Decidability of ordered fragments of FOL via modal translation

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Abstract

We present a simplification and a modification of a method introduced by Herzig to prove the decidability of Quine's ordered fragment of first-order logic. The method consists in an interpretation of quantifiers as modal operators. We show that our modification yields the decidability of two new ordered fragments of first-order logic, called the grooved fragment and the loosely grooved fragment, whose expressive power lies between Quine's ordered fragment and the fluted fragment.

Keywords

Fragments of first-order logic, Decidability, Modal logic, Tree model property

1. Introduction

The *ordered fragments* of first-order logic (FOL) are those fragments in which an ordering associated with the set of variables imposes restrictions on their occurrences in atomic formulas, as well as on scopes of quantifiers [1]. The simplest of such fragments is due to Quine [2] and results from the following ideas:

1. an atomic formula can be formed only by giving variables $x_1, ..., x_n$ (in this order) as arguments to an n-ary predicate;
2. a complex formula is either obtained by applying sentential connectives to formulas with the same free variables, or by quantifying over the free variable with the largest index in a formula.

For example, the following sentence belongs to Quine's ordered fragment:

$$\forall x_1(Px_1 \to \exists x_2(Rx_1x_2 \wedge \forall x_3 Sx_1x_2x_3)).$$

Herzig [3] provides a translation from this fragment into propositional modal language in such a way that any input formula is satisfied in a first-order model iff its translation is satisfied in a Kripke model over a *serial* frame. Accordingly, one can employ decision procedures for the modal logic **KD** (which is semantically characterized by the class of all serial frames) to decide the satisfiability problem for Quine's ordered fragment.

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Herzig's translation relies on an intuitive reading of quantifiers as modal operators: $\forall$ corresponds to $\Box$ and $\exists$ corresponds to $\Diamond$. In other words, the claim that ϕ is the case for every individual amounts to the claim that ϕ is necessary, whereas the claim that ϕ is the case for some individual amounts to the claim that ϕ is possible. We stress that such a connection between quantifiers and modalities is simpler than the one employed in the *standard translation* of modal logic into (the guarded fragment of) first-order logic.

In the present article we provide a simplification and a modification of Herzig's method. By doing so, we obtain decidability and direct model construction for two other ordered fragments of FOL called the *grooved fragment* and the *loosely grooved fragment*, both of which lie between Quine's ordered fragment and the *fluted fragment* [4, 5]. More precisely, we have the following chain of (strict) inclusion for the mentioned fragments of FOL:

$$\textit{Quine's} \subset \textit{grooved} \subset \textit{loosely grooved} \subset \textit{fluted}$$

The rest of the article is arranged as follows. In Section 2 we introduce the modal translation of Quine's ordered fragment, followed by our simplified proof of satisfiability-invariance under the translation. In Section 3 we define the grooved fragment and use a modified translation to address the satisfiability problem. The loosely grooved fragment is defined in Section 4, where we show that its sentences can be rewritten into satisfiability-equivalent ones in the grooved fragment. We conclude the article with some remarks on potential applications of the ordered fragments, and on the relation between the fragments analysed here and some other fragments of FOL.

2. Quine's ordered fragment

The first-order language we are considering here is denoted by $\mathcal{L}_{FOL}$. It consists of a countable set $Pred$ of predicates (each with an arity $n \geq 1$), a countable set Var of individual variables, $\neg$, $\wedge$, $\forall$, and parentheses. (Other logical symbols can be introduced by definition in the usual way.) Elements of $Pred$ are denoted by P_1, P_2, etc., whereas those of Var by x_1, x_2, etc. The set of well-formed formulas of $\mathcal{L}_{FOL}$, denoted by $Form(\mathcal{L}_{FOL})$, is constructed in the usual way.

Now we start by specifying Quine's ordered fragment. For the sake of a more concise exposition, we associate each formula of the fragment with a *level* (a natural number).

Definition 1 (Ordered formulas). The set of ordered formulas $Form_{ord}(\mathcal{L}_{FOL})$ is the smallest subset of $Form(\mathcal{L}_{FOL})$ that satisfies the following conditions:

1. For any n-ary predicate P_i ($n \geq 1$), $P_i x_1 \ldots x_n$ is an ordered formula of level n.
2. If ϕ and ψ are ordered formulas of level n, so are $\neg\phi$ and $(\phi \wedge \psi)$.
3. If ϕ is an ordered formula of level n ($n > 0$), then $\forall x_n \phi$ is an ordered formula of level $n - 1$.

Note that an ordered formula of level 0 is an ordered *sentence* since it does not contain any free variables.

In the analysis of ordered formulas we will employ a simplified definition of the *satisfaction* relation. Recall that a model for $\mathcal{L}_{FOL}$ is an ordered pair $\mathcal{M} = \langle D, I\rangle$, where D is a non-empty set and I is an interpretation function s.t. for any n-ary predicate P_i, $I(P_i) \subseteq D^n$. For an ordered formula of level n, since its free variables are exactly $x_1, \dots, x_n$, we do not need to distinguish assignments which differ only on the value of other variables. Thus, we can use an n-tuple $\langle a_1, \dots, a_n\rangle$, where $a_i \in D$, to denote any assignment which assigns a_i to x_i. In particular, we can use the empty tuple ϵ for any assignment.

Definition 2 (Satisfaction for ordered formulas)**.** Let $\mathcal{M} = \langle D, I\rangle$ be a model for $\mathcal{L}_{FOL}$ and D^* be the set of all finite tuples of elements of D. We write σ_n for an element of D^n (where $D^n \subset D^*$) and ϕ_n, ψ_n for ordered formulas of level n. The *satisfaction relation* $\vDash$ is defined as follows (where $\sigma_{n-1}a \in D^n$ is the concatenation of $\sigma_{n-1} \in D^{n-1}$ and $a \in D$):

- $\mathcal{M}, \sigma_n \vDash P_i x_1 \dots x_n$ iff $\sigma_n \in I(P_i)$
- $\mathcal{M}, \sigma_n \vDash \neg\phi_n$ iff it is not the case that $\mathcal{M}, \sigma_n \vDash \phi_n$
- $\mathcal{M}, \sigma_n \vDash \phi_n \wedge \psi_n$ iff $\mathcal{M}, \sigma_n \vDash \phi_n$ and $\mathcal{M}, \sigma_n \vDash \psi_n$
- $\mathcal{M}, \sigma_{n-1} \vDash \forall x_n \phi_n$ iff for all $a \in D$, $\mathcal{M}, \sigma_{n-1}a \vDash \phi_n$

In particular, when an ordered *sentence* ϕ is true in $\mathcal{M}$, we write $\mathcal{M} \vDash \phi$ instead of $\mathcal{M}, \epsilon \vDash \phi$.

This definition can be trivially generalized to cover also the case where ϕ_n, a formula of level n, is evaluated with a tuple of length $n + m$ (where $m \geq 1$). Since the elements of D assigned to variables $x_{n+1}...x_{n+m}$ are irrelevant to the satisfaction of ϕ_n.

The propositional modal language $\mathcal{L}_{PML}$ used to translate Quine's ordered fragment consists of a countable set $Prop$ of propositional variables, $\neg$, $\wedge$, and the modal operator $\Box$. (Other logical symbols can be introduced by definition in the usual way.) The set $Prop$ is assumed to be equinumerous with $Pred$, and elements of $Prop$ will be denoted by p_1, p_2, etc. We say that a propositional variable corresponds to a predicate (and vice versa) if they have the same index. The set of formulas of $\mathcal{L}_{PML}$ is constructed as usual and will be denoted by $Form(\mathcal{L}_{PML})$.

We employ the standard relational semantics for $\mathcal{L}_{PML}$. A model for $\mathcal{L}_{PML}$ (henceforth also $\mathcal{L}_{PML}$-model or Kripke model) is an ordered triple $\mathfrak{M} = \langle W, R, V\rangle$ where: W is a non-empty set; R is a binary relation on W; and $V : Prop \longrightarrow \wp(W)$ is a function. Formulas of $\mathcal{L}_{PML}$ are evaluated relative to elements of W. In particular, $\mathfrak{M}, w \vDash \Box\phi$ iff $\mathfrak{M}, v \vDash \phi$ for all $v \in W$ s.t. Rwv. We will assume that the elements W, R and V define an $\mathcal{L}_{PML}$-model $\mathfrak{M}$, the elements W', R' and V' define an $\mathcal{L}_{PML}$-model $\mathfrak{M}'$, etc. The *frame* of an $\mathcal{L}_{PML}$-model $\mathfrak{M} = \langle W, R, V\rangle$ is the pair $\langle W, R\rangle$. A frame is said to be *serial* iff $(\forall w \in W)(\exists v \in W)Rwv$.

Some model-theoretic concepts will become relevant later, so we also mention them here for reference.

Definition 3 (Tree unravelling)**.** Given an $\mathcal{L}_{PML}$-model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$, the tree unravelling of $\mathfrak{M}$ at w is the model $\mathfrak{M}' = \langle W', R', V'\rangle$ defined as follows:

- W' is the set of all sequences $(v_1, \dots, v_n) \in W^n$ (where $n \geq 1$) s.t.:
  - $v_1 = w$,

- for $1 \leq i < n$, $Rv_i v_{i+1}$;

- $R'(v_1, \ldots, v_n)(u_1, \ldots, u_m)$ iff
  - $m = (n + 1)$,
  - for $1 \leq i \leq n$, $v_i = u_i$,
  - $Rv_n u_{n+1}$;
- for $p \in Prop$, $(v_1, \ldots, v_n) \in V'(p)$ iff $v_n \in V(p)$.

Definition 4 (Bounded morphism)**.** A bounded morphism from an $\mathcal{L}_{PML}$-model $\mathfrak{M}$ to an $\mathcal{L}_{PML}$-model $\mathfrak{M}'$ is a function $f : W \longrightarrow W'$ s.t.:

- for $p \in Prop$ and $w \in W$, $w \in V(p)$ iff $f(w) \in V'(p)$;
- if Rwv, then $R'f(w)f(v)$;
- if $R'f(w)v'$, then Rwu for some $u \in W$ s.t. $f(u) = v'$.

If f is a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then, for each $\phi \in Form(\mathcal{L}_{PML})$ and each $w \in W$, it holds that $\mathfrak{M}, w \vDash \phi$ iff $\mathfrak{M}', f(w) \vDash \phi$. Also, if $\mathfrak{M}'$ is the tree-unraveling of $\mathfrak{M}$ at w, there is a bounded morphism from the former to the latter.

Moreover, *filtration* is one of the standard techniques for constructing finite models in modal logic. (A detailed discussion can be found in [6].) Specifically, it allows one to prove that if $\phi \in Form(\mathcal{L}_{PML})$ is satisfiable in a class of $\mathcal{L}_{PML}$-models $\mathcal{C}$, then it is satisfiable in a finite model in $\mathcal{C}$. In our case $\mathcal{C}$ will be the class of models over serial frames or a specified subclass of this.

This ends the preliminaries. Now we move on to the translation of $Form_{ord}(\mathcal{L}_{FOL})$ into $Form(\mathcal{L}_{PML})$, which is due to Herzig [3].

Definition 5 (Translation)**.** The translation function, $tr : Form_{ord}(\mathcal{L}_{FOL}) \longrightarrow Form(\mathcal{L}_{PML})$, is defined recursively as follows:

- $tr(P_i x_1 \ldots x_n) = p_i$
- $tr(\neg\phi) = \neg tr(\phi)$
- $tr(\phi \wedge \psi) = tr(\phi) \wedge tr(\psi)$
- $tr(\forall x \phi) = \Box tr(\phi)$

Now we present a way to construct a Kripke model (i.e. a model for $\mathcal{L}_{PML}$) based on a first-order model (i.e. a model for $\mathcal{L}_{FOL}$).

Let $\mathcal{M} = \langle D, I \rangle$ be a model for $\mathcal{L}_{FOL}$. Then the $\mathcal{L}_{PML}$-*analogue* of $\mathcal{M}$, $\mathfrak{M} = \langle W, R, V \rangle$, is a model for $\mathcal{L}_{PML}$ such that:

- $W = D^*$ (the set of all finite tuples of elements of D)
- for any $\sigma, \tau \in W$, $R\sigma\tau$ iff $\tau = \sigma a$ for some $a \in D$
- $V(p_i) = I(P_i)$

Note that the frame $\langle W, R \rangle$ specified here is a tree where the root is the empty tuple. Also note that the frame is serial and $\mathfrak{M}$ is thus a **KD**-model.

Proposition 1 (Satisfiability invariance for $\mathcal{L}_{PML}$-analogues)**.** *Let* $\mathcal{M} = \langle D, I\rangle$ *be a model for* $\mathcal{L}_{FOL}$ *and* $\mathfrak{M} = \langle W, R, V\rangle$ *its* $\mathcal{L}_{PML}$*-analogue. For* $n \geq 0$*, if* ϕ_n *is an ordered formula of level* n *and* σ_n *is an* n*-tuple in* D^**, then:* $\mathcal{M}, \sigma_n \vDash \phi_n$ *iff* $\mathfrak{M}, \sigma_n \vDash tr(\phi_n)$*.*

Proof. By induction on ordered formulas. We only analyse the case where $\phi_n = \forall x_{n+1}\psi$: $\mathcal{M}, \sigma_n \vDash \forall x_{n+1}\psi$ iff $\mathcal{M}, \sigma_n a \vDash \psi$ for every $a \in D$ iff (by I.H.) $\mathfrak{M}, \sigma_n a \vDash tr(\psi)$ for every $a \in D$ iff $\mathfrak{M}, \sigma_n \vDash \Box tr(\psi)$. □

So far we have shown that, if we have a first-order model which satisfies an ordered *sentence* ϕ, we can build a **KD**-model over a tree in which $tr(\phi)$ is satisfied at the root.

For the other direction, namely to show that when we have a **KD**-model satisfiying $tr(\phi)$ we can build a first-order model satisfying ϕ, we begin with the following observations. First, given a **KD**-model where $tr(\phi)$ is satisfied, a filtration of the model through the set of subformulas of $tr(\phi)$ preserves the satisfiability of $tr(\phi)$ as well as the seriality of the frame, and therefore we can restrict our attention to **KD**-models with a bounded size. Second, for a finite **KD**-model with a bounded size, its tree unravelling is a **KD**-model where each node has a bounded number of children.

Let us fix some terminology before moving on. We say that a tree is m*-ary* iff each of its nodes has at most m children. A *perfect* m-ary tree is one in which each node has exactly m children. A tree is *serial* iff every one of its nodes has at least one child. Clearly, for $m > 0$, a perfect m-ary tree is serial. Moreover, each node in a tree is assigned a natural number as its *height*, defined as follows:

- if v is the root, $height(v) = 0$;
- if v' is a child of v, $height(v') = height(v) + 1$.

We say that a node w is lower than a node v just in case $height(w) < height(v)$.

We proceed in two steps. First, we show that each serial m-ary tree model can be expanded to a perfect m-ary tree model which is invariant w.r.t. the satisfiability of modal formulas.

Proposition 2 (From a tree to a perfect tree)**.** *Let* $\mathfrak{M} = \langle W, R, V\rangle$ *be a model over a serial* m*-ary tree (*$m > 0$*) with root* w_0*. Then there is a model* $\mathfrak{M}' = \langle W', R', V'\rangle$ *over a* perfect m*-ary tree with root* w'_0 *and a surjective bounded morphism* $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ *s.t.* $f(w'_0) = w_0$*.*

Proof. We describe a systematic procedure for constructing $\mathfrak{M}'$ and f.

Stage 0 Set $\mathfrak{M}_0 = \langle W_0, R_0, V_0\rangle = \mathfrak{M}$, and $f_0 = id_W$ (the identity function on W).

Stage n+1 If the frame in $\mathfrak{M}_n = \langle W_n, R_n, V_n\rangle$ is not a perfect tree, choose the lowest node $w \in W_n$ having less than m children; if there are multiple such nodes, choose one of them. Then pick a $v \in W_n$ s.t. $R_n wv$, and let

$$U = \{u \in W_n : R_n^* vu\} \text{ (} R_n^* \text{ is the reflexive and transitive closure of } R_n\text{)}$$

Suppose w has $m - k$ children ($0 < k < m$). For $1 \leq i \leq k$, let U_i be a set of *fresh* nodes (i.e. $U_i \cap W_n = \emptyset$ and for $1 \leq j < i$, $U_i \cap U_j = \emptyset$) s.t. $|U_i| = |U|$, and $g_i : U_i \longrightarrow U$ be a bijection. Let $S_i \subseteq U_i^2$ and $T_i : Prop \longrightarrow \mathcal{P}(U_i)$ be as follows:

for any $u, u' \in U_i$, $S_i uu'$ iff $R_n g_i(u) g_i(u')$
for any $u \in U_i$ and any $p \in Prop$, $u \in T_i(p)$ iff $g_i(u) \in V_n(p)$

Then, set $\mathfrak{M}_{n+1} = \langle W_{n+1}, R_{n+1}, V_{n+1} \rangle$ where

$W_{n+1} = W_n \cup \bigcup_{1 \leq i \leq k} U_i$
$R_{n+1} = R_n \cup \{(w, g_i^{-1}(v)) : 1 \leq i \leq k\} \cup \bigcup_{1 \leq i \leq k} S_i$
for any $p \in Prop$, $V_{n+1}(p) = V_n(p) \cup \bigcup_{1 \leq i \leq k} T_i(p)$

Also, let $f_{n+1} : \mathfrak{M}_{n+1} \longrightarrow \mathfrak{M}_n$ be the function such that

$$f_{n+1}(w) = \begin{cases} w & w \in W_n \\ g_i(w) & w \in U_i \end{cases}$$

This procedure yields the desired tree model $\mathfrak{M}' = \langle W', R', V' \rangle$ where:

$W' = \bigcup W_n$;
$R' = \bigcup R_n$;
for any $p \in Prop$, $V'(p) = \bigcup V_n(p)$.

Moreover, we have the function $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ such that

$f(w) = f_0 \circ f_1 \circ \cdots \circ f_n(w)$, if w first appears in stage n.

This is a surjective bounded morphism, and, obviously, $f(w'_0) = w_0$. $\square$

Now we can proceed to the second step of our construction, which consists in deriving a first-order model from a Kripke model over a perfect m-ary tree.

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model for $\mathcal{L}_{PML}$ in which $\langle W, R \rangle$ is a perfect m-ary tree ($m > 0$). Let D be a set with $|D| = m$, $E \subseteq D^* \times D^*$ be a relation such that: for $\sigma, \tau \in D^*$, $E\sigma\tau$ iff $\sigma = \tau a$ for some $a \in D$; accordingly, $\langle D^*, E \rangle$ is isomorphic to $\langle W, R \rangle$. Let $h : \langle D^*, E \rangle \longrightarrow \langle W, R \rangle$ be an isomorphism, and I be the interpretation function on $\mathcal{L}_{FOL}$ such that: for any n-ary predicate P_i, $I(P_i) = \{\sigma \in D^n : h(\sigma) \in V(p_i)\}$. Then $\mathcal{M} = \langle D, I \rangle$ is a model for $\mathcal{L}_{FOL}$, and we call it an *$\mathcal{L}_{FOL}$-analogue* of $\mathfrak{M}$. Notice that there is a unique $\mathcal{L}_{FOL}$-analogue for each $\mathcal{L}_{PML}$-model, up to isomorphism.

Proposition 3 (Satisfiability invariance for $\mathcal{L}_{FOL}$-analogues)**.** *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a perfect m-ary ($m > 0$) tree model for $\mathcal{L}_{PML}$, and $\mathcal{M} = \langle D, I \rangle$ be an $\mathcal{L}_{FOL}$-analogue of $\mathfrak{M}$. Then: for $n \geq 0$, $\mathfrak{M}, h(\sigma_n) \vDash tr(\phi_n)$ iff $\mathcal{M}, \sigma_n \vDash \phi_n$, where ϕ_n is an ordered formula of level n, and σ_n is an n-tuple from D^*.*

Proof. By induction on ordered formulas. $\square$

Proposition 4 (Satisfiability invariance under tr)**.** *Let ϕ be an ordered sentence. Then: ϕ is satisfiable iff $tr(\phi)$ is* **KD***-satisfiable. Therefore, the satisfiability problem for Quine's ordered fragment is decidable.*

Proof. By Propositions 1, 2, and 3. □

We also stress that the construction provided above indicates that one can build a *conservative extension* of both Quine's fragment of FOL and $\mathbf{KD}$ thanks to function tr, as discussed in [7]. We can define a conservative extension of Quine's fragment of FOL and $\mathbf{KD}$ as a system S of first-order modal logic (hence, whose language results from the combination of $\mathcal{L}_{FOL}$ and $\mathcal{L}_{PML}$) such that (i) S contains all theorems of each of the two systems at issue, and (ii) the schema $\forall x\phi \leftrightarrow \psi$ is derivable in S for some formula $\psi \in \mathcal{L}_{PML}$. In order to see this point, consider the semantic procedure described below.

Combine any $\mathcal{L}_{PML}$-model $\mathfrak{M} = \langle W, R, V\rangle$ over a perfect m-ary tree ($m > 0$) and an $\mathcal{L}_{FOL}$-analogue $\mathcal{M} = \langle D, I\rangle$ of $\mathfrak{M}$ thanks to the bijective function h used in the construction of $\mathcal{M}$ (see Proposition 2). The result is a hybrid structure $M = \langle W, R, V, D, I, h\rangle$ where $\mathcal{L}_{PML}$ and $\mathcal{L}_{FOL}$ are simultaneously interpreted. Use the bijective function h to assign a *label* to each element of W, i.e. for $\sigma \in D^*$, if $h(\sigma) = w \in W$, then $\sigma = label(w)$. Moreover, put together the definition of the satisfaction relation $\vDash$ in Kripke models and in first-order models and let $M, w \vDash \phi$ become interchangeable with $M, label(w) \vDash \phi$. Once this is done, it is immediate to see that $\phi \leftrightarrow tr(\phi)$ is valid in M. *A fortiori*, $\forall x\phi \leftrightarrow \Box tr(\phi)$ is valid in M.[1] Furthermore, notice that $\Box tr(\phi) \in \mathcal{L}_{PML}$. Finally, let $\mathcal{C}$ be the class of all hybrid models defined as above and $Th(\mathcal{C})$ the set of first-order modal formulas that are valid in $\mathcal{C}$. Then, any system of first-order modal logic S whose set of theorems contains $Th(\mathcal{C})$ is a conservative extension of both Quine's fragment of FOL and $\mathbf{KD}$.

3. The grooved fragment

In this section we present a modification of the method used above. We consider an ordered fragment which is larger than Quine's. We name it the *grooved fragment*. The additional expressiveness of this new fragment results from allowing unary predicates to take a variable x_n, for any $n > 0$.

Definition 6 (Grooved formulas)**.** The set of grooved formulas $Form_{gro}(\mathcal{L}_{FOL})$ is the smallest subset of $Form(\mathcal{L}_{FOL})$ that satisfies the following conditions:

1. If P is an n-ary predicate ($n > 1$), $Px_1 \dots x_n$ is a grooved formula of level n.
2. If P is a *unary* predicate, Px_n is a grooved formula of level n ($n > 0$).
3. If ϕ and ψ are grooved formulas of level n, so are $\neg\phi$ and $(\phi \wedge \psi)$.
4. If ϕ is a grooved formula of level n ($n > 0$), then $\forall x_n\phi$ is a grooved formula of level $n - 1$.

The identification of the grooved fragment is inspired by works on the *relational syllogistic*, the extension of the classical syllogistic with relational terms. Logical systems in that context feature, for example, the following sentences:

[1]For instance, consider the following case: $\forall x_1 P_i x_1 \leftrightarrow \Box p_i$. Let M be a hybrid model and w a state in its domain. It holds that $M, w \vDash \forall x_1 P_i x_1$ iff $M, label(w) \vDash \forall x_1 P_i x_1$ iff $M, label(w)a \vDash P_i x_1$ for every $a \in D$ iff $M, v \vDash p_i$ for every v s.t. Rwv iff $M, w \vDash \Box p_i$. Therefore, $M, w \vDash \forall x_1 P_i x_1 \leftrightarrow \Box p_i$.

No student admires every professor
$\forall x_1(Sx_1 \to \neg\forall x_2(Px_2 \to Ax_1x_2))$
No lecturer introduces any professor to every student
$\forall x_1(Lx_1 \to \neg\exists x_2(Px_2 \wedge \forall x_3(Sx_3 \to Ix_1x_2x_3)))$

Clearly, such sentences are not in the ordered fragment defined in the previous section, since they typically contain atoms of the form Px_n, where $n > 1$, whereas this is now accommodated by the grooved fragment. In fact, the grooved fragment is more expressive than many systems for the relational syllogistic. See e.g. [8] and [9] for a detailed comparison.

Given the existence of formulas of the form Px_n, we modify the satisfaction relation as follows.

Definition 7 (Satisfaction for grooved formulas)**.** Let $\mathcal{M} = \langle D, I\rangle$ be a model for $\mathcal{L}_{FOL}$, σ_n be an n-tuple from D^*, $last(\sigma_n)$ be the last element of σ_n, and ϕ_n, ψ_n be grooved formulas of level n. Then:

- $M, \sigma_n \vDash Px_1 \ldots x_n$ iff $\sigma_n \in I(P)$ (P is not unary)
- $M, \sigma_n \vDash Px_n$ iff $last(\sigma_n) \in I(P)$ (P is unary)
- $M, \sigma_n \vDash \neg\phi_n$ iff it is not the case that $M, \sigma_n \vDash \phi_n$
- $M, \sigma_n \vDash \phi_n \wedge \psi_n$ iff $M, \sigma_n \vDash \phi_n$ and $M, \sigma_n \vDash \psi_n$
- $M, \sigma_{n-1} \vDash \forall x_n\phi_n$ iff for all $a \in D$, $M, \sigma_{n-1}a \vDash \phi_n$

The translation function for the grooved fragment is slightly different from the one in Definition 5. We will call the new function tr as well since no ambiguity will arise.

Definition 8 (Translation)**.** The translation function, $tr : Form_{gro}(\mathcal{L}_{FOL}) \longrightarrow Form(\mathcal{L}_{PML})$, is defined recursively as follows:

- $tr(P_ix_1 \ldots x_n) = p_i$ (P_i is not unary)
- $tr(P_ix_n) = p_i$ (P_i is unary)
- $tr(\neg\phi) = \neg tr(\phi)$
- $tr(\phi \wedge \psi) = tr(\phi) \wedge tr(\psi)$
- $tr(\forall x\phi) = \Box tr(\phi)$

Now we present a modified way to construct a Kripke model from a first-order one. Let $\mathcal{M} = \langle D, I\rangle$ be a model for $\mathcal{L}_{FOL}$. Then the *$\mathcal{L}_{PML}$-analogue* of $\mathcal{M}$, $\mathfrak{M} = \langle W, R, V\rangle$, is a model for $\mathcal{L}_{PML}$ defined as below:

- $W = D^*$
- for any $\sigma, \tau \in W$, $R\sigma\tau$ iff $\tau = \sigma a$ for some $a \in D$
- for non-unary P_i, $V(p_i) = I(P_i)$
- for unary P_i, $V(p_i) = \{\sigma \in D^* \setminus \{\epsilon\} : last(\sigma) \in I(P_i)\}$ (ϵ is the empty tuple)

$\mathfrak{M}$ is obviously a **KD**-model over a tree.

Proposition 5 (Satisfiability invariance for $\mathcal{L}_{PML}$-analogues)**.** *Let $\mathcal{M}$ be a model for $\mathcal{L}_{FOL}$ and $\mathfrak{M}$ its $\mathcal{L}_{PML}$-analogue. For $n \geq 0$, if ϕ_n is a grooved formula of level n and σ_n is an n-tuple in D^*, then: $\mathcal{M}, \sigma_n \vDash \phi_n$ iff $\mathfrak{M}, \sigma_n \vDash tr(\phi_n)$.*

Proof. As in Proposition 1. $\square$

Thus, in particular, a grooved *sentence* ϕ is true in $\mathcal{M}$ exactly when $tr(\phi)$ is true at the root of $\mathfrak{M}$.

Given a grooved sentence ϕ, let $\mathbf{S}(\phi)$ be the set of propositional variables corresponding to the *unary* predicates in ϕ. Let $\Gamma(\phi)$ be the set of maximal consistent sets of literals (i.e. propositional variables or their negation) formed by elements of $\mathbf{S}(\phi)$, and let

$$\Psi(\phi) = \left\{ \bigwedge \Sigma : \Sigma \in \Gamma(\phi) \right\}$$

$$\Upsilon(\phi) = \left(\bigwedge_{\psi \in \Psi(\phi)} (\Diamond\psi \to \Box\Diamond\psi) \right) \wedge \left(\bigwedge_{\psi \in \Psi(\phi)} (\neg\Diamond\psi \to \Box\neg\Diamond\psi) \right)$$

Proposition 6. *Let $\mathcal{M} = \langle D, I \rangle$ be a model for $\mathcal{L}_{FOL}$, $\mathfrak{M} = \langle W, R, V \rangle$ be the $\mathcal{L}_{PML}$-analogue of $\mathcal{M}$. Then $\Upsilon(\phi)$ is valid (globally true) in $\mathfrak{M}$.*

Proof. We observe that for any $\sigma, \tau \in D^* \setminus \{\epsilon\}$, if $last(\sigma) = last(\tau)$ then: for any $s \in \mathbf{S}(\phi)$, $\sigma \in V(s)$ iff $\tau \in V(s)$. $\square$

So far, we have seen that if a grooved sentence ϕ is satisfiable, then its translation $tr(\phi)$ is satisfied in a **KD**-model where $\Upsilon(\phi)$ is valid. For the opposite direction we start from the following observations.

Given a **KD**-model where $\Upsilon(\phi)$ is valid and $tr(\phi)$ is satisfied, a filtration of the model through the set of subformulas of $\Upsilon(\phi)$ or $tr(\phi)$ preserves the satisfiability of $tr(\phi)$ as well as the validity (global truth) of $\Upsilon(\phi)$. Also, since the filtration remains a **KD**-model and has a bounded size, its tree unravelling at the node satisfying $tr(\phi)$ is a **KD**-model in which each node has a bounded number of children.

For simplicity, from now on we always assume the restriction of $\mathcal{L}_{FOL}$ to the predicates occurring in ϕ, and, correspondingly, the restriction of $\mathcal{L}_{PML}$ to the propositional variables occurring in $tr(\phi)$.

Given a Kripke model $\mathfrak{M} = \langle W, R, V \rangle$, let the *sort* of each $w \in W$, written $srt(w)$, be as follows:

$$srt(w) = \{ s \in \mathbf{S}(\phi) : w \in V(s) \}$$

Proposition 7. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a tree model for $\mathcal{L}_{PML}$, with $w_0 \in W$ its root, such that $\Upsilon(\phi)$ is valid in $\mathfrak{M}$. For any $w, u, v \in W$, if Rwu then there is $u' \in W$ s.t. Rvu' and $srt(u) = srt(u')$.*

Proof. Suppose $s_1, \ldots, s_n$ are all the members of $\mathbf{S}(\phi)$. Let $\bigwedge_{i=1}^{n} \pm s_i$ be the conjunction of *literals* true at u. Then $\Diamond \bigwedge_{i=1}^{n} \pm s_i$ is true at w. Since $\Upsilon(\phi)$ is valid in the model, in particular we have that

$$\Diamond \bigwedge_{i=1}^{n} \pm s_i \to \Box\Diamond \bigwedge_{i=1}^{n} \pm s_i$$

and

$$\neg\Diamond\bigwedge_{i=1}^{n}\pm s_i \rightarrow \Box\neg\Diamond\bigwedge_{i=1}^{n}\pm s_i$$

are valid, from which we can show that $\Diamond\bigwedge_{i=1}^{n}\pm s_i$ is true (false) at the root w_0 iff it is globally true (false). Since $\Diamond\bigwedge_{i=1}^{n}\pm s_i$ is true at w, it must also be true at w_0, and therefore globally true. Thus, for any $v \in W$, there is $u' \in W$ s.t. Rvu' and $\bigwedge_{i=1}^{n}\pm s_i$ is true there. $\Box$

Given a Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ and $w \in W$, let

$$\mathbf{Srt}(w) = \{srt(u) : u \in W \text{ and } Rwu\}$$

Then, if $\mathfrak{M}$ is a tree model and $\Upsilon(\phi)$ is valid in $\mathfrak{M}$, by Proposition 7 we have that, for any $w, v \in W$, $\mathbf{Srt}(w) = \mathbf{Srt}(v)$. We thus call $\mathfrak{M}$ a *well-sorted* tree model, and let

$$\mathbf{Srt}(\mathfrak{M}) = \{srt(w) : w \in W\}$$

Also, if $\langle W, R\rangle$ is an m-ary tree, we have $|\mathbf{Srt}(\mathfrak{M})| \leq m$.

For a well-sorted tree model $\mathfrak{M} = \langle W, R, V\rangle$, we denote the members of $\mathbf{Srt}(\mathfrak{M})$ by $\mathbf{Q}_1, \ldots, \mathbf{Q}_{|\mathbf{Srt}(\mathfrak{M})|}$, and then, for each $w \in W$, let

$$\mathbf{Q}_i(w) = \{u \in W : Rwu \text{ and } srt(u) = \mathbf{Q}_i\}$$

Let $\mu(\mathbf{Q}_i)$ be the maximum number of children of sort $\mathbf{Q}_i$ that a node of the tree can have, i.e.

$$\mu(\mathbf{Q}_i) = max\{|\mathbf{Q}_i(w)| : \ w \in W\}$$

Obviously, if $\langle W, R\rangle$ is an m-ary tree, then $\mu(\mathbf{Q}_i) \leq m$.

In a well-sorted tree model $\mathfrak{M} = \langle W, R, V\rangle$, a node $w \in W$ is *fulfilled* iff for $1 \leq i \leq |\mathbf{Srt}(\mathfrak{M})|$, $|\mathbf{Q}_i(w)| = \mu(\mathbf{Q}_i)$. A well-sorted tree model is fulfilled iff all of its nodes are fulfilled. Clearly, the frame of a fulfilled tree model is a perfect tree.

We next proceed, as in Section 2, by showing how to expand a serial tree to a perfect tree. This time, though, the number of children of each node in the resulting tree may be higher than the maximum number of children of a node in the original tree.

Proposition 8 (From a well-sorted tree model to a fulfilled tree model)**.** *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a well-sorted tree model over a serial m-ary tree ($m > 0$) with root w_0. Then there is a fulfilled tree model $\mathfrak{M}' = \langle W', R', V'\rangle$ over a perfect $\left(\sum_{i=1}^{|\mathbf{Srt}(\mathfrak{M})|}\mu(\mathbf{Q}_i)\right)$-ary tree with root w_0', and a surjective bounded morphism $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ s.t. $f(w_0') = w_0$.*

Proof. The following procedure constructs $\mathfrak{M}'$ and f.

Stage 0 Set $\mathfrak{M}_0 = \langle W_0, R_0, V_0\rangle = \mathfrak{M}$, and $f_0 = id_W$.

Stage n+1 If $\mathfrak{M}_n = \langle W_n, R_n, V_n\rangle$ is not fulfilled, choose the lowest node $w \in W_n$ which is not fulfilled; if there are multiple such nodes, choose one. For the least i s.t. $|\mathbf{Q}_i(w)| < \mu(\mathbf{Q}_i)$:

Pick up a $v \in W_n$ s.t. R_nwv and $srt(v) = \mathbf{Q}_i$, and let

$U = \{u \in W_n : R_n^* vu\}$ (R_n^* is the reflexive and transitive closure of R_n)

Let $l = \mu(\mathbf{Q}_i) - |\mathbf{Q}_i(w)|$. For $1 \leq j \leq l$, let U_j be a set of *fresh* nodes s.t. $|U_j| = |U|$; and let $g_j : U_j \longrightarrow U$ be a bijection. Let $S_j \subseteq U_j^2$ and $T_j : Prop \longrightarrow \mathcal{P}(U_j)$ be as follows:

for any $u, u' \in U_j$, $S_j uu'$ iff $R_n g_j(u) g_j(u')$

for any $u \in U_j$ and any $p \in Prop$, $u \in T_j(p)$ iff $g_j(u) \in V_n(p)$

Then, set $\mathfrak{M}_{n+1} = \langle W_{n+1}, R_{n+1}, V_{n+1} \rangle$ where

$W_{n+1} = W_n \cup \bigcup_{1 \leq j \leq l} U_j$

$R_{n+1} = R_n \cup \{(w, g_j^{-1}(v)) : 1 \leq j \leq l\} \cup \bigcup_{1 \leq j \leq l} S_j$

for any $p \in Prop$, $V_{n+1}(p) = V_n(p) \cup \bigcup_{1 \leq j \leq l} T_j(p)$

Also, let $f_{n+1} : \mathfrak{M}_{n+1} \longrightarrow \mathfrak{M}_n$ be the function such that

$$f_{n+1}(w) = \begin{cases} w & w \in W_n \\ g_j(w) & w \in U_j \end{cases}$$

As before, we end up with the desired tree model $\mathfrak{M}' = \langle W', R', V' \rangle$ where: $W' = \bigcup W_n$; $R' = \bigcup R_n$; for any $p \in Prop$, $V'(p) = \bigcup V_n(p)$. Obviously, each node in $\mathfrak{M}'$ has exactly $\sum_{i=1}^{|\mathbf{Srt}(\mathfrak{M})|} \mu(\mathbf{Q}_i)$ children. We can also define the surjective bounded morphism $f : \mathfrak{M}' \longrightarrow \mathfrak{M}$ in the same way as in the proof of Proposition 2. $\square$

Now, with a fulfilled tree model we can build a first-order model. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a fulfilled tree model for $\mathcal{L}_{PML}$, where $\langle W, R \rangle$ is a perfect m-ary tree ($m > 0$). Let D be an arbitrary set s.t. $|D| = m$, and $E \subseteq D^* \times D^*$ be a relation s.t., for $\sigma, \tau \in D^*$, $E\sigma\tau$ iff $\sigma = \tau a$ for some $a \in D$; accordingly, $\langle D^*, E \rangle$ is isomorphic to $\langle W, R \rangle$. Let $h : \langle D^*, E \rangle \longrightarrow \langle W, R \rangle$ be an isomorphism such that

for $\sigma, \sigma' \in D^* \setminus \{\epsilon\}$, if $last(\sigma) = last(\sigma')$ then $srt(h(\sigma)) = srt(h(\sigma'))$.

Let I be an interpretation function on $\mathcal{L}_{FOL}$ such that: for any n-ary predicate P_i, $I(P_i) = \{\sigma \in D^n : h(\sigma) \in V(p_i)\}$. Then $\mathcal{M} = \langle D, I \rangle$ is a model for $\mathcal{L}_{FOL}$, and we call it an $\mathcal{L}_{FOL}$*-analogue* of $\mathfrak{M}$.

Proposition 9 (Satisfiability invariance for $\mathcal{L}_{FOL}$-analogues). *Let $\mathfrak{M}$ be a fulfilled tree model for $\mathcal{L}_{PML}$ and $\mathcal{M}$ its $\mathcal{L}_{FOL}$-analogue. Then: for $n \geq 0$, $\mathfrak{M}, h(\sigma_n) \vDash tr(\phi_n)$ iff $\mathcal{M}, \sigma_n \vDash \phi_n$, where ϕ_n is a grooved formula of level n, and σ_n is an n-tuple from D^*.*

Proof. By induction on grooved formulas. $\square$

Proposition 10 (Satisfiability invariance under tr). *Let ϕ be a grooved sentence. Then: ϕ is satisfiable iff $tr(\phi)$ is satisfied in a* $\mathbf{KD}$*-model in which $\Upsilon(\phi)$ is valid. Therefore, the satisfiability problem for the grooved fragment is decidable.*

Proof. By Propositions 5, 6,7, 8, and 9. □

Let $\mathbf{KD}+\Upsilon(\phi)$ be the system obtained by adding $\Upsilon(\phi)$ to an axiomatic basis for **KD**. The construction provided in this section indicates that we can build a *conservative extension* of both the grooved fragment and $\mathbf{KD}+\Upsilon(\phi)$. To see this, one can proceed as at the end of Section 2.

4. The loosely grooved fragment

In this section we define an ordered fragment more expressive than the grooved fragment. We call it the loosely grooved fragment. We will show that each sentence in this fragment can be rewritten into a satisfiability-equivalent grooved sentence.

Definition 9 (Loosely grooved formulas)**.** The set of loosely grooved formulas $Form_{gro'}(\mathcal{L}_{FOL})$ is the smallest subset of $Form(\mathcal{L}_{FOL})$ that satisfies the following conditions:

1. For each $(n-m+1)$-ary predicate P $(m \leq n)$, $Px_m \dots x_n$ is a loosely grooved formula of level n.
2. If ϕ is a loosely grooved formulas of level n, then so is $\neg\phi$.
3. If $\phi(x_l, \dots, x_n)$ and $\psi(x_m, \dots, x_n)$ are loosely grooved formulas of level n, whose free variables are exactly those in the parentheses respectively, and one of the following conditions holds, then so is $(\phi \wedge \psi)$:
   - $l = m$, i.e. ϕ and ψ have the same free variables
   - $l = n$ or $m = n$, i.e. one of ϕ and ψ has exactly x_n free
   - $l > n$ or $m > n$, i.e. one of them has no free variable
4. If ϕ is a loosely grooved formula of level n $(n > 0)$, then $\forall x_n \phi$ is a grooved formula of level $n-1$.

Note that if, for example, P is a ternary predicate and Q is a binary predicate, then $Px_2x_3x_4 \wedge Qx_3x_4$ is *not* a formula of the loosely grooved fragment (even though both $Px_2x_3x_4$ and Qx_3x_4 are loosely grooved formulas of level 4), since the two conjuncts have different numbers of free variables and both of them have more than one free variable.

Before describing a general procedure for rewriting a loosely grooved sentence into a satisfiability-equivalent grooved one, let us take a look at an example. The following sentence is not grooved but loosely grooved:

$$\forall x_1(Px_1 \to \forall x_2(\forall x_3(Px_3 \to Rx_2x_3) \to \neg Rx_1x_2))$$

Since the atom Rx_2x_3 is not grooved. Notice that, if we introduce a fresh unary predicate, say Q, substitute Qx_2 for $\forall x_3(Px_3 \to Rx_2x_3)$, and conjoin the result with the formula $\forall x_1(Qx_1 \leftrightarrow \forall x_2(Px_2 \to Rx_1x_2))$, we get

$$\forall x_1(Px_1 \to \forall x_2(Qx_2 \to \neg Rx_1x_2)) \wedge \forall x_1(Qx_1 \leftrightarrow \forall x_2(Px_2 \to Rx_1x_2))$$

which is satisfiability-equivalent to the original formula. Notice also that Qx_2 is a grooved formula of level 2, and Qx_1 and $\forall x_2(Px_2 \to Rx_1x_2)$ are grooved formulas of level 1, so the whole formula is indeed a grooved sentence.

In the following we write $\forall x_{i+1}\phi(x_i, x_{i+1})$ for a formula in that form where ϕ has exactly x_i and x_{i+1} free (so the whole formula has only x_i free). We say a loosely grooved formula is *bad* if it is not grooved *and* is of the form $\forall x_{i+1}\phi(x_i, x_{i+1})$, where $i > 1$.

Proposition 11. *Let φ be a loosely grooved sentence. We can effectively construct a loosely grooved sentence φ' which is (i) satisfiability-equivalent to φ and (ii) free of* bad *subformulas.*

Proof. The following procedure constructs the sentence we desire. It starts by setting $\varphi_0 := \varphi$. Then, for each loosely grooved sentence φ_k, if φ_k contains a bad subformula $\forall x_{i+1}\phi(x_i, x_{i+1})$ of which no proper-subformula is bad, choose a unary predicate Q not occurring in φ_k, and let

$$\varphi_{k+1} := \varphi_n[\forall x_{i+1}\phi(x_i, x_{i+1})/Qx_i]$$

$$\psi_{k+1} := \forall x_1(Qx_1 \leftrightarrow \forall x_1\phi(x_1, x_2))$$

where $\forall x_1\phi(x_1, x_2)$ is the result of decreasing all indices of variables in $\forall x_{i+1}\phi(x_i, x_{i+1})$ by $i - 1$.

Observe that Qx_i is a loosely grooved formula of level i, and Qx_1, $\forall x_1\phi(x_1, x_2)$ are loosely grooved formulas of level 1, so φ_{n+1} and ψ_{n+1} are both loosely grooved sentences. Also, since $\forall x_{i+1}\phi(x_i, x_{i+1})$ contains no non-grooved subformula of the form $\forall x_{j+1}\psi(x_j, x_{j+1})$ ($j > i$), we observe that $\forall x_1\phi(x_1, x_2)$ contains no bad subformulas. Therefore, the procedure will terminate on a loosely grooved sentence φ_n free from bad subformulas, together with a sequence of formulas $\psi_1, \ldots, \psi_n$, all of which are free from bad subformulas, too. Finally, let

$$\varphi' := \varphi_n \wedge \psi_1 \wedge \cdots \wedge \psi_n$$

Clearly, φ' is a loosely grooved sentence satisfiability-equivalent to φ. □

The following result shows that the procedure indeed gives us a grooved sentence.

Proposition 12. *A loosely grooved sentence φ free from bad subformulas is a grooved sentence.*

Proof. Looking at Definition 9, we observe that the definition of grooved formulas is the special case where in clause 1 we only allow that $m = 1$ or $m = n$. (Since in that case the conditions on forming conjunctions are automatically satisfied.) In other words, a loosely grooved formula is grooved if all of its atomic subformulas are of the form $Px_1 \ldots x_n$ or Px_n.

Thus, we only need to show that φ contains no atomic formulas of the form $Px_m \ldots x_n$ ($1 < m < n$). Suppose to the contrary that φ contains $Px_m \ldots x_n$ ($1 < m < n$). Given the constraints on forming conjunctions, we observe that a superformula of $Px_m \ldots x_n$ having at least x_m and x_{m+1} free cannot form a conjunction with any formula having also x_l free, where $l < m$. Let $\phi(x_m, x_{m+1})$ be the largest superformula of $Px_m \ldots x_n$ which has exactly x_m and x_{m+1} free. Since ϕ is the largest such subformula of φ, we know that neither $\neg\phi(x_m, x_{m+1})$ nor $\phi(x_m, x_{m+1}) \wedge \psi$ (where ψ has at most x_m and x_{m+1} free) are subformulas of ϕ. Thus, $\forall x_{m+1}\phi(x_m, x_{m+1})$ is a subformula of φ. Since $\forall x_{m+1}\phi(x_m, x_{m+1})$ is a superformula of $Px_m \ldots x_n$, it is not grooved and hence bad, contradicting the assumption that φ has no such subformula. □

5. Final remarks

Ordered fragments of FOL can be used to describe properties of data structures like lists, stacks or queues, given that information is stored in a sequential way in these structures. Different fragments allow one to capture different properties of data structures. We mention two simple examples of this use in the case of the *grooved fragment*.

Suppose that we read the predicate S^n (for $n \geq 1$) as "form(s) a stack with n elements" and the predicate R as "is removed". Then, consider the following grooved formula:

$$\exists x_{n+1}(S^{n+1}x_1 \dots x_{n+1} \wedge Rx_{n+1}) \to S^n x_1 \dots x_n$$

This may be used to say that if an object is at the top of a stack and it is removed, we get a smaller stack. In other words, if we have a stack of n objects where x_n is the latest object (i.e. the one at the top) and we perform the operation of *removing* x_n, then we get a stack with $n-1$ elements (i.e. $x_1, \dots, x_{n-1}$).

Moreover, suppose that we read the predicate A as "is added". Then, consider the following grooved formula:

$$S^n x_1 \dots x_n \to \exists x_{n+1}(Ax_{n+1} \wedge S^{n+1}x_1 \dots x_{n+1})$$

This may be used to say that an object can always be added on top of a stack,. In other words, if $x_1, \dots, x_n$ form a stack with n elements, then there is some object x_{n+1} s.t. if one performs the action of *adding* x_{n+1}, one gets a stack with $n+1$ elements (i.e. $x_1, \dots, x_n, x_{n+1}$).

From a theoretical point of view, the motivation for introducing the loosely grooved fragment, the strongest of the fragments analysed in this article, has two main sources. First, some extended systems of the relational syllogistic feature sentences that are not accommodated by the grooved fragment. For example,

> Everything which is Related to something which is Related to every Q is not P.
> $\forall x_1(\exists x_2(\forall x_3(Qx_3 \to Rx_2x_3) \wedge Rx_1x_2) \to \neg Px_1)$

So, for a generalization to such systems of the relational syllogistic, we need an expansion in more or less the same spirit as the loosely grooved fragment. In fact, the formulation of the loosely grooved fragment allows for much more sentences than the relational syllogistic, as most languages in that context only allow for unary and binary predicates, and Boolean operations are highly restricted. (Again, see [8] and [9] for a detailed comparison.)

Second, the loosely grooved fragment is, from a different aspect, a generalization of what we can call the 'modal fragment' of first-order logic, i.e. the fragment in which all formulas are the standard translation of some modal formula. Note that the standard translation of basic modal formulas are all loosely grooved formulas, provided that variables are suitably chosen. Given a modal formula χ, we can define the standard translation st for its subformulas as follows:

- $st(p) = Px_{n+1}$, if p is in the scope of exactly n $\Box$'s
- $st(\neg\phi) = \neg st(\phi)$
- $st(\phi \wedge \psi) = st(\phi) \wedge st(\psi)$
- $st(\Box\phi) = \forall x_{n+1}(Rx_nx_{n+1} \to st(\phi))$, where x_{n+1} is the free variable in $st(\phi)$

Observe that if a subformula is in the scope of exactly n $\Box$'s, its translation has exactly x_{n+1} free, so st always outputs a loosely grooved formula of level 1.

Meanwhile, the loosely grooved fragment, and, indeed, all ordered fragments mentioned in this paper, are not comparable with the guarded fragment of FOL. Recall the example,

No student admires every professor
$\forall x_1(Sx_1 \to \exists x_2(Px_2 \wedge \neg Ax_1x_2))$

Clearly, the subformula $\exists x_2(Px_2 \wedge \neg Ax_1x_2)$ is not guarded.

The fluted fragment (see [4, 5]) can be seen as a generalization of the loosely grooved fragment by allowing conjunction between any two formulas of the same level in clause 3 of Definition 9. One direction of our future work is to investigate the modal translation of the fluted fragment.

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