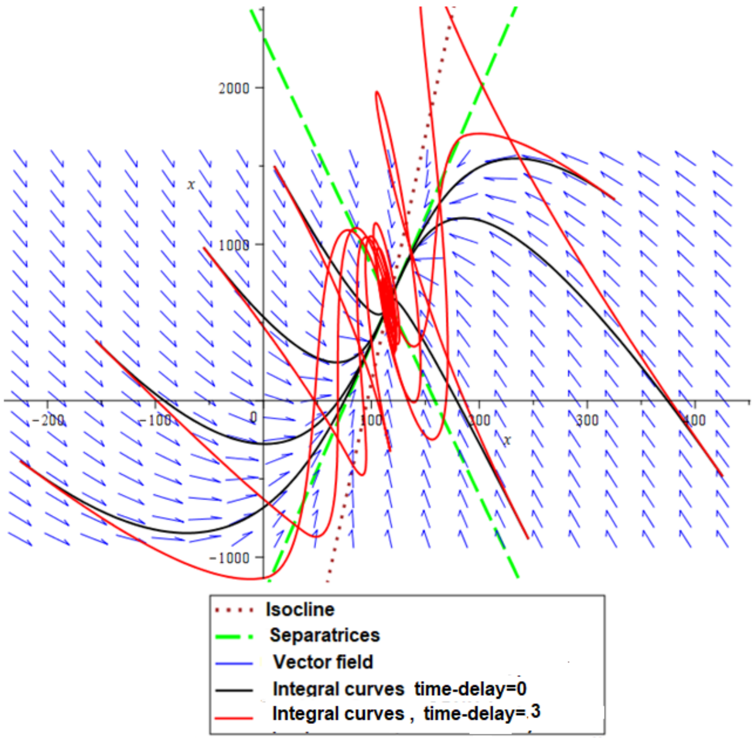

Figure 4. The behavior of the trajectories for model (6) with a delay value of *τ=3* against the background of a normal phase portrait of model (3)

As can be seen from Fig.4, with a delay value of 3 units, we can already see a distortion of the direction of movement of the integral curves, which no longer corresponds to the behavior of the equilibrium point "node", but also does not make it possible to clearly classify the obtained portrait.

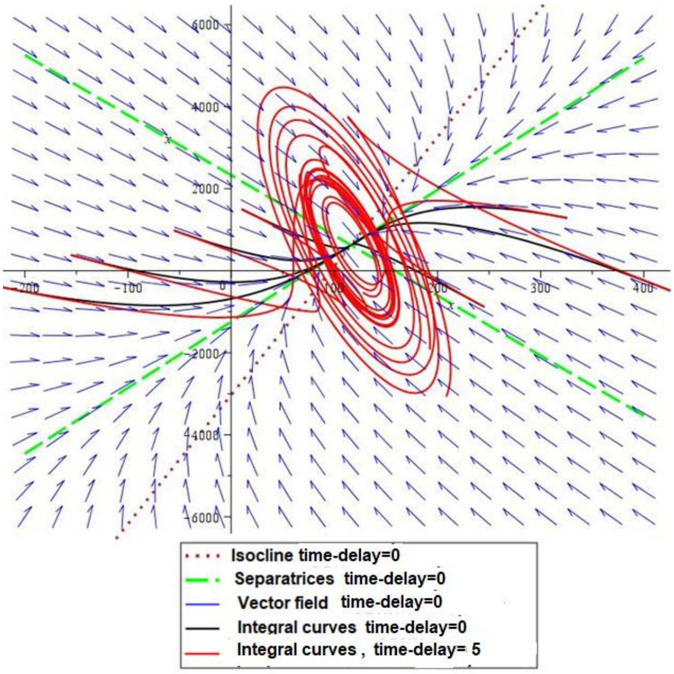

Figure 5. The behavior of the trajectories for model (6) with a delay value of *τ=5* against the background of a normal phase portrait of model (3)

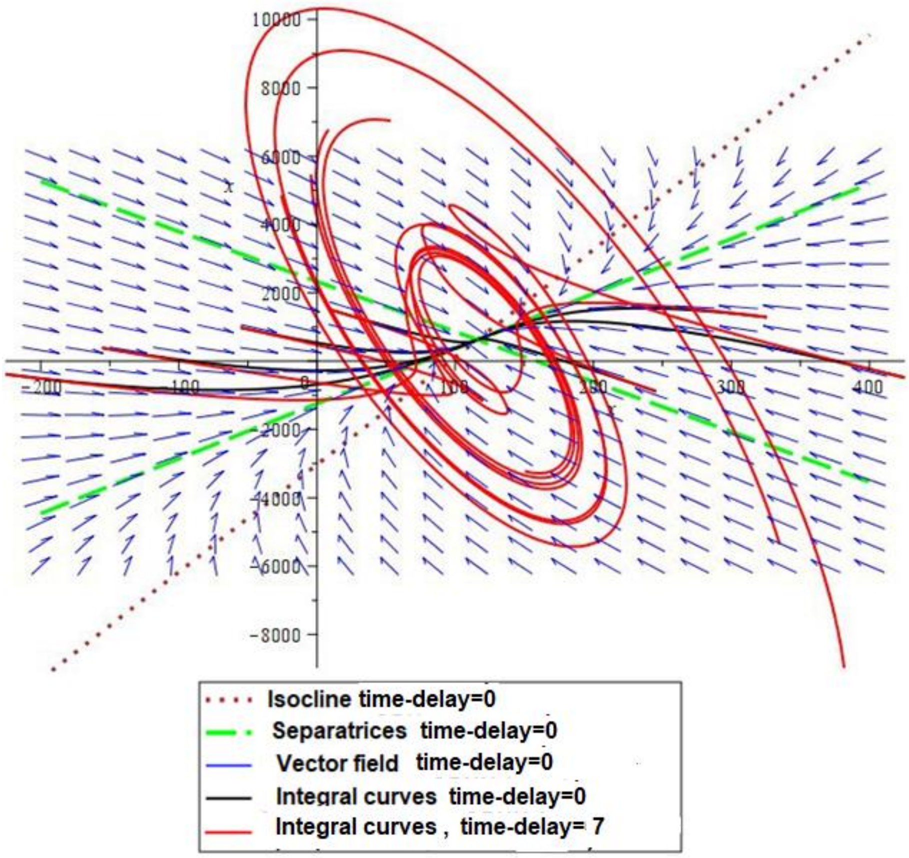

Figure 6. The behavior of the trajectories for model (6) with a delay value of *τ=5* against the background of a normal phase portrait of model (3)

As we can see, at a time-delay value of 5 units, the bifurcation of the phase portrait of the dynamic system under study occurred. The asymptotically stable equilibrium point of the "node"-type has finally disappeared. The behavior of the trajectories around this point clearly resembles the "focus"-type point.

By increasing the delay value of the argument to 7 units (see Fig.6), it becomes even more noticeable that the behavior of the integral curves has turned into a "focus". Moreover, a couple of additional points around which the trajectories begin to wind become noticeable.

Therefore, with a significant increase in the value of the delay in the dynamic system, a complete bifurcation of the phase portrait occurs.

These numerical calculations accurately confirmed the well-known theoretical results regarding complex dynamic systems described in terms of functional differential equations.

4. Conclusion

The problem of the arms race is very urgent today, few countries currently dare to directly escalate the conflict, at the same time they continue to actively build up their military forces, spending large percentages of their own budgets. In the framework of this study, Richardson's model of the arms race is considered, together with the history of its creation and progress. Based on statistical data from [16], which were calculated based on the GDP of countries and their defense spending, a mathematical model of the arms race between Ukraine and Russia starting in 2014 (Conflict in Donbas) was built. An accurate comparison was somewhat hindered by the fact that at the beginning of the conflict, the currencies of the respective countries experienced large inflationary fluctuations, and therefore the data in US dollars is not quite adequate, and the GDP and defense expenditures of the given countries are very different, which also makes the analysis of the arms race more difficult and more unpredictable.

Thanks to the created software using the MAPLE package, numerical calculations of the models were carried out (without argument deviation and with different gradually increasing time-delay values). They fully demonstrated the confirmation of theoretical mathematical results from the qualitative theory of differential equations.

In the future, trying to expand the scope of research, and with the aim of applying the obtained results in related fields, it is possible to partially use the results presented, for example, in works [21-24]. This can make it possible to modernize the conducted research by connecting the latest achievements in the field of artificial intelligence.

Finally, as for the conflicts related to the arms race in general, which can be described by Richardson's models, the main conclusion is one, and it is quite obvious, the delay in response significantly affects the development of the confrontation and is a destabilizing factor.

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