Transitional Modal Logics of Quasiary Predicates with Equality and Sequent Calculi for these Logics

Oksana Shkilniak[1], Stepan Shkilniak[1]

[1] Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, 01601, Ukraine

Abstract

The paper explores new classes of program-oriented logical formalisms of the modal type –pure first-order modal logics of partial quasiary predicates with the monotonicity condition removed and enriched with equality predicates. The apparatus of modal logics is used for the description and modeling of various subject areas, artificial intelligence systems, and information and software systems. The limitations of classical predicate logic, on which traditional modal logics are based, underscore the relevance of developing new program-oriented logical formalisms. Such are transitional modal logics of quasiary predicates (TML), which reflect the aspect of change and development in subject areas. They synthesize the capabilities of traditional modal logics and the logics of partial quasiary predicates. Pure first-order TML are called TMLQ. We propose two types of TMLQ with equality: with strong equality predicates ≡xy, called TMLQ≡, and with weak equality predicates =xy, called TMLQ=. The characteristic features of these logics include the use of extended renomination compositions and special indicator predicates that denote the presence of a component with the corresponding subject name in the input data, which are necessary for the quantifier elimination in non-monotonic predicate logics. The work describes the semantic models and languages of TMLQ≡ and TMLQ=. Attention is focused on properties related to equality predicates, and the features of substitution of equals in these logics are described. A number of logical consequence relations for sets of formulas specified with states is defined, and their main properties are described. Based on this semantic foundation, calculi of sequent type are proposed for the investigated logics. Various types of such calculi for different logical consequence relations are described, basic sequent forms for these calculi are presented, and the closedness conditions for sequents are provided. The construction of derivations in the proposed calculi is described, and the soundness and completeness theorems for them are proven.

Keywords

Modal logic, transitional modal system, partial predicate, equality predicate, logical consequence relation, sequent calculus

1. Introduction

Modal logics are used with great success to describe a dynamic world that changes and evolves. The exceptional flexibility of modal logics allows them to be applied to analyze and model a wide variety of human activities. The apparatus of modal logics is utilized for the description and modeling of artificial intelligence systems, information and software systems (see, for example [1, 2, 3, 4]). Temporal and epistemic logics have found the most application in practical fields. Temporal logics are successfully used for software specification and verification [2, 5, 6, 7], and for modeling complex dynamic systems. Epistemic logics are used to describe artificial intelligence systems, information, and expert systems. Traditional modal logics are based on classical predicate logic. At

14th International Scientific and Practical Conference on Programming UkrPROG'2024, May 14-15, 2024, Kyiv, Ukraine
* Corresponding author Oksana Shkilniak
† These authors contributed equally.
oksana.sh@knu.ua (Oksana Shkilniak); ss.sh@knu.ua (Stepan Shkilniak)
0000-0003-4139-2525 (Oksana Shkilniak); 0000-0001-8624-5778 (Stepan Shkilniak)

the same time, classical logic has several limitations (see [8]), which complicates its application. This makes the task of developing new, program-oriented modal logics highly relevant. Such are composition nominative modal logics (CNML), which combine the capabilities of traditional modal logics [9, 3] and composition nominative logics of partial quasiary predicates [8, 10]. CNML are built on the basis of the composition nominative approach, common to both logic and programming. The most important class of CNML is the transitional modal logics (TML); they reflect the aspect of change and development in subject areas. These logics have been studied, in particular, in [11, 12]. It should be noted that traditional modal logics can be naturally considered within the framework of TML.

The aim of this work is to study new classes of program-oriented modal logics − pure first-order TML of partial quasiary predicates with the monotonicity condition (equitonicity) removed and enriched with equality predicates. Two types of such predicates have been distinguished [10]: weak equality =xy and strong equality ≡xy. Pure first-order TML without the monotonicity restriction will be called TMLQ; TMLQ with strong equality predicates will be called TMLQ≡, while TMLQ with weak equality predicates will be called TMLQ=. TMLQ without equality predicates have been studied in [11, 12]. TMLQ≡ and TMLQ= are considered in this paper. The semantic models and languages of these logics are described, along with the features of the substitution of equals in TMLQ≡ and TMLQ=. We define a number of logical consequence relations for sets of formulas specified with states of the language, and properties of these relations are provided.

One of the most important applications of mathematical logic is the automation of proof search. Efficient proof search is essential for successfully solving a number of problems that arise in computer science and programming. A powerful tool for constructing proofs is Gentzen-type calculi, also known as sequent calculi. These calculi formalize the fundamental notion of logical consequence. In this work, we propose such calculi for TMLQ with equality predicates. The semantic basis for constructing sequent calculi for TML is the properties of logical consequence relations for sets of formulas specified with states. Varieties of these calculi for different logical consequence relations are described, and basic sequent forms and conditions for the closedness of sequents are provided for them. The soundness and completeness theorems for the proposed calculi are proved.

Concepts not defined here are interpreted in the sense of works [8, 10, 11, 13].

2. Transitional Modal Systems

At the core of the CNML concept lies the notion of a compositional nominative modal system (CNMS). Such systems serve as models for the possible worlds in modal logics.

CNMS is the object $\boldsymbol{M} = (Cms, Ds, Im)$, where:

– *Cms* is a composition modal system which defines semantic aspects of the world;

– *Ds* is a descriptive system which defines standard descriptions: usually a set *Fm* of formulas of the CNML language;

− *Dns* is a denotation system which determines values of standard descriptions on semantic models: usually an interpretation mapping *Im* of formulas on states of the world.

Composition modal system is the object $Cms = (St, \boldsymbol{R}, Pr, C)$, where:

– *St* is a set of states of the world;

– $\boldsymbol{R}$ is a set of relations on St of the form $R \subseteq St \times St^n$;

– *Pr* is a set of predicates on *St*;

– *C* is a set of compositions on *Pr*.

Thus, CMS are relational-type semantic models.

In expanded form, we will further define CNMS as follows: $\boldsymbol{M} = ((St, \boldsymbol{R}, Pr, C), Fm, Im)$.

For the first-order CNMS, the set St is specified as a set of algebraic systems (structures) $\alpha = (A_\alpha, Pr_\alpha)$, where A_α is a set of basic data of the state α, Pr_α is a set of quasiary predicates ${}^{V}A_\alpha \to \{T, F\}$.

The predicates $Pr\alpha$ are called *predicates of the state* α.

The predicates ${}^{V}A \to \{T, F\}$, where $A = \bigcup_{\alpha \in S} A_\alpha$, will be called *global*.

Transitional modal logics (TML), an important class of CNML, reflect the aspect of change and development in subject areas, describing transitions from one state of the world to another. Central to TML is the concept of a transitional modal system (TMS), which can be considered the most important class of CNMS.

We specify TMS as CNMS in which the set $\boldsymbol{R}$ consists of relations of the form $R \subseteq St \times St$. These relations are treated as state transition relations, hence the name.

Traditional varieties of TMS include general transitional, temporal, and multimodal systems (see [11, 12]).

TMS, in which $\boldsymbol{R}$ consists of a single binary relation $>$, and the basic modal composition is $\Box$ ("necessary"), are called *general* (GMS).

TMS, in which $\boldsymbol{R}$ consists of a single binary relation $>$, and the basic modal compositions are $\Box_\uparrow$ ("it will always be the case") i $\Box_\downarrow$ ("it has always been the case"), are called *temporal* (TmMS).

TMS with the set of relations $\boldsymbol{R} = \{ >_i \mid i \in I \}$, and basic modal compositions M_i, $i \in I$, in which each $>_i \in \boldsymbol{R}$ is matched with the corresponding modal composition M_i, are called multimodal (MMS). In MMS, each Mi acts as $\Box$, but only with respect to its own relation $>_i$, $i \in I$. In this sense, GMS is a special case of MMS.

For GMS, the derivative composition $\Diamond$ ("possibly") is traditionally defined as: $\Diamond P$ means $\neg \Box \neg P$.

For TmMS, we specify the derivative compositions $\Diamond_\uparrow$ ("it will sometimes be the case") and $\Diamond_\downarrow$ ("it was sometime the case"): $\Diamond_\uparrow P$ means $\neg \Box_\uparrow \neg P$, while $\Diamond_\downarrow P$ means $\neg \Box_\downarrow \neg P$.

Pure first-order TMS will be called TMSQ. The corresponding notation GMS^Q, $TmMS^Q$, MMS^Q will be used for pure first-order GMS, TmMS, MMS respectively.

Basic logical compositions of for TMSQ are logical connectives $\neg$ and $\vee$, renomination $R^{\bar{v},\bar{u}}_{\bar{x},\perp}$ and existential quantification $\exists x$. For TMS^Q with equality we add special 0-ary compositions – equality predicates. TMS^Q with strong equality predicates $\equiv_{xy}$ will be called $TMS^{Q\cdot}$, and TMS^Q with weak equality predicates =xy will be called $TMS^{Q=}$.

V-A-quasiary predicate [8] is a partial function Q: VA → {T, F}, where VA is a set of all V-A-nominative sets, {T, F} is the set of truth values; *V* and *A* are interpreted as sets of subject names (variables) and subject values respectively.

V-A-nominative set (V-A-NS) is defined [5] as a single-valued function of the form $V \to A$. We represent *V-A-NS* as $[v_i \mapsto a_i]_{i \in I}$, where $v_i \in V$, $a_i \in A$, $v_i \neq v_j$ when $i \neq j$.

For V-A-NS, we introduce the operations of projection $\|Z$ and $\|_{-Z}$, where $Z \subseteq V$, overlay ∇, and (extended) renomination $r^{\bar{v},\bar{u}}_{\bar{x},\perp}$ (see [8, 13, 14]).

Note that in this work, we use extended renomination operations $r^{\bar{v},\bar{u}}_{\bar{x},\perp}$ and the corresponding extended renomination compositions $R^{\bar{v},\bar{u}}_{\bar{x},\perp}$. Traditional renomination operations $r^{\bar{v}}_{\bar{x}}$ and renomination compositions $R^{\bar{v}}_{\bar{x}}$ are their particular cases which were used, for example, in [8, 11, 12].

Each *V-A*-quasiary predicate Q is determined by two sets: its *truth* domain $T(Q) = \{d \mid Q(d) = T\}$ and its *falsity* domain $F(Q) = \{d \mid Q(d) = F\}$.

Predicate Q is *single-valued*, or *P-predicate*, if $T(Q) \cap F(Q) = \varnothing$.

In this work, only *single-valued V-A*-quasiary predicates will be considered.

Predicate Q is *irrefutable*, if $F(Q) = \varnothing$;

Predicate Q is *satisfiable*, if $T(Q) \neq \varnothing$.

In the class of P-predicates, we have 3 constant predicates:

– Q is identically *true* (denoted by T), if $F(Q) = \varnothing$ and $T(Q) = {}^{V}A$;

– Q is *identically false* (denoted by F), if $T(Q) = \varnothing$ and $F(Q) = {}^{V}A$;

– Q is *totally undefined* (denoted by $\perp$), if $T(Q) = F(Q) = \varnothing$.

P-predicate Q is *equitone*, if $(Q(d)\downarrow$ and $d \subseteq d') \Rightarrow Q(d')\downarrow = Q(d)$.

Subject name $x \in V$ is *unessential* for the predicate Q, if $d_1 \|_{-x} = d_2 \|_{-x} \Rightarrow Q(d_1) = Q(d_2)$.

The basic logical compositions $\neg$, $\vee$, $\exists x$, $R^{\bar{v},\bar{u}}_{\bar{x},\perp}$ for quasiary predicates are specified in [13].

Equality predicates are treated as special 0-ary compositions, considering their general logical status. Two varieties of these predicates are distinguished [10]: weak (up to definability) equality predicates $=_{\{x,y\}}$ and strong (strict) equality predicates $\equiv_{\{x,y\}}$. They are defined as follows:

$T(=_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) = d(y)\}$,

$F(=_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) \neq d(y)\}$;

$T(\equiv_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) = d(y)\} \cup \cup \{d \mid d(x)\uparrow \text{ and } d(y)\uparrow\}$,

$F(\equiv_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow, d(x) \neq d(y)\} \cup \cup \{d \mid d(x)\downarrow, d(y)\uparrow \text{ or } d(x)\uparrow, d(y)\downarrow\}$.

Specific cases of $=_{\{x,y\}}$ and $\equiv_{\{x,y\}}$, when x and y coincide, are $=_{\{x\}}$ and $\equiv_{\{x\}}$.

The predicates $=_{\{x,y\}}$, $=_{\{x\}}$ and $\equiv_{\{x,y\}}$, $\equiv_{\{x\}}$ will be more conventionally denoted as $=_{xy}$, $=_{xx}$ and $\equiv_{xy}$, $\equiv_{xx}$.

Thus, $=_{xy}$ and $=_{yx}$ represent the same predicate, as do $\equiv_{xy}$ and $\equiv_{yx}$, respectively.

The predicates $\equiv_{xy}$ and $\equiv_{xx}$ are total and non-monotonic; the predicates $=_{xy}$ and $=_{xx}$ are partial and equitone.

For quantifier elimination in the logics of non-monotonic predicates, special 0-ary compositions –predicates-indicators which detect whether a component with a corresponding name has a value in the input data – are needed. The use of such indicator predicates is a characteristic feature of TMLQ. Total predicates-indicators determine the presence or absence of a component with a given name, while partial predicates-indicators only detect the presence of such a component.

Total predicates-indicators Ez are non-monotonic; they are defined as follows (see [8]):

$T(Ez) = \{d \mid d(z)\downarrow\}$;

$F(Ez) = \{d \mid d(z)\uparrow\}$.

Total indicator predicates Ez were used, in particular, in [8, 10, 11, 13].

Partial predicates-indicators are already present in $TMS^{Q=}$ as the equitone predicates $=_{zz}$.

Indeed, we have $T(=_{zz}) = \{d \mid d(z)\downarrow\} = T(Ez)$ and $F(=_{zz}) = \{d \mid d(z)\downarrow \text{ and } d(z) \neq d(z)\} = \varnothing$.

Note that the predicates Ez can be expressed as $\exists y \equiv_{xy}$, but it is more appropriate to explicitly define them as special 0-ary compositions, which is done in TMLQ≡. At the same time, partial indicator predicates, such as the predicates $=_{xx}$, are explicitly present in $TML^{Q=}$.

Therefore, we have the following varieties of TMS^{Q} with equality: $GMS^{Q\cdot}$, $TmMS^{Q\cdot}$, $MMS^{Q\cdot}$ for $TMS^{Q\cdot}$, and $GMS^{Q=}$, $TmMS^{Q=}$, $MMS^{Q=}$ for $TMS^{Q=}$.

3. Languages of Transitional Modal Systems

Let us describe a language of $GMS^{Q\cdot}$. The alphabet: a set V of subject names (variables); a set Ps of predicate symbols; the set $\{\neg, \vee, R^{\bar{v},\bar{u}}_{\bar{x},\perp}, \exists x, \equiv_{xy}, Ex\}$ of basic logical compositions' symbols; the set $Ms = \{\Box\}$ of basic modal compositions' symbols.

The set Fr of formulas of the language is determined as follows:

Fa) $Ps \subseteq Fr$;

F≡) $\{Ex \mid x \in V\} \subseteq Fr$ and $\{\equiv_{xy} \mid x, y \in V\} \subseteq Fr$;

Fp) $\Phi, \Psi \in Fr \Rightarrow \neg\Phi \in Fr$ and $\vee\Phi\Psi \in Fr$;

FR) $\Phi \in Fr \Rightarrow R^{\bar{v},\bar{u}}_{\bar{x},\perp}\Phi \in Fr$;

F∃) $\Phi \in Fr \Rightarrow \exists x\Phi \in Fr$;

F□) $\Phi \in Fr \Rightarrow \Box\Phi \in Fr$.

Formulas of the form $p \in Ps$, Ex, $\equiv_{xy}$ will be called *atomic*.

Atomic formulas and formulas of the form $R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)$, where $p \in Ps$, will be called *primitive*.

To write formulas, we will traditionally use the prefix notation and the symbols for derived compositions $\rightarrow$, &, $\forall x$, $\Diamond$. Thus, the formulas $\vee\neg\Phi\Psi$, $\neg\vee\neg\Phi\neg\Psi$, $\neg\exists x\neg\Phi$, and $\neg\Box\neg\Phi$ will be abbreviated $\Phi\rightarrow\Psi$, $\Phi\&\Psi$, $\forall x\Phi$, and $\Diamond\Phi$, respectively.

Sets of guaranteed to be unessential names for formulas are specified by a function $\nu : Fr \rightarrow 2^V$ (see [8]). At the same time, we define $\nu(\Box\Phi) = \nu(\Phi)$.

The type of GMS^{Q-} is determined by the extended signature $\sigma = (Ps, \nu)$ and properties of the relation $>$.

Let us define an interpretation mapping *Im* of formulas on states of the world. First, we specify $Im : Ps \times St \rightarrow Pr$, with condition $Im(p, \alpha) \in Pr_\alpha$ (basic predicates are predicates of states). Compositions' symbols are interpreted as corresponding compositions (in particular, the symbols *Ex* and ≡xy are interpreted as the corresponding predicates-indicators and equality predicates). The mapping *Im* is continued to $Fm \times St \rightarrow Pr$ in the following fashion:

Ip) $Im(\neg, \alpha) = \neg(Im(\Phi, \alpha))$;

$Im(\vee\Phi\Psi, \alpha) = \vee(Im(\Phi, \alpha), Im(\Psi, \alpha))$;

IR) $Im(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\alpha) = \mathrm{R}^{\bar{v},\bar{u}}_{\bar{x},\perp}(Im(\Phi,\alpha))$;

$$\text{I}\exists)\ Im(\exists x\Phi, \alpha)(d) = \begin{cases} T, \text{ if exists } a \in A_\alpha : Im(\Phi,\alpha)(d\nabla x \text{ a } a) = T, \\ F, \text{ if } Im(\Phi,\alpha)(d\nabla x \text{ a } a) = F \text{ for all } a \in A_\alpha, \\ \text{else undefined.} \end{cases}$$

$$\text{I}\Box)\ Im(\Box\Phi, \alpha)(d) = \begin{cases} T, \text{ if } Im(\Phi,\delta)(d) = T \text{ for all } \delta \in S : \alpha > \delta, \\ F, \text{ if exists } \delta \in S : \alpha > \delta \text{ and } Im(\Phi,\delta)(d) = F, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha > \beta$, then we define $\mathrm{I\tau}(\Box\Phi, \alpha)(d)\uparrow$ for all $d \in {}^V A$.

For abbreviations of formulas of the form $\forall x\Phi$ and $\Diamond\Phi$, the mapping *Im* is specified as follows:

$$\text{I}\exists)\ Im(\forall x\Phi, \alpha)(d) = \begin{cases} T, \text{ if } Im(\Phi,\alpha)(d\nabla x \text{ a } a) = T \text{ for all } a \in A_\alpha, \\ F, \text{ if exists } a \in A_\alpha : Im(\Phi,\alpha)(d\nabla x \text{ a } a) = F, \\ \text{else undefined.} \end{cases}$$

$$\text{I}\Diamond)\ Im(\Diamond\Phi, \alpha)(d) = \begin{cases} T, \text{ if exists } \delta \in St \text{ such that } \alpha > \delta \text{ and } Im(\Phi,\delta)(d) = T, \\ F, \text{ if } Im(\Phi,\delta)(d) = F \text{ for all } \delta \in St \text{ such that } \alpha > \delta, \\ \text{else undefined.} \end{cases}$$

Given for $\alpha \in St$ there is no β such that $\alpha > \beta$, then we define $Im(\Diamond\Phi, \alpha)(d)\uparrow$ for all $d \in {}^V A$.

Predicates that are values of modalized formulas, belong to global predicates.

We specify TMS as $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$.

The following definitions are given identically for all the described variants of TMS^Q.

Predicate $Im(\Phi, \alpha)$, which is a value of the formula Φ on state α, is denoted by Φ_α.

Formula Φ is *irrefutable on state* α (denoted by $\alpha \models \Phi$), if Φ_α is a irrefutable predicate.

Formula Φ is irrefutable in TMS M (denoted by M $\models \Phi$), if for all $\alpha \in$ St, $\Phi\alpha$ is irrufutable.

Let □ be a TMS class of a given type.

Formula Φ is □-*irrefutable* (denoted by □ $\models \Phi$), if $\boldsymbol{M} \models \Phi$ for all TMS $\boldsymbol{M} \in$ □.

Depending on conditions imposed on the relation $>$, different classes of GMS^{Q-} can be specified. Traditionally, we can consider cases of reflexive, symmetric or transitive $>$, or their combinations: then we add the corresponding symbol *R*/*T*/*S* to the GMS^{Q-} name. Thus, the following classes are obtained: *R*-GMS^{Q-}, *T*-GMS^{Q-}, *S*-GMS^{Q-}, *RT*-GMS^{Q-}, *RS*-GMS^{Q-}, *TS*-GMS^{Q-}, *RTS*-GMS^{Q-}.

The language of GMS$^{Q=}$ is defined similarly to the language of GMS^{Q-} with the following differences. The set of symbols of basic logical compositions is $\{\neg, \vee, R^{\bar{v},\bar{u}}_{\bar{x},\perp}, \exists x, =_{xy}\}$. In the definition of the set of formulas, instead of the item F≡ we have {=xy | x, y∈V} ⊆ Fr; the interpretation mapping is defined accordingly.

Let us describe the TmMS^{Q-} language. The alphabet is identical to the alphabet of GMS^{Q-} with the set of basic modal compositions' symbols specified as *Ms* = $\{\Box_\uparrow, \Box_\downarrow\}$. The set *Fm* of formulas of the language is determined according to the items Fa, F≡, Fp, FR, F∃ for the language of GMS^{Q}-, but instead of F□ we have:

F□↑↓) Φ∈Fr ⇒ □↑Φ∈Fr and $\Box_{\downarrow}$Φ∈*Fr*.

When we define the mapping Iт, instead of I□ we have the following item $I\Box_{\uparrow\downarrow}$ (see [11, 12]):

$$I\Box_{\uparrow\downarrow})\ Im(\Box_{\uparrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for all } \delta\in St \text{ such that } \alpha>\delta,\\ F, \text{ if there exists } \delta\in St \text{ such that } \alpha>\delta \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined;}\end{cases}$$

$$Im(\Box_{\downarrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for all } \delta\in St \text{ such that } \delta>\alpha,\\ F, \text{ if there exists } \delta\in St \text{ such that } \delta>\alpha \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined.}\end{cases}$$

Given for α∈St there is no β such that $\alpha>\beta$, then we define $Im(\Box_{\uparrow}\Phi,\alpha)(d)\uparrow$ for all $d\in{}^{V}A$.

Given for α∈St there is no β such that $\beta>\alpha$, then we define $Im(\Box_{\downarrow}\Phi,\alpha)(d)\uparrow$ for all $d\in{}^{V}A$.

For the abbreviated formulas $\Diamond_{\uparrow}\Phi$ and $\Diamond_{\downarrow}\Phi$, we have the following interpretation mapping *Im*:

$$I\Diamond_{\uparrow\downarrow})\ Im(\Diamond_{\uparrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if there exists } \delta\in S, \text{ such that } \alpha>\delta \text{ and } Im(\Phi,\delta)(d)=T,\\ F, \text{ if } Im(\Phi,\delta)(d)=F \text{ for all } \delta\in S, \text{ such that } \alpha>\delta,\\ \text{else undefined;}\end{cases}$$

$$Im(\Diamond_{\downarrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if there exists } \delta\in S, \text{ such that } \delta>\alpha \text{ and } Im(\Phi,\delta)(d)=T,\\ F, \text{ if } Im(\Phi,\delta)(d)=F \text{ for all } \delta\in S, \text{ such that } \delta>\alpha,\\ \text{else undefined.}\end{cases}$$

Given for α∈*St* there is no β such that $\alpha>\beta$, then we define $Im(\Diamond_{\uparrow}\Phi,\alpha)(d)\uparrow$ for all $d\in{}^{V}A$.

Given for α∈*St* there is no β such that $\beta>\alpha$, then we define $Im(\Diamond_{\downarrow}\Phi,\alpha)(d)\uparrow$ for all $d\in{}^{V}A$.

The language of $TmMS^{Q=}$ is defined similarly to the language of $TmMS^{Q}$-, with the differences corresponding to those between the languages of $GMS^{Q=}$ and GMS^{Q}-.

Depending on the conditions imposed on >, we define different classes of $GMS^{Q=}$, $TmMS^{Q}$-, and $TmMS^{Q=}$ as done for GMS^{Q}-.

Similarly, we define the languages of MMS^{Q}≡ and $MMS^{Q=}$.

Depending on how the value $\Phi_{\delta}(d)$ is set in case $d\notin{}^{V}A_{\delta}$, two types of TMS are distinguished ([12]): with *strong* condition of undefinedness on states and with *general* condition of undefinedness on states. The strong condition is specified as follows: under the condition $d\notin{}^{V}A_{\delta}$ we have $\Phi_{\delta}(d)\uparrow$. Hence: $(\Box\Phi)_{\alpha}(d)=T \Rightarrow d\in{}^{V}A_{\delta}$ for all δ such that α⊳δ. This implies that basic objects cannot disappear when transitioning to a successor state, which imposes too strong a restriction on semantic models. The strong condition also does not preserve [12] equitonicity of predicates with modalities.

The general condition of undefinedness on states does not have these drawbacks; it is defined as follows:

for all $d\in{}^{V}A$ and δ∈*St*, we have $\Phi_{\delta}(d)=\Phi_{\delta}(d_{\delta})$.

Here d_{δ} denotes the name set $[v \mapsto a \in d \mid a\in A_{\delta}]$.

Given $d\notin{}^{V}A_{\delta}$, we have $\Phi_{\delta}(d)=\Phi_{\delta}(d_{\delta})$, meaning that predicates on states δ "perceive" only components $v \mapsto a$ with basic data $a\in A_{\delta}$.

The interaction of modal compositions with renominations and quantifiers has been studied in [12]. Let us briefly describe it for GMS^{Q}.

Theorem 1. For all Φ∈*Fr*, $d\in{}^{V}A$ we have $R_{\bar{x},\perp}^{\bar{v},\bar{u}}(✹\Phi)_{\alpha}(d) = ✹(R_{\bar{x},\perp}^{\bar{v},\bar{u}}(\Phi))_{\alpha}(d)$.

Therefore, symbols of *Ms* can be carried through renomination symbols.

Theorem 2. Formulas ∃x□Φ → □∃xΦ, □∀xΦ →∀x□Φ, ◇∀xΦ → ∀*x*◇Φ, ∃*x*◇Φ → ◇∃*x*Φ are irrefutable in GMSQ of equitone predicates.

Theorem 3. Formulas ∃*x*□Φ → □∃*x*Φ, □∀*x*Φ →∀*x*□Φ are refuted in GMS^{Q}.

Corrolary 1. Formulas ◇∀*x*Φ → ∀*x*◇Φ, ∃*x*◇Φ → ◇∃*x*Φ are refuted in GMS^{Q}.

Theorem 4. Formulas □∃*x*Φ → ∃*x*□Φ, ∀*x*□Φ → □∀*x*Φ are refuted in GMS^{Q} of equitone predicates.

Corollary 2. Formulas $\forall x\Diamond\Phi \to \Diamond\forall x\Phi$, $\Diamond\exists x\Phi \to \exists x\Diamond\Phi$ are refuted in GMSQ of equitone predicates.

Examples of GMS^Q, in which the formulas specified in Theorems 3 and 4 are refuted, are given in the work [12].

Let's consider the specific properties of GMS^Q, related to equality predicates. For $TmMS^Q$ and MMS^Q, these properties are formulated similarly.

Assertion 1. 1) for every $GMS^{Q\cdot}$ $\boldsymbol{M}$ we have $\boldsymbol{M} \models \equiv_{xx}$ and $\boldsymbol{M} \models \Box\equiv_{xx}$;

2) for every $GMSQ^{=}$ $\boldsymbol{M}$ we have $\boldsymbol{M} \models =_{xx}$ and $\boldsymbol{M} \models \Box=_{xx}$.

Indeed, $F(\equiv_{xx}) = \varnothing$ and $F(=_{xx}) = \varnothing$.

Example **1**. Let us consider $GMS^{Q\cdot}$ with $St = \{\alpha, \beta\}$ and $\boldsymbol{R} = \{\alpha > \beta\}$. Since there is no state η such that $\beta > \eta$, then $(\Box\equiv_{xx})_\beta(d)\uparrow$ for each d.

Therefore, $\Box\equiv_{xx}$ is not always interpreted as the constant predicate T.

Theorem 5. Formulas $=_{xy} \to \Box=_{xy}$ and $\Box=_{xy} \to =_{xy}$ are irrefutable in $GMS^{Q=}$.

In particular, $=_{xx} \to \Box=_{xx}$ and $\Box=_{xx} \to =_{xx}$ are irrefutable in $GMS^{Q=}$. At the same time, we have

Example 2. Formulas $\equiv_{xy} \to \Box\equiv_{xy}$ and $\Box\equiv_{xy} \to \equiv_{xy}$ are refuted in the following GMSQ·.

Let $St = \{\alpha, \beta\}$, $\boldsymbol{R} = \{\alpha > \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b\}$.

Let $d = [x \mapsto b, z \mapsto a] \Rightarrow d_\alpha = [z \mapsto a]$, $d_\beta = [x \mapsto b]$. Hence $d_\alpha(x)\uparrow$, $d_\alpha(y)\uparrow$, $d_\beta(x) = b$, $d_\beta(y)\uparrow$. We have $(\equiv_{xy})_\alpha(d) = (\equiv_{xy})_\alpha(d_\alpha) = T$.

At the same time, $(\equiv_{xy})_\beta(d) = (\equiv_{xy})_\beta(d_\beta) = F \Rightarrow (\Box\equiv_{xy})_\alpha(d) = (\Box\equiv_{xy})_\alpha(d_\alpha) = F$.

Therefore, $(\equiv_{xy} \to \Box\equiv_{xy})_\alpha(d) = F \Rightarrow \alpha \not\models \equiv_{xy} \to \Box\equiv_{xy}$.

Let us take $h = [x \mapsto a, z \mapsto b] \Rightarrow h_\alpha = [x \mapsto a]$, $h_\beta = [z \mapsto b]$. Hence $h_\alpha(x) = a$, $h_\alpha(y)\uparrow$, $h_\beta(x)\uparrow$, $h_\beta(y)\uparrow$. We have $(\equiv_{xy})_\alpha(h) = (\equiv_{xy})_\alpha(h_\alpha) = F$.

At the same time, $(\equiv_{xy})_\beta(h) = (\equiv_{xy})_\beta(h_\beta) = T \Rightarrow (\Box\equiv_{xy})_\alpha(h) = (\Box\equiv_{xy})_\alpha(h_\alpha) = T$.

Thus, $(\Box\equiv_{xy} \to \equiv_{xy})_\alpha(h) = F \Rightarrow \alpha \not\models \Box\equiv_{xy} \to \equiv_{xy}$.

Example 3. Formulas $Ex \to \Box Ex$ and $\Box Ex \to Ex$ are refuted in $GMS^{Q\cdot}$.

Let us consider the $GMS^{Q\cdot}$ from Example 2.

Let us take $h = [x \mapsto a, z \mapsto b] \Rightarrow h_\alpha = [x \mapsto a]$, $h_\beta = [z \mapsto b]$. From this, $Ex_\alpha(h) = T$, $Ex_\beta(h) = F$, which gives $\Box Ex_\alpha(h) = F$. Therefore, $(Ex \to \Box Ex)_\alpha(h) = F$, whence $\alpha \not\models Ex \to \Box Ex$.

Let us take $d = [x \mapsto b, z \mapsto a] \Rightarrow d_\alpha = [z \mapsto a]$, $d_\beta = [x \mapsto b]$. From this, $Ex_\alpha(d) = F$, $Ex_\beta(d) = T$, which gives $\Box Ex_\alpha(d) = T$. Therefore, $(\Box Ex \to Ex)_\alpha(h) = F$, whence $\alpha \not\models \Box Ex \to Ex$.

Theorem 5 and Example 2 demonstrate significant differences between GMSQ≡ and $GMS^{Q=}$.

Another confirmation of this is provided below by Theorem 6 and Example 4.

Theorem 6. Formula $=_{xy} \,\&\, \Box=_{xz} \to \Box=_{yz}$ is irrefutable in $GMS^{Q=}$.

Example 4. Formula $\equiv_{xy} \,\&\, \Box\equiv_{xz} \to \Box\equiv_{yz}$ is refuted in the following $GMS^{Q\cdot}$.

Let $St = \{\alpha, \beta\}$, $\boldsymbol{R} = \{\alpha > \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b, c\}$.

Let us take $d = [x \mapsto c, y \mapsto b, z \mapsto c, s \mapsto a] \Rightarrow d_\alpha = [s \mapsto a]$, $d_\beta = [x \mapsto c, y \mapsto b, z \mapsto c]$.

Hence $d_\alpha(x)\uparrow$, $d_\alpha(y)\uparrow$; $d_\beta(x) = c$, $d_\beta(y) = b$, $d_\beta(z) = c$.

From this, $(\equiv_{xy})_\alpha(d) = (\equiv_{xy})_\alpha(d_\alpha) = T$; $(\equiv_{xz})_\beta(d) = (\equiv_{xz})_\beta(d_\beta) = T$, $(\equiv_{yz})_\beta(d) = (\equiv_{yz})_\beta(d_\beta) = F$.

We obtain $(\equiv_{xz})_\beta(d) = T$ and $\alpha \rhd \beta \Rightarrow (\Box\equiv_{xz})_\alpha(d) = T$; $(\equiv_{yz})_\beta(d) = F$ and $\alpha \rhd \beta \Rightarrow (\Box\equiv_{yz})_\alpha(d) = F$.

Therefore, $(\equiv_{xy} \,\&\, \Box\equiv_{xz} \to \Box\equiv_{yz})_\alpha(d) = F$.

4. Logical consequence relations for sets of formulas specified with states

We will define logical consequence relations in TMS on a set of formulas specified with names of states, or simply, specified with states.

Formula specified with a name of the state has the form Φ^α, where Φ is a formula of the language, $\alpha \in S$ – its specification, S – a set of names of states of the world.

Let us call a set of formulas specified with states Σ with a specifications' set S *consistent* with TMS $M = (St, \boldsymbol{R}, A, Im)$, provided that an injection of S into *St* is defined.

On sets of formulas specified with states, we introduce the relations of irrefutability (*IR*-consequence), truth (*T*-consequence), falsity (*F*-consequence), and strong (*TF*-consequence) logical consequence. These relations correspond to the similarly named relations in logics of quasiary predicates (see [8, 10, 13]).

Let Δ and Γ be sets of formulas specified with states. Further on, the notation $\Gamma\ {}_{M}|\!=_{*} \Delta$ by default assumes that Γ and Δ are consistent with TMS *M*.

Δ is a *IR*-consequence of Γ in a consistent with them TMS ***M*** (denoted $\Gamma\ {}_{M}|\!=_{IR} \Delta$), if for all $d \in {}^{V}\!A$ we have: $\Phi_{\alpha}(d) = T$ for all $\Phi^{\alpha} \in \Gamma \Rightarrow \Psi_{\beta}(d) \neq F$ for some $\Psi^{\beta} \in \Delta$.

Δ is a *logical IR-consequence* of Γ with respect to a TMS of a type □ (denoted $\Gamma\ {}^{□}|\!=_{IR} \Delta$), if $\Gamma\ {}_{M}|\!=_{IR} \Delta$ for all $M \in □$.

Δ is a *T*-consequence of Γ in a consistent with them TMS ***M*** (denoted $\Gamma\ {}_{M}|\!=_{T} \Delta$), if for all $d \in {}^{V}\!A$ we have: $\Phi_{\alpha}(d) = T$ for all $\Phi^{\alpha} \in \Gamma \Rightarrow \Psi_{\beta}(d) = T$ for some $\Psi^{\beta} \in \Delta$.

Δ is a *logical T-consequence* of Γ with respect to a TMS of a type □ (denoted $\Gamma\ {}^{□}|\!=_{T} \Delta$), if $\Gamma\ {}_{M}|\!=_{T} \Delta$ for all $M \in □$.

Δ is an *F*-consequence of Γ in a consistent with them TMS ***M*** (denoted $\Gamma\ {}_{M}|\!=_{F} \Delta$), if for all $d \in {}^{V}\!A$ we have: $\Psi_{\beta}(d) = F$ for all $\Psi^{\beta} \in \Delta \Rightarrow \Phi_{\alpha}(d) = F$ for some $\Phi^{\alpha} \in \Gamma$.

Δ is a *logical F-consequence* of Γ with respect to a TMS of a type □ (denoted $\Gamma\ {}^{□}|\!=_{F} \Delta$), if $\Gamma\ {}_{M}|\!=_{F} \Delta$ for all $M \in □$.

Δ is a *TF*-consequence of Γ in a consistent with them TMS ***M*** (denoted $\Gamma\ {}_{M}|\!=_{TF} \Delta$), if $\Gamma\ {}_{M}|\!=_{T} \Delta$ and $\Gamma\ {}_{M}|\!=_{F} \Delta$.

Δ is a *logical TF-consequence* of Γ with respect to a TMS of a type □ (denoted $\Gamma\ {}^{□}|\!=_{TF} \Delta$), if $\Gamma\ {}_{M}|\!=_{TF} \Delta$ for all $M \in □$.

Therefore, we have: $\Gamma\ {}^{□}|\!=_{TF} \Delta \Leftrightarrow \Gamma\ {}^{□}|\!=_{T} \Delta$ and $\Gamma\ {}^{□}|\!=_{F} \Delta$.

In logics with weak equality predicates, the relations of types *T*, *F* and *TF* are incorrect (see [10]), so in $\mathrm{GMS}^{Q=}$ we consider only relations of the *IR* type. In $\mathrm{GMS}^{Q\equiv}$, all the above-defined relations can be considered.

The non-modal properties of the relations for sets of formulas specified with states repeat the corresponding properties of the same-named relations for sets of formulas of the traditional logic of quasiary predicates described in [8, 10, 13, 14]. These are such properties.

1) Properties of formulas decomposition $\neg\neg_{L}$, $\neg\neg_{R}$, $\vee_{L}$, $\vee_{R}$, $\neg\vee_{L}$, $\neg\vee_{R}$, and also properties $\neg_{L}$,and $\neg_{R}$ for $|\!=_{IR}$ (see [8]).

2) Properties of simplification and equivalent transformations related to renominations, induced by the predicates properties R, $R_{\perp}I$, $R_{\perp}U$, $R_{\perp}R$, $R_{\perp}\neg$, $R_{\perp}\vee$, R↑ (see [13, 14]).

3) Properties of simplification related to renomination of the *Ez* predicates, induced by the predicates properties $R_{\perp E}$ and $R_{\perp Ev}$, and the property El_{RE} of elimination of the $_{F}$-formula $R^{\bar{v},\bar{u},z}_{\bar{x},\perp,\perp}(Ez)$(see [13]).

4) Auxiliary properties in $\mathrm{GMS}^{Q\equiv}$: elimination of $\neg$ when carrying a formula from the left side of the consequence relation to the right and vice versa for the symbols *Ez*, $\equiv_{xy}$, and their renominations.

5) Properties related to the quantifier elimination; in $\mathrm{GMS}^{Q\equiv}$, they repeat the properties $\exists R_{\perp L}$, $\neg\exists R_{\perp R}$, $\exists R_{\perp}\vee_{R}$, $\neg\exists R_{\perp}\vee_{L}$ (see [8]), in $\mathrm{GMS}^{Q=}$, we have the analogous properties $\exists R_{\perp L=}$, $\neg\exists R_{\perp R=}$, $\exists R_{\perp}\vee_{R=}$, $\neg\exists R_{\perp}\vee_{L=}$, where instead of *Ez*, we use $=_{zz}$; moreover, we add the properties of *E*-distribution Ed and of primary definition Ev in $\mathrm{GMS}^{Q\equiv}$ (see [14]), and similarly, the properties $\downarrow_{=}$d and $\downarrow_{=}$v with $=_{zz}$ instead of *Ez* in $\mathrm{GMS}^{Q=}$.

6) Properties in $\mathrm{GMS}^{Q\equiv}$ related to the $\equiv_{xy}$ predicates; these are simplification properties induced by the predicates properties $R_{\perp}\equiv_{xx}$, $R_{\perp}\equiv_{0}$, $R_{\perp}\equiv_{1}$, $R_{\perp}\equiv_{2}$, $R_{\perp}\equiv_{1E}$, $R_{\perp}\equiv_{2E}$; properties of elimination of the $_{T}$-formulas $\equiv_{xx}$ and $R^{\bar{v},\bar{u},x,y}_{\bar{w},\perp,\perp,\perp}(\equiv_{xy})$, elimination of the pair of equals in a renomination ≡elR, transitivity Tr≡ (see [13]). For example, let us consider the transitivity property (here and further on the symbol * denotes one of the *IR*, *T*, *F*, *TF*):

Tr≡) $\equiv_{xy}{}^{\alpha}, \equiv_{yz}{}^{\alpha}, \Gamma\ {}_{M}|\!=_{*} \Delta \Leftrightarrow \equiv_{xy}{}^{\alpha}, \equiv_{yz}{}^{\alpha}, \equiv_{xz}{}^{\alpha}, \Gamma\ {}_{M}|\!=_{*} \Delta$.

We add to them the properties of substitution of equals in $\mathrm{GMS}^{Q\equiv}$ described below.

7) Properties in $GMS^{Q=}$ related to the $=_{xy}$ and $=_{xx}$ predicates; these are simplification properties induced by the predicates properties $R_{\perp}{=}_{zz}$, $R_{\perp}{=}_{zz0}$ $R_{\perp}{=}_{0}$, $R_{\perp}{=}_{1}$, $R_{\perp}{=}_{2}$; the properties $El_{R=L}$ and $El_{R=R}$ of elimination of the $_{\perp}$-formulas and the property $El_{=L}$ of elimination of the $_{pT}$-formulas $=_{zz}$; the properties of elimination of the pair of equals in a renomination =elR, transitivity Tr= and substitution of equals $=R_{\perp}r_{L}$, $=R_{\perp}r_{R}$ (see [13]).

Let us describe the properties that guarantee the considered consequence relations in GMS^{Q-}.

For all such relations we have the basic property *C* and property *CF*:

C) $\Phi^{\alpha}, \Gamma\ {}_{M}|{=}_{*}\ \Delta, \Phi^{\alpha}$;

CF) $R^{\bar{v},\bar{u},z}_{\bar{x},\perp,\perp}(Ez), \Gamma \mid{=}_{*}\ \Delta$.

Additionally, the corresponding consequence relation is guaranteed by one of the following properties:

CL) $\Phi^{\alpha}, \neg\Phi^{\alpha}, \Gamma\ {}_{M}|{=}_{T}\ \Delta$;

CR) $\Gamma\ {}_{M}|{=}_{F}\ \Delta, \Phi^{\alpha}, \neg\Phi^{\alpha}$;

CLR) $\Phi^{\alpha}, \neg\Phi^{\alpha}, \Gamma\ {}_{M}|{=}_{TF}\ \Delta, \Psi^{\alpha}, \neg\Psi^{\alpha}$.

Based on the properties of equality, we have the properties of the presence of each of the relations:

C_{Rf-}) $\Gamma\ {}_{M}|{=}_{*}\ \Delta, \equiv_{xx}^{\alpha}$;

$C_{E\text{-}L}$) $\equiv_{xy}^{\alpha}, Ex^{\alpha}, \Gamma \mid{=}_{*}\ Ey^{\alpha}, \Delta$;

$C_{E\text{-}R}$) $\Gamma \mid{=}_{*}\ \equiv_{xy}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Delta$;

CT_{-}) $\Gamma\ {}_{M}|{=}_{*}\ \Delta, R^{\bar{v},\bar{u},x,z}_{\bar{x},\perp,\perp,\perp}(\equiv_{xz})^{\alpha}$.

Based on the *properties* of guaranteed presence of one of the relations $\Gamma\ {}_{M}|{=}_{IR}\ \Delta$, $\Gamma\ {}_{M}|{=}_{T}\ \Delta$, $\Gamma\ {}_{M}|{=}_{IF}\ \Delta$, $\Gamma\ {}_{M}|{=}_{TF}\ \Delta$, we have the corresponding *conditions* that *guarantee* this relation.

C) there exists a formula Φ: $\Phi^{\alpha}\in\Gamma$ and $\Phi^{\alpha}\in\Delta$ – for all relations;

CF) there exists a formula $R^{\bar{v},\bar{u},z}_{\bar{x},\perp,\perp}(Ez)^{\alpha}\in\Gamma$ – for all relations;

CL) there exists a formula Φ^{α}: $\Phi^{\alpha}\in\Gamma$ and $\neg\Phi^{\alpha}\in\Gamma$ – for the relation ${}_{M}|{=}_{T}$;

CR) there exists a formula Φ^{α}: $\Phi^{\alpha}\in\Delta$ and $\neg\Phi^{\alpha}\in\Delta$ – for the relation ${}_{M}|{=}_{F}$; ;

CLR) there exist formulas Φ and Ψ: $\Phi^{\alpha}, \neg\Phi^{\alpha}\in\Gamma$ and $\Psi^{\alpha}, \neg\Psi^{\alpha}\in\Delta$ – for the relation ${}_{M}|{=}_{TF}$;

C_{Rf-}) there exists a formula $\equiv_{xx}^{\alpha}\in\Delta$ – for all relations;

$C_{E\text{-}L}$) there exist formulas $\equiv_{xy}$, Ex and Ey: $\equiv_{xy}^{\alpha}, Ex^{\alpha}\in\Gamma$ and $Ey^{\alpha}\in\Delta$ – for all relations;

$C_{E\text{-}R}$) there exist formulas $\equiv_{xy}$, Ex and Ey: $\equiv_{xy}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}\in\Delta$ – for all relations;

CT_{-}) there exists a formula $R^{\bar{v},\bar{u},x,z}_{\bar{x},\perp,\perp,\perp}(\equiv_{xz})$: $R^{\bar{v},\bar{u},x,z}_{\bar{x},\perp,\perp,\perp}(\equiv_{xz})^{\alpha}\in\Delta$ – for all relations.

This provides the complete conditions that guarantee each of the corresponding relations $\Gamma\ {}_{M}|{=}_{*}\ \Delta$ in GMS^{Q-}.

The relation ${}_{M}|{=}_{IR}$: $C \vee CF \vee C_{Rf-} \vee C_{E\text{-}L} \vee C_{E\text{-}R} \vee CT_{-}$.

The relation ${}_{M}|{=}_{T}$: $C \vee CL \vee CF \vee C_{Rf-} \vee C_{E\text{-}L} \vee C_{E\text{-}R} \vee CT_{-}$.

The relation ${}_{M}|{=}_{F}$: $C \vee CR \vee CF \vee C_{Rf-} \vee C_{E\text{-}L} \vee C_{E\text{-}R} \vee CT_{-}$.

The relation ${}_{M}|{=}_{TF}$: $C \vee CLR \vee CF \vee C_{Rf-} \vee C_{E\text{-}L} \vee C_{E\text{-}R} \vee CT_{-}$.

Let us describe the properties of guaranteed presence of the relation $\Gamma\ {}_{M}|{=}_{IR}\ \Delta$ in $GMS^{Q=}$. These are the basic property *C* and the following ones:

$C_{Rf=}$) $\Gamma\ {}_{M}|{=}_{IR}\ \Delta, =_{xx}^{\alpha}$;

$C\perp_{L}$) $R^{\bar{v},\bar{u},x}_{\bar{w},\perp,\perp}(=_{xy})^{\alpha}, \Gamma\ {}_{M}|{=}_{IR}\ \Delta$;

$C\perp_{R}$) $\Gamma\ {}_{M}|{=}_{IR}\ \Delta, R^{\bar{v},\bar{u},x}_{\bar{w},\perp,\perp}(=_{xy})^{\alpha}$.

Hence, the same-named conditions that guarantee the relation $\Gamma\ {}_{M}|{=}_{IR}\ \Delta$ in $GMS^{Q=}$.

$C_{Rf=}$) there exists a formula $=_{xx}^{\alpha}\in\Delta$;

$C\perp_{L}$) there exists a formula $R^{\bar{v},\bar{u},x}_{\bar{w},\perp,\perp}(=_{xy})^{\alpha}\in\Gamma$;

$C\perp_{R}$) there exists a formula $R^{\bar{v},\bar{u},x}_{\bar{w},\perp,\perp}(=_{xy})^{\alpha}\in\Delta$.

Thus, we obtain the complete condition of guaranteed presence of the relation $\Gamma\ {}_{M}|{=}_{IR}\ \Delta$ in $GMS^{Q=}$:

$C \vee C_{Rf=} \vee C\perp_{L} \vee C\perp_{R}$.

To conclude the description of the non-modal properties of consequence relations for sets of formulas specified with states in GMS^{Q}-, we will examine properties related to the substitution of equals.

Example 5. The property $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!=_{IR} \Box\equiv_{yz}{}^{\alpha}$ is refuted in the following GMS^{Q}-.

In the GMS $\boldsymbol{M}$ from Example 4 for $d = [x\,a\;\; c, y\,a\;\; b, z\,a\;\; c, s\,a\;\; a]$ we have $(\equiv_{xy})_{\alpha}(d) = T$, $(\Box\equiv_{xz})_{\alpha}(d) = T$, $(\Box\equiv_{yz})_{\alpha}(d) = F$; hence $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!\neq_{IR} \Box\equiv_{yz}{}^{\alpha}$.

Thus, $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!\neq_{*} \Box\equiv_{yz}{}^{\alpha}$, where the symbol * denotes one of the *IR*, *T*, *F*, *TF*.

At the same time, we have $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}, \Box\equiv_{yz}{}^{\alpha} |\!=_{*} \Box\equiv_{yz}{}^{\alpha}$ by the property *C* of guaranteed presence of each of the logical consequence relations.

Assertion 2. The following statements are not equivalent:

$$\equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*} \Delta \text{ and } \equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*} \Delta.$$

We have $R^{v}_{x}(\Box\equiv_{vz})^{\alpha} = \Box\equiv_{xz}{}^{\alpha}$ and $R^{v}_{y}(\Box\equiv_{vz})^{\alpha} = \Box\equiv_{yz}{}^{\alpha}$, by Example 5 $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!\neq_{*} \Box\equiv_{yz}{}^{\alpha}$, hence $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Box\equiv_{vz})^{\alpha}\ {}_M|\!\neq_{*} R^{v}_{y}(\Box\equiv_{vz})^{\alpha}$. However, $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Box\equiv_{vz})^{\alpha}, R^{v}_{y}(\Box\equiv_{vz})^{\alpha} |\!=_{*} R^{v}_{y}(\Box\equiv_{vz})^{\alpha}$, therefore the statements $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*} \Delta$ and $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Phi)^{\alpha}, R^{v}_{y}(\Phi)^{\alpha}\Gamma\ {}_M|\!=_{*} \Delta$ are not equivalent; in thegeneral case, the statements $\equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*} \Delta$ and $\equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},v}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*} \Delta$ are not equivalent.

Thus, the condition validity for $\equiv_{xy}{}^{\alpha}$ is insufficient for the substitution of equals.

Assertion 3. We have $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Box\equiv_{xz}{}^{\alpha}, |\!=_{*} \Box\equiv_{yz}{}^{\alpha}$.

Let us prove this by contradiction for the case of $|\!=_{IR}$, similarly, it is proven for $|\!=_{T}$, $|\!=_{F}$, and $|\!=_{TF}$.

Let us assume the opposite: suppose for some GMS $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$ and $\alpha \in St$, $d \in {}^{V}\!A$ we have $(\equiv_{xy})_{\alpha}(d) = T$, $Ex_{\alpha}(d) = T$, $Ey_{\alpha}(d) = T$, $(\Box\equiv_{xz})_{\alpha}(d) = T$ and $(\Box\equiv_{yz})_{\alpha}(d) = F$; from this $d_{\alpha}(x)\!\downarrow\, = d_{\alpha}(y)\!\downarrow\, = a \in A_{\alpha}$ for some $a \in A \Rightarrow d(x) = d(y) = a$. According to $(\Box\equiv_{yz})_{\alpha}(d) = F$ we have $(\equiv_{yz})_{\beta}(d) = F$ for some β such that $\alpha > \beta$. According to $(\Box\equiv_{xz})_{\alpha}(d) = T$ we have $(\equiv_{xz})_{\beta}(d) = T$, which gives us two possible cases:

1) $d_{\beta}(x)\!\uparrow$ and $d_{\beta}(z)\!\uparrow$; hence $d(x) \notin d_{\beta}$, and since $d(x) = d(y) = a$ we have $d_{\beta}(y)\!\uparrow$, so $(\equiv_{yz})_{\beta}(d) = T$, which contradicts $(\equiv_{yz})_{\beta}(d) = F$.

2) $d_{\beta}(x)\!\downarrow\, = d_{\beta}(z)\!\downarrow$; since $d(x) = d(y) = a$ then $d_{\beta}(z) = d_{\beta}(x) = d_{\beta}(y) = a$, which contradicts $(\equiv_{yz})_{\beta}(d) = F$.

In both cases, we obtained a contradiction.

Assertion 4. We have $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha} |\!=_{*} \Box\equiv_{xy}{}^{\alpha}$, where the symbol $*$ denotes either *IR* or *F*.

It is sufficient to prove for the case $|\!=_{F}$. Let us assume the opposite: suppose for some GMS $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$ and $\alpha \in St$, $d \in {}^{V}\!A$ we have $(\Box\equiv_{xy})_{\alpha}(d) = F$, $(\equiv_{xy})_{\alpha}(d) \neq F$, $Ex_{\alpha}(d) \neq F$, $Ey_{\alpha}(d) \neq F$. Since the predicates *Ex* and $\equiv_{xy}$ are total, we have $(\equiv_{xy})_{\alpha}(d) = T$, $Ex_{\alpha}(d) = T$, $Ey_{\alpha}(d) = T$. Hence $d_{\alpha}(x)\!\downarrow$, $d_{\alpha}(y)\!\downarrow$, and since $(\equiv_{xy})_{\alpha}(d) = T$, therefore $d_{\alpha}(x) = d_{\alpha}(y) = a$ for some $a \in A_{\alpha} \subseteq A$, so $d(x) = d(y) = a$. Given $(\Box\equiv_{xy})_{\alpha}(d) = F$, we have $(\equiv_{xy})_{\beta}(d) = F$ for some β such that $\alpha > \beta$. We obtain three possible cases:

1) $d_{\beta}(x)\!\downarrow$, $d_{\beta}(y)\!\downarrow$ and $d_{\beta}(x) \neq d_{\beta}(y)$; since $A_{\beta} \subseteq A$ then $d(x)\!\downarrow$, $d(y)\!\downarrow$ and $d(x) \neq d\,(y)$; at the same time we have $d(x) = d(y) = a$, which gives us a contradiction;

2) $d_{\beta}(x)\!\downarrow$ and $d_{\beta}(y)\!\uparrow$; since $A_{\beta} \subseteq A$ then $d_{\beta}(x) = d(x) = d(y) = a$, so $a \in A_{\beta}$, hence $d_{\beta}(y) = a$, which contradicts $d_{\beta}(y)\!\uparrow$;

3) the case $d_{\beta}(x)\!\uparrow$ and $d_{\beta}(y)\!\downarrow$ is treated similarly to 2).

Assertion 5. In the general case we have $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha} |\!\neq_{T} \Box\equiv_{xy}{}^{\alpha}$. Indeed, for GMS $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$, where no β exists such that $\alpha > \beta$, we have $(\Box\equiv_{xy})_{\alpha}(d)\!\uparrow$ for all $d \in {}^{V}\!A$. However, for $d = [x\,a\;\; a, y\,a\;\; a]$ with $a \in A_{\alpha}$ we have $(\equiv_{xy})_{\alpha}(d) = T$, $Ex_{\alpha}(d) = T$, $Ey_{\alpha}(d) = T$.

At the same time, by adding $\Box\equiv_{xx}$ to the left side, $|\!=_{F}$ and $|\!=_{IR}$ are preserved, and $|\!=_{T}$ is guaranteed even in the absence of β such that $\alpha > \beta$. Indeed, then $(\Box\equiv_{xy})_{\alpha}(d)\!\uparrow$ and $(\Box\equiv_{xx})_{\alpha}(d)\!\uparrow$ for all $d \in {}^{V}\!A$. Hence, we get:

Assertion 6. We have $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Box\equiv_{xx}{}^{\alpha} |\!=_{*} \Box\equiv_{xy}{}^{\alpha}$, where $*$ denotes one of the *IR*, *F*, *T*, *TF*.

Note that the reflexivity of the relation $>$ guarantees $|\!=_{T}$ and $|\!=_{TF}$ in Assertion 6.

Adding the validity conditions Ex^{α} та Ey^{α} to the validity condition $\equiv_{xy}{}^{\alpha}$ allows for the substitution of equals. Finally, we have the following properties of substitution of equals in GMS^{Q}:

≡rp$_L$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_* \Delta \Leftrightarrow \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_* \Delta$;

≡rp$_R$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_* R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Delta \Leftrightarrow \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_* R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Delta$;

≡¬rp$_L$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_* \Delta \Leftrightarrow \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_* \Delta$;

≡¬rp$_R$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_* \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Delta \Leftrightarrow \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_* \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Delta$.

Let us describe the properties related to modal compositions. In TMS of non-monotonic predicates with equality, these properties are generally analogous to the corresponding properties of TMS without equality predicates (see, for example, [12]). These are properties of carrying modalities over renominations, which belong to the properties of equivalent transformations, and properties of modality elimination. We will present the properties of carrying □ over renomination in GMS.

R□$_L$) $\Gamma, R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)^{\alpha}\ {}_M|\!=_* \Delta \Leftrightarrow \Gamma,\ \Box R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi)^{\alpha}\ {}_M|\!=_* \Delta$;

¬R□$_L$) $\Gamma, \neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)^{\alpha}\ {}_M|\!=_* \Delta \Leftrightarrow \Gamma,\ \neg\Box R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi)^{\alpha}\ {}_M|\!=_* \Delta$;

R□$_R$) $\Gamma\ {}_M|\!=_* \Delta, R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!=_* \Delta,\ \Box R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi)^{\alpha}$;

¬R□$_R$) $\Gamma\ {}_M|\!=_* \Delta, \neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!=_* \Delta,\ \neg\Box R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi)^{\alpha}$.

For the relation ${}_M|\!=_{IR}$, it is sufficient to have the properties R□$_L$ and R□$_R$.

The properties of elimination of the modal composition □ in TMS are as follows:

□$_L$) $\Box\Phi^{\alpha}, \Gamma\ {}_M|\!=_*\Delta \Leftrightarrow \{\Phi^{\beta} \mid \alpha > \beta\} \cup \Gamma\ {}_M|\!=_*\Delta$;

¬□$_R$) $\Gamma\ {}_M|\!=_*\Delta, \neg\Box\Phi^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!=_*\Delta \cup \{\neg\Phi^{\beta} \mid \alpha > \beta\}$;

□$_R$) $\Gamma\ {}_M|\!=_*\Delta, \Box\Phi^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!=_*\Delta, \Phi^{\beta}$ for all states $\beta \in S$ such that $\alpha > \beta$;

¬□$_L$) $\neg\Box\Phi^{\alpha}, \Gamma\ {}_M|\!=_*\Delta \Leftrightarrow \neg\Phi^{\beta}, \Gamma\ {}_M|\!=_*\Delta$ for all states $\beta \in S$ such that $\alpha > \beta$.

For the relation ${}_M|\!=_{IR}$, it is sufficient to have the properties □$_L$ and □$_R$.

Similar properties related to modal compositions are formulated for TmMS та MMS.

For the properties of the relation ${}_M|\!=_*$, the corresponding dual properties of the relation ${}_M|\!\neq_*$ can be specified (see [11]), using which the sequent forms of the calculus that formalizes ${}_M|\!=_*$ can be directly formulated. In particular, the properties of modality elimination for ${}_M|\!\neq_*$ in GMS have the form:

n□$_L$) $\Box\Phi^{\alpha}, \Gamma\ {}_M|\!\neq_*\Delta \Leftrightarrow \{\Phi^{\beta} \mid \alpha > \beta\} \cup \Gamma\ {}_M|\!\neq_*\Delta$;

n¬□$_R$) $\Gamma\ {}_M|\!\neq_*\Delta, \neg\Box\Phi^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!\neq_*\Delta \cup \{\neg\Phi^{\beta} \mid \alpha > \beta\}$;

n□$_R$) $\Gamma\ {}_M|\!\neq_*\Delta, \Box\Phi^{\alpha} \Leftrightarrow \Gamma\ {}_M|\!\neq_*\Delta, \Phi^{\beta}$ for some $\beta \in S$ such that $\alpha > \beta$;

n¬□$_L$) $\neg\Box\Phi^{\alpha}, \Gamma\ {}_M|\!\neq_*\Delta \Leftrightarrow \neg\Phi^{\beta}, \Gamma\ {}_M|\!\neq_*\Delta$ for some $\beta \in S$ such that $\alpha > \beta$.

With additional conditions imposed on the relation $>$, the properties of modality elimination are correspondingly modified. For the cases where the relation can be either transitive, reflexive, symmetric, or their combinations, these properties are described, in particular, in [11].

5. Sequent Calculi for Transitional Modal Logics

Properties of logical consequence relations on sets of formulas specified with states are the semantic foundation for construction of the sequent type calculi for TML. Specifications take the form of $\alpha|-$ or $\alpha-|$, where α is the name of the state. Sequents are treated as sets of such formulas. By singling out $|-$-formulas and $-|$-formulas, we denote sequents by $|-\Gamma-|\Delta$.

Sequents are enriched with sets of relations on states obtained at the time of derivation. Let M be such a reachability relation constructed for the names of states i.e. world model schema. The enriched sequents are denoted by $\Sigma\ //\ M$.

The sets of defined, undefined, and undistributed names of a state α of the sequent Σ are defined as:

$val(\Sigma_{\alpha}) = \{u \mid {}_{\alpha|-}Eu \in \Sigma\}$; $unv(\Sigma_{\alpha}) = \{u \mid {}_{\alpha-|}Eu \in \Sigma\}$; $ud(\Sigma_{\alpha}) = nm(\Sigma_{\alpha}) \setminus (val(\Sigma_{\alpha}) \cup unv(\Sigma_{\alpha}))$.

We propose sequent calculi that formalize the relation $M_{=}\models_{IR}$ in $GMS^{Q=}$ and the relations $_{M}\models_{IR}$, $_{M}\models_{T}$, $_{M}\models_{F}$, $_{M}\models_{TF}$ in $GMS^{Q\equiv}$. The calculus for the relation $_{M=}\models_{IR}$ in $GMS^{Q=}$ will be called $C^{GQ=IR}$, and the calculi for $_{M}\models_{IR}$, $_{M}\models_{T}$, $_{M}\models_{F}$, $_{M}\models_{TF}$ in $GMS^{Q\equiv}$ will be called $C^{GQ\equiv IR}$, $C^{GQ\equiv T}$, $C^{GQ\equiv F}$, $C^{GQ\equiv TF}$.

In this paper, we will restrict ourselves to describing the calculi $C^{GQ\equiv IR}$, $C^{GQ\equiv T}$, $C^{GQ\equiv F}$, $C^{GQ\equiv TF}$. The calculus CGQ=IR is generally similar to CGQ≡IR, but with =xx serving as predicates-indicators.

Sequent calculus is defined by basic sequent forms and closedness conditions for sequents.

The derivation in sequent calculi has the form of a tree, the vertices of which are sequents; such trees are called sequent trees. Inference rules in sequent calculi are called sequent forms; they are induced by properties of logical consequence relations for sets of formulas specified with states. Closed sequents are axioms of the sequent calculus. The closedness of $_{\vdash}\Gamma_{\dashv}\Delta$ must guarantee $\Gamma \models \Delta$. The sequent tree is closed if every its leaf is a closed sequent. The sequent Σ is derivable if there exists a closed sequent tree with the root Σ, called a derivation of the sequent Σ.

The closedness conditions for $_{\vdash}\Gamma_{\dashv}\Delta$ are defined by the specified above conditions that guarantee the corresponding logical consequence relation $\Gamma\ _{M}\models_{*}\Delta$.

Although the calculi $C^{GQ\equiv T}$, $C^{GQ\equiv F}$, $C^{GQ\equiv TF}$ have the same basic sequent forms, their closedness conditions for sequents are different.

We have the following *closedness conditions* for a sequent Σ:

C) there exists a formula Φ: $_{\alpha\vdash}\Phi\in\Sigma$ and $_{\alpha\dashv}\Phi\in\Sigma$;

CF) there exists a formula $_{\alpha\vdash}R^{\bar{v},\bar{u},z}_{\bar{x},\perp,\perp}(Ez)\in\Sigma$;

CL) there exists a formula Φ: $_{\alpha\vdash}\Phi\in\Sigma$ and $_{\alpha\vdash}\neg\Phi\in\Sigma$;

CR) there exists a formula Φ: $_{\alpha\dashv}\Phi\in\Sigma$ and $_{\alpha\dashv}\neg\Phi^{\alpha}\in\Sigma$;

CLR) there exist formulas Φ and Ψ: $_{\alpha\vdash}\Phi\in\Sigma$, $_{\alpha\vdash}\neg\Phi\in\Sigma$ and $_{\alpha\dashv}\Psi\in\Sigma$, $_{\alpha\dashv}\neg\Psi\in\Sigma$;

$C_{Rf\equiv}$) there exists a formula $_{\alpha\dashv}\equiv_{xx}{}^{\alpha}\in\Sigma$;

$C_{E\equiv L}$) there exist formulas $\equiv_{xy}$, Ex and Ey: $_{\alpha\vdash}\equiv_{xy}\in\Sigma$, $_{\alpha\vdash}Ex\in\Sigma$, $_{\alpha\dashv}Ey\in\Sigma$;

$C_{E\equiv R}$) there exist formulas $\equiv_{xy}$, Ex and Ey: $_{\alpha\dashv}\equiv_{xy}\in\Sigma$, $_{\alpha\dashv}Ex\in\Sigma$, $_{\alpha\dashv}Ey\in\Sigma$;

$CT_{\equiv}$) there exists a formula $R^{\bar{v},\bar{u},x,z}_{\bar{x},\perp,\perp,\perp}(\equiv_{xz})$: $_{\alpha\dashv}R^{\bar{v},\bar{u},x,z}_{\bar{x},\perp,\perp,\perp}(\equiv_{xz})\in\Sigma$.

Thus, we get the following closedness conditions for a sequent Σ in a certain calculus:

– for the calculus $MC^{Q\equiv T}$: the condition $C \vee CF \vee CL \vee C_{Rf\equiv} \vee C_{E\equiv L} \vee C_{E\equiv R} \vee CT_{\equiv}$;

– for the calculus $MC^{Q\equiv F}$: the condition $C \vee CF \vee CR \vee C_{Rf\equiv} \vee C_{E\equiv L} \vee C_{E\equiv R} \vee CT_{\equiv}$;

– for the calculus $MC^{Q\equiv TF}$: the condition $C \vee CF \vee CLR \vee C_{Rf\equiv} \vee C_{E\equiv L} \vee C_{E\equiv R} \vee CT_{\equiv}$.

The closedness condition for the calculus $MC^{Q\equiv IR}$: $C \vee CF \vee C_{Rf\equiv} \vee C_{E\equiv L} \vee C_{E\equiv R} \vee CT_{\equiv}$.

Let us present basic sequent forms for the calculi $MC^{Q\equiv T}$, $MC^{Q\equiv F}$, $MC^{Q\equiv TF}$.

Simplification forms (types R, RI, RU, R↑1, R↑2, and ≡elR):

$_{\vdash}$R $\dfrac{_{\alpha\vdash}\Phi,\Sigma\ //\ M}{_{\alpha\vdash}R(\Phi),\Sigma\ //\ M}$; $_{\dashv}$R $\dfrac{_{\alpha\dashv}\Phi,\Sigma\ //\ M}{_{\alpha\dashv}R(\Phi),\Sigma\ //\ M}$;

$_{\vdash}\neg$R $\dfrac{_{\alpha\vdash}\neg\Phi,\Sigma\ //\ M}{_{\alpha\vdash}\neg R(\Phi),\Sigma\ //\ M}$; $_{\dashv}\neg$R $\dfrac{_{\alpha\dashv}\neg\Phi,\Sigma\ //\ M}{_{\alpha\dashv}\neg R(\Phi),\Sigma\ //\ M}$;

$_{\vdash}$RI $\dfrac{_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\vdash}R^{z,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$; $_{\dashv}$RI $\dfrac{_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\dashv}R^{z,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$;

$_{\vdash}\neg$RI $\dfrac{_{\alpha\vdash}\neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\vdash}\neg R^{z,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$; $_{\dashv}\neg$RI $\dfrac{_{\alpha\dashv}\neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\dashv}\neg R^{z,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$;

$_{\vdash}$RU $\dfrac{_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\vdash}R^{y,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$, $y\in\nu(\Phi)$; $_{\dashv}$RU $\dfrac{_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\dashv}R^{y,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$, $y\in\nu(\Phi)$;

$_{\vdash}\neg$RU $\dfrac{_{\alpha\vdash}\neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\vdash}\neg R^{y,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$, $y\in\nu(\Phi)$; $_{\dashv}\neg$RU $\dfrac{_{\alpha\dashv}\neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\Sigma\ //\ M}{_{\alpha\dashv}\neg R^{y,\bar{v},\bar{u}}_{z,\bar{x},\perp}(\Phi),\Sigma\ //\ M}$, $y\in\nu(\Phi)$;

$$_{\vdash}\mathrm{R}{\uparrow}1\ \frac{{}_{\alpha\vdash}R^{\bar{v},\bar{u},y}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},y}_{\bar{x},\bot,z}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps; \qquad {}_{\dashv}\mathrm{R}{\uparrow}1\ \frac{{}_{\alpha\dashv}R^{\bar{v},\bar{u},y}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},y}_{\bar{x},\bot,z}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps;$$

$$_{\vdash}\neg\mathrm{R}{\uparrow}1\ \frac{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{u},y}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{u},y}_{\bar{x},\bot,z}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps; \qquad {}_{\dashv}\neg\mathrm{R}{\uparrow}1\ \frac{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{u},y}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{u},y}_{\bar{x},\bot,z}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps;$$

$$_{\vdash}\mathrm{R}{\uparrow}2\ \frac{{}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},z}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps; \qquad {}_{\dashv}\mathrm{R}{\uparrow}2\ \frac{{}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},z}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps;$$

$$_{\vdash}\neg\mathrm{R}{\uparrow}2\ \frac{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{u}}_{\bar{x},\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{u},z}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps; \qquad {}_{\dashv}\neg\mathrm{R}{\uparrow}2\ \frac{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{u}}_{\bar{x},\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{u},z}_{\bar{x},\bot,\bot}(p),\ {}_{\alpha\dashv}Ez,\Sigma\ //\ M},\ p\in Ps;$$

$$_{\vdash}{\equiv}\mathrm{elR}\ \frac{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\vdash}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,y}(\Phi),\Sigma\ //\ M}; \qquad {}_{\dashv}{\equiv}\mathrm{elR}\ \frac{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\dashv}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,y}(\Phi),\Sigma\ //\ M};$$

$$_{\vdash}{\equiv}\mathrm{el}\neg\mathrm{R}\ \frac{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\vdash}\neg R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\vdash}\neg R^{\bar{v},\bar{u},x}_{\bar{w},\bot,y}(\Phi),\Sigma\ //\ M}; \qquad {}_{\dashv}{\equiv}\mathrm{el}\neg\mathrm{R}\ \frac{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\dashv}\neg R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}\equiv_{xy},\ {}_{\alpha\dashv}\neg R^{\bar{v},\bar{u},x}_{\bar{w},\bot,y}(\Phi),\Sigma\ //\ M}.$$

Forms of equivalemt transformations – renomination of predicates-indicators:

$$_{\vdash}\mathrm{R}_{\bot\mathrm{E}}\ \frac{{}_{\alpha\vdash}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(Ez),\Sigma\ //\ M},\ z\notin\{\bar{v},\bar{u}\}; \qquad {}_{\dashv}\mathrm{R}_{\bot\mathrm{E}}\ \frac{{}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(Ez),\Sigma\ //\ M},\ z\notin\{\bar{v},\bar{u}\};$$

$$_{\vdash}\mathrm{R}_{\bot\mathrm{Ev}}\ \frac{{}_{\alpha\vdash}Ey,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},z}_{\bar{x},\bot,y}(Ez),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}_{\bot\mathrm{Ev}}\ \frac{{}_{\alpha\dashv}Ey,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},z}_{\bar{x},\bot,y}(Ez),\Sigma\ //\ M}.$$

Forms of equivalemt transformations – renomination of equality predicates:

$$_{\vdash}\mathrm{R}{\equiv}_{xx}\ \frac{{}_{\alpha\vdash}\equiv_{xx},\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\equiv_{xx}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{xx}\ \frac{{}_{\alpha\dashv}\equiv_{xx},\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\equiv_{xx}),\Sigma\ //\ M};$$

$$_{\vdash}\mathrm{R}{\equiv}_{0}\ \frac{{}_{\alpha\vdash}\equiv_{xy},\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\equiv_{xy}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{0}\ \frac{{}_{\alpha\dashv}\equiv_{xy},\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{w},\bot}(\equiv_{xy}),\Sigma\ //\ M}.$$

Condition for the forms $_{\vdash}\mathrm{R}{\equiv}_0$ and $_{\dashv}\mathrm{R}{\equiv}_0$: $x, y\notin\{\bar{u},\bar{v}\},\ x\neq y$.

$$_{\vdash}\mathrm{R}{\equiv}_{1}\ \frac{{}_{\alpha\vdash}\equiv_{zy},\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,z}(\equiv_{xy}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{1}\ \frac{{}_{\alpha\dashv}\equiv_{zy},\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,z}(\equiv_{xy}),\Sigma\ //\ M};$$

$$_{\vdash}\mathrm{R}{\equiv}_{1\mathrm{E}}\ \frac{{}_{\alpha\dashv}Ey,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,\bot}(\equiv_{xy}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{1\mathrm{E}}\ \frac{{}_{\alpha\vdash}Ey,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},x}_{\bar{w},\bot,\bot}(\equiv_{xy}),\Sigma\ //\ M}.$$

Condition for the forms $_{\vdash}\mathrm{R}{\equiv}_1$, $_{\dashv}\mathrm{R}{\equiv}_1$, $_{\vdash}\mathrm{R}{\equiv}_{1\mathrm{E}}$ and $_{\dashv}\mathrm{R}{\equiv}_{1\mathrm{E}}$: $y\notin\{\bar{u},\bar{v}\},\ x\neq y$.

$$_{\vdash}\mathrm{R}{\equiv}_{2}\ \frac{{}_{\alpha\vdash}\equiv_{zs},\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},x,y}_{\bar{w},\bot,z,s}(\equiv_{xy}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{2}\ \frac{{}_{\alpha\dashv}\equiv_{zs},\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},x,y}_{\bar{w},\bot,z,s}(\equiv_{xy}),\Sigma\ //\ M};$$

$$_{\vdash}\mathrm{R}{\equiv}_{2\mathrm{E}}\ \frac{{}_{\alpha\dashv}Ez,\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u},x,y}_{\bar{w},\bot,z,\bot}(\equiv_{xy}),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}{\equiv}_{2\mathrm{E}}\ \frac{{}_{\alpha\vdash}Ez,\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u},x,y}_{\bar{w},\bot,z,\bot}(\equiv_{xy}),\Sigma\ //\ M}.$$

Forms of equivalemt transformations – renomination of compositions:

$$_{\vdash}\mathrm{RR}\ \frac{{}_{\alpha\vdash}R^{\bar{v},\bar{z}}_{\bar{y},\bot}\circ^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{z}}_{\bar{y},\bot}(R^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi)),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{RR}\ \frac{{}_{\alpha\dashv}R^{\bar{v},\bar{z}}_{\bar{y},\bot}\circ^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{z}}_{\bar{y},\bot}(R^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi)),\Sigma\ //\ M};$$

$$_{\vdash}\neg\mathrm{RR}\ \frac{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{z}}_{\bar{y},\bot}\circ^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{z}}_{\bar{y},\bot}(R^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi)),\Sigma\ //\ M}; \qquad {}_{\dashv}\neg\mathrm{RR}\ \frac{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{z}}_{\bar{y},\bot}\circ^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{z}}_{\bar{y},\bot}(R^{\bar{u},\bar{t}}_{\bar{x},\bot}(\Phi)),\Sigma\ //\ M};$$

$$_{\vdash}\mathrm{R}\neg\ \frac{{}_{\alpha\vdash}\neg R^{\bar{v},\bar{u}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\vdash}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(\neg\Phi),\Sigma\ //\ M}; \qquad {}_{\dashv}\mathrm{R}\neg\ \frac{{}_{\alpha\dashv}\neg R^{\bar{v},\bar{u}}_{\bar{x},\bot}(\Phi),\Sigma\ //\ M}{{}_{\alpha\dashv}R^{\bar{v},\bar{u}}_{\bar{x},\bot}(\neg\Phi),\Sigma\ //\ M};$$

$$_{\vdash}\neg R\neg\ \ \frac{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\vdash}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\neg\Phi),\Sigma\,//\,M};\qquad _{\dashv}\neg R\neg\ \ \frac{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\neg\Phi),\Sigma\,//\,M};$$

$$_{\vdash}R\vee\ \ \frac{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi)\vee R^{\bar v,\bar u}_{\bar x,\bot}(\Psi),\Sigma\,//\,M}{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi\vee\Psi),\Sigma\,//\,M};\qquad _{\dashv}R\vee\ \ \frac{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi)\vee R^{\bar v,\bar u}_{\bar x,\bot}(\Psi),\Sigma\,//\,M}{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi\vee\Psi),\Sigma\,//\,M};$$

$$_{\vdash}\neg R\vee\ \ \frac{_{\alpha\vdash}\neg(R^{\bar v,\bar u}_{\bar x,\bot}(\Phi)\vee R^{\bar v,\bar u}_{\bar x,\bot}(\Psi)),\Sigma\,//\,M}{_{\alpha\vdash}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\Phi\vee\Psi),\Sigma\,//\,M};\qquad _{\dashv}\neg R\vee\ \ \frac{_{\alpha\dashv}\neg(R^{\bar v,\bar u}_{\bar x,\bot}(\Phi)\vee R^{\bar v,\bar u}_{\bar x,\bot}(\Psi)),\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\Phi\vee\Psi),\Sigma\,//\,M};$$

$$_{\vdash}R\Box\ \ \frac{_{\alpha\vdash}\text{}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar x,\bot}(\text{}\Phi),\Sigma\,//\,M};\qquad _{\dashv}R\Box\ \ \frac{_{\alpha\dashv}\text{}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar x,\bot}(\text{}\Phi),\Sigma\,//\,M};$$

$$_{\vdash}\neg R\Box\ \ \frac{_{\alpha\vdash}\neg\text{}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\vdash}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\text{}\Phi),\Sigma\,//\,M};\qquad _{\dashv}\neg R\Box\ \ \frac{_{\alpha\dashv}\neg\text{}R^{\bar v,\bar u}_{\bar x,\bot}(\Phi),\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar x,\bot}(\text{}\Phi),\Sigma\,//\,M}.$$

Forms of elimination for the constant $_{\top}$-formula:

$$\mathrm{El}_{\mathrm{RE}}\ \ \frac{\Sigma\,//\,M}{_{\alpha\vdash}R^{\bar v,\bar u,z}_{\bar x,\bot,\bot}(Ez),\Sigma\,//\,M};\qquad \mathrm{El}\equiv\ \ \frac{\Sigma\,//\,M}{_{\alpha\vdash}\equiv_{xx},\Sigma\,//\,M};\qquad \mathrm{ElR}\equiv\ \ \frac{\Sigma\,//\,M}{_{\alpha\vdash}R^{\bar v,\bar u,x,y}_{\bar w,\bot,\bot,\bot}(\equiv_{xy}),\Sigma\,//\,M}.$$

Forms of elimination of $\neg$ in formulas related to *TS*-predicates:

$$_{\vdash}\neg_{E}\equiv\ \ \frac{_{\alpha\dashv}Ez,\Sigma\,//\,M}{_{\alpha\vdash}\neg Ez,\Sigma\,//\,M};\qquad _{\dashv}\neg_{E}\ \ \frac{_{\alpha\vdash}Ez,\Sigma\,//\,M}{_{\alpha\dashv}\neg Ez,\Sigma\,//\,M};$$

$$_{\vdash}\neg_{RE}\ \ \frac{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar w,\bot}(Ez),\Sigma\,//\,M}{_{\alpha\vdash}\neg R^{\bar v,\bar u}_{\bar w,\bot}(Ez),\Sigma\,//\,M};\qquad _{\dashv}\neg_{RE}\ \ \frac{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar w,\bot}(Ez),\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar w,\bot}(Ez),\Sigma\,//\,M};$$

$$_{\vdash}\neg\equiv\ \ \frac{_{\alpha\dashv}\equiv_{xy},\Sigma\,//\,M}{_{\alpha\vdash}\neg\equiv_{xy},\Sigma\,//\,M};\qquad _{\dashv}\neg\equiv\ \ \frac{_{\alpha\vdash}\equiv_{xy},\Sigma\,//\,M}{_{\alpha\dashv}\neg\equiv_{xy},\Sigma\,//\,M};$$

$$_{\vdash}\neg_{R}\equiv\ \ \frac{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar w,\bot}(\equiv_{xy}),\Sigma\,//\,M}{_{\alpha\vdash}\neg R^{\bar v,\bar u}_{\bar w,\bot}(\equiv_{xy}),\Sigma\,//\,M};\qquad _{\dashv}\neg_{R}\equiv\ \ \frac{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar w,\bot}(\equiv_{xy}),\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar w,\bot}(\equiv_{xy}),\Sigma\,//\,M}.$$

Forms of formulas' decomposition:

$$_{\vdash}\neg\neg\ \ \frac{_{\alpha\vdash}\Phi,\Sigma\,//\,M}{_{\alpha\vdash}\neg\neg\Phi,\Sigma\,//\,M};\qquad _{\dashv}\neg\neg\ \ \frac{_{\alpha\dashv}\Phi,\Sigma\,//\,M}{_{\alpha\dashv}\neg\neg\Phi,\Sigma\,//\,M};$$

$$_{\vdash}\vee\ \ \frac{_{\alpha\vdash}\Phi,\Sigma\,//\,M\qquad _{\alpha\vdash}\Psi,\Sigma\,//\,M}{_{\alpha\vdash}\Phi\vee\Psi,\Sigma\,//\,M};\qquad _{\dashv}\vee\ \ \frac{_{\alpha\dashv}\Phi,\ _{\alpha\dashv}\Psi,\Sigma\,//\,M}{_{\alpha\dashv}\Phi\vee\Psi,\Sigma\,//\,M};$$

$$_{\vdash}\neg\vee\ \ \frac{_{\alpha\vdash}\neg\Phi,\ _{\alpha\vdash}\neg\Psi,\Sigma\,//\,M}{_{\alpha\vdash}\neg(\Phi\vee\Psi),\Sigma\,//\,M};\qquad _{\dashv}\neg\vee\ \ \frac{_{\alpha\dashv}\neg\Phi,\Sigma\,//\,M\qquad _{\alpha\dashv}\neg\Psi,\Sigma\,//\,M}{_{\alpha\dashv}\neg(\Phi\vee\Psi),\Sigma\,//\,M}.$$

Forms of quantifier elimination:

$$_{\vdash}\exists\ \ \frac{_{\alpha\vdash}R^{x}_{z}(\Phi),\ _{\alpha\vdash}Ez,\Sigma\,//\,M}{_{\alpha\vdash}\exists x\Phi,\Sigma\,//\,M};\qquad _{\dashv}\neg\exists\ \ \frac{_{\alpha\dashv}\neg R^{x}_{z}(\Phi),\ _{\alpha\vdash}Ez,\Sigma\,//\,M}{_{\alpha\dashv}\neg\exists x\Phi,\Sigma\,//\,M};$$

$$_{\vdash}\exists R\ \ \frac{_{\alpha\vdash}R^{\bar v,\bar u,x}_{\bar w,\bot,z}(\Phi),\ _{\alpha\vdash}Ez,\Sigma\,//\,M}{_{\alpha\vdash}R^{\bar v,\bar u}_{\bar w,\bot}(\exists x\Phi),\Sigma\,//\,M};\qquad _{\dashv}\neg\exists R\ \ \frac{_{\alpha\dashv}\neg R^{\bar v,\bar u,x}_{\bar w,\bot,z}(\Phi),\ _{\alpha\vdash}Ez,\Sigma\,//\,M}{_{\alpha\dashv}\neg R^{\bar v,\bar u}_{\bar w,\bot}(\exists x\Phi),\Sigma\,//\,M};$$

$$_{\dashv}\exists\vee\ \ \frac{_{\alpha\dashv}\exists x\Phi,\ _{\alpha\dashv}R^{x}_{y}(\Phi),\ _{\alpha\vdash}Ey,\Sigma\,//\,M}{_{\alpha\dashv}\exists x\Phi,\ _{\alpha\vdash}Ey,\Sigma\,//\,M};\qquad _{\vdash}\neg\exists\vee\ \ \frac{_{\alpha\vdash}\neg\exists x\Phi,\ _{\alpha\vdash}\neg R^{x}_{y}(\Phi),\ _{\alpha\vdash}Ey,\Sigma\,//\,M}{_{\alpha\vdash}\neg\exists x\Phi,\ _{\alpha\vdash}Ey,\Sigma\,//\,M};$$

$$_{\dashv}\exists R\vee\ \ \frac{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar w,\bot}(\exists x\Phi),\ _{\alpha\dashv}R^{\bar v,\bar u,x}_{\bar w,\bot,y}(\Phi),\ _{\alpha\vdash}Ey,\Sigma\,//\,M}{_{\alpha\dashv}R^{\bar v,\bar u}_{\bar w,\bot}(\exists x\Phi),\ _{\alpha\vdash}Ey,\Sigma\,//\,M};$$

$$_{|-}\neg\exists\text{Rv}\ \frac{{}_{\alpha|-}\neg R^{\bar{v},\bar{u}}_{\bar{w},\perp}(\exists x\Phi),\ {}_{\alpha|-}\neg R^{\bar{v},\bar{u},x}_{\bar{w},\perp,y}(\Phi),\ {}_{\alpha|-}Ey, \Sigma\ //\ M}{{}_{\alpha|-}\neg R^{\bar{v},\bar{u}}_{\bar{w},\perp}(\exists x\Phi),\ {}_{\alpha|-}Ey, \Sigma\ //\ M}.$$

Condition for $|{-}\exists$ and $_{-|}\neg\exists$: $z\in fu(\Sigma, \exists x\Phi)$; condition for $_{|-}\exists$R and $_{-|}\neg\exists$R: $z\in fu(R^{\bar{v},\bar{u}}_{\bar{w},\perp}(\exists x\Phi))$.

The forms $_{|-}\exists$, $_{-|}\neg\exists$, $_{|-}\exists$R and $_{-|}\neg\exists$R will be called $\exists_T$-forms; the forms $_{-|}\exists$v, $_{|-}\neg\exists$v, $_{-|}\exists$Rv and $_{|-}\neg\exists$Rv will be called $\exists_F$-forms.

Forms of *E*-distribution and primary definition:

$$\text{Ed}\ \frac{{}_{\alpha|-}Ex,\ \Sigma\ //\ M \qquad {}_{\alpha-|}Ex,\ \Sigma\ //\ M}{\Sigma},\ \text{where } {}_{\alpha|-}Ex, {}_{\alpha-|}Ex \notin \Sigma.$$

$$\text{Ev}\ \frac{{}_{\alpha|-}Ez,\ \Sigma\ //\ M}{\Sigma\ //\ M}\quad \text{given } z\in fu(\Sigma).$$

Forms of transitivity and substitution of equals:

$$\text{Tr}{\equiv}\ \frac{{}_{\alpha-|}\equiv_{xy},\ {}_{\alpha|-}\equiv_{yz},\ {}_{\alpha|-}\equiv_{xz}, \Sigma\ //\ M}{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}\equiv_{yz}, \Sigma\ //\ M};$$

$$_{|-}{\equiv}\text{rp}\ \frac{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha|-}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi),\ {}_{\alpha|-}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi), \Sigma\ //\ M}{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha|-}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi), \Sigma\ //\ M};$$

$$_{-|}{\equiv}\text{rp}\ \frac{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha-|}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi),\ {}_{\alpha-|}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi), \Sigma\ //\ M}{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha-|}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi), \Sigma\ //\ M};$$

$$_{|-}\neg{\equiv}\text{rp}\ \frac{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha|-}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi),\ {}_{\alpha|-}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi), \Sigma\ //\ M}{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha|-}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi), \Sigma\ //\ M};$$

$$_{-|}\neg{\equiv}\text{rp}\ \frac{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha-|}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi),\ {}_{\alpha-|}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi), \Sigma\ //\ M}{{}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey,\ {}_{\alpha-|}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi), \Sigma\ //\ M}.$$

The forms of *modal operator elimination* depend on the properties of the relation $>$. We will describe these forms in the general case where no additional conditions are imposed on the relation.

The names of the calculi $MC^{Q\text{-}T}$, $MC^{Q\text{-}F}$ and $MC^{Q\text{-}TF}$ correspond to this case.

If at the moment of applying the form to $_{\alpha|-}\Box\Phi$ or $_{\alpha-|}\neg\Box\Phi$ we have states $\beta_1,\ldots, \beta_n$ such that $\alpha>\beta_1,\ldots,\alpha>\beta_n$, then we apply the corresponding form to $_{\alpha|-}\Box\Phi$ or $_{\alpha|-}\neg\Box\Phi$:

$$_{|-}\Box\ \frac{{}_{\alpha|-}\Box\Phi,\ {}_{\beta_1|-}\Phi,\ldots, {}_{\beta_n|-}\Phi,\ \Sigma\ //\ M}{{}_{\alpha|-}\Box\Phi, \Sigma\ //\ M}; \qquad _{-|}\neg\Box\ \frac{{}_{\alpha-|}\neg\Box\Phi,\ {}_{\beta_1-|}\neg\Phi,\ldots, {}_{\beta_n-|}\neg\Phi,\ \Sigma\ //\ M}{{}_{\alpha-|}\neg\Box\Phi, \Sigma\ //\ M}.$$

If there are no such γ such that $\alpha>\gamma$, then we apply the form to $_{\alpha|-}\Box\Phi$ or $_{\alpha-|}\neg\Box\Phi$:

$$_{|-}\Box\text{f}\ \frac{{}_{\alpha|-}\Box\Phi,\ {}_{\beta|-}\Phi, \Sigma\ //\ M\cup\{\alpha>\beta\}}{{}_{\alpha|-}\Box\Phi,\ \Sigma\ //\ M},\ \beta \text{ is a new state;}$$

$$_{-|}\neg\Box\text{f}\ \frac{{}_{\alpha-|}\neg\Box\Phi,\ {}_{\beta-|}\neg\Phi, \Sigma\ //\ M\cup\{\alpha>\beta\}}{{}_{\alpha-|}\neg\Box\Phi,\ \Sigma\ //\ M},\ \beta \text{ is a new state.}$$

The elimination form applied to $_{\alpha-|}\Box\Phi$ or $_{\alpha|-}\neg\Box\Phi$ is:

$$_{-|}\Box\ \frac{{}_{\beta-|}\Phi, \Sigma\ //\ M\cup\{\alpha>\beta\}}{{}_{\alpha-|}\Box\Phi,\ \Sigma\ //\ M},\ \beta \text{ is a new state;} \qquad _{|-}\neg\Box\ \frac{{}_{\beta|-}\neg\Phi, \Sigma\ //\ M\cup\{\alpha>\beta\}}{{}_{\alpha|-}\neg\Box\Phi,\ \Sigma\ //\ M},\ \beta \text{ is a new state.}$$

Let us describe basic sequent forms for the sequent calculus $MC^{Q\text{-}IR}$. There is no need to list the sequent forms for external negation on renomination, which results in the following basic forms.

Simplification forms $_{|-}$R, $_{-|}$R, $_{|-}$RI, $_{-|}$RI, $_{|-}$RU, $_{-|}$RU, $_{-}$R↑1, $_{-|}$R↑1, $_{|-}$R↑2, $_{-|}$R↑2, $_{|-}{\equiv}$elR, $_{-|}{\equiv}$elR.

Forms of equivalemt transformations: $_{|-}$RR, $_{-|}$RR, $_{|-}$R¬, $_{-|}$R¬, $_{|-}$R∨, $_{-|}$R∨, $_{|-}$R□, $_{-|}$R□; $_{|-}$R$_{\perp E}$, $_{-|}$R$_{\perp E}$, $_{|-}$R$_{\perp Ev}$, $_{-|}$R$_{\perp Ev}$; $_{|-}$R$\equiv_{xx}$, $_{-|}$R$\equiv_{xx}$, $_{|-}$R$\equiv_0$, $_{-|}$R$\equiv_0$, $_{|-}$R$\equiv_1$, $_{-|}$R$\equiv_1$, $_{|-}$R$\equiv_2$,$_{-|}$R$\equiv_2$, $_{|-}$R$\equiv_{1E}$, $_{-|}$R$\equiv_{1E}$, $_{|-}$R$\equiv_{2E}$, $_{-|}$R$\equiv_{2E}$.

Forms of elimination for the constant $_T$-formula El$_{RE}$, El≡, ElR≡.

Forms of decomposition of the formulas $_{|-}$∨ and $_{-|}$∨, to which the forms $_{|-}$¬ and $_{-|}$¬ are added:

$$\vdash\neg\ \frac{{}_{\alpha\dashv}\Phi,\Sigma\ //\ M}{{}_{\alpha\vdash}\neg\Phi,\Sigma\ //\ M}; \qquad\qquad {}_{\dashv}\neg\ \frac{{}_{\alpha\vdash}\Phi,\Sigma\ //\ M}{{}_{\alpha\dashv}\neg\Phi,\Sigma\ //\ M}.$$

Forms of quantifier elimination ${}_{\vdash}\exists$, ${}_{\vdash}\exists$R, ${}_{\dashv}\exists$v, ${}_{\dashv}\exists$Rv; *E*-distribution Ed, and primary definition Ev.

Forms of transitivity and substitution of equals Tr≡, ${}_{\vdash}$≡rp, ${}_{\dashv}$≡rp.

Forms of modal operator elimination ${}_{\vdash}\Box$, ${}_{\vdash}\Box$f, ${}_{\dashv}\Box$ (in the absence of additional conditions on $>$).

We have the following groups of basic forms for the calculi $MC^{Q_{\text{-}}T}$, $MC^{Q_{\text{-}}F}$, $MC^{Q_{\text{-}}TF}$, $MC^{Q_{\text{-}}IR}$:

– *auxiliary* simplification forms: types R, RI, RU, R↑1, R↑2, and ≡elR;

– forms of *E*-distribution and primary definition: Ed та Ev;

– *main* forms: all other basic sequent forms.

The main property of the listed basic sequent forms of TML is described by

Theorem 7. Let $\frac{{}_{\vdash}\Lambda\,{}_{\dashv}\mathrm{K}\ //\ \mathrm{M}}{{}_{\vdash}\Gamma\,{}_{\dashv}\Delta\ //\ \mathrm{M}}$ and $\frac{{}_{\vdash}\Lambda\,{}_{\dashv}\mathrm{K}\ //\ \mathrm{M}\quad {}_{\vdash}\mathrm{X}\,{}_{\dashv}\mathrm{Z}\ //\ \mathrm{M}}{{}_{\vdash}\Gamma\,{}_{\dashv}\Delta\ //\ \mathrm{M}}$ be sequent forms. Then:

1) $\Lambda \models_* \mathrm{K} \Leftrightarrow \Gamma \models_* \Delta$; $\Lambda \models_* \mathrm{K}$ and $\mathrm{X} \models_* \mathrm{Z} \Leftrightarrow \Gamma \models_{*'} \Delta$;

2) $\Gamma \not\models_* \Delta \Leftrightarrow \Lambda \not\models_* \mathrm{K}$; $\Gamma \not\models_* \Delta \Leftrightarrow \Lambda \not\models_* \mathrm{K}$ and $\mathrm{X} \not\models_* \mathrm{Z}$.

Let us briefly describe the construction of derivations in sequent calculi for TML. The step-by-step construction of a sequent tree for countable sequents in TML calculi is similar to that for calculi of quasiary predicate logics (see [9, 13]). The process is carried out in parallel with the formation of the world model schema. This schema is updated with each application of the appropriate forms of modality elimination, which add new states.

The construction of the tree begins from its root − the initial sequent Σ. Each application of a sequent form is performed on a finite set of formulas available at the moment.

At the start of each stage, an access step is performed: one formula from each of the lists of |–-formulas and –|-formulas is added to the list of available formulas. At the beginning of the construction, a pair of the first formulas from these lists is available (either a single |–-formula or –|-formula, if one of the lists is empty).

After applying each sequent form, we check the resulting sequent Ω for closedness. If a closed sequent is obtained, no further forms can be applied to it, and the construction of the tree on this path terminates. If all leaves of the constructed tree are closed, then we have a closed sequent tree, and the proof construction is completed *successfully*.

If the construction is not completed, for each non-closed leaf ξ, we proceed with the next access step and then extend a finite subtree with root ξ as follows. We activate all available (except primitive) formulas of ξ. Next, we apply the corresponding sequent forms to each active formula. Whenever appropriate, we perform simplifications using the necessary *auxiliary* forms of the types R, RI, RU, R↑1, R↑2, and ≡elR; forms of the types R↑1 and R↑2 are applied to primitive formulas and their negations, yielding primitive *Un*-forms (see [13]) with the set $Un = \{x \mid {}_{\alpha\dashv}Ex \in \Theta\}$, where Θ is the set of formulas on the path from the root Σ to the given sequent. After applying the main form and performing simplifications, the formulas generated on this stage become passive; at this stage, the main sequent forms cannot be applied to such formulas.

In the application of the main sequent forms, the process proceeds as follows. First, all non-modalized forms are executed. The application of $\exists_T$-forms precedes the application of ∃F-forms. When applying an ∃T-form, we always select a new totally non-essential z that does not appear on the path from the root to the sequent where the ∃T-form is applied. Each $\exists_F$-form is applied multiple times for each assigned component y from formulas on the path from Σ to the given sequent η. Let Ξ be the set of available sequent formulas on the path from Σ to η. For each α, we define $Ud_\alpha = ud(\Xi_\alpha)$. If, when transitioning to the application of an $\exists_F$-form, $Ud_\alpha \neq \varnothing$, then there are *undistributed* names of the state α, so using Ed, we perform all possible distributions of names from Ud_α into assigned and unassigned ones. This results in constructing a subtree of height $|Ud_\alpha|$ with the subroot η, which produces $m = 2^{|Ud_\alpha|}$ successors of η – sequents $\eta_1,\ldots,\eta_m$ with sets $Vn_{\alpha k} \subseteq Ud_\alpha$ of new assigned names. If $val(\Xi_\alpha) = \varnothing$, then for η_j, where $Vn_{\alpha j} = \varnothing$, we perform the initial assignment – adding ${}_{\alpha\vdash}Ez$ for a new totally unessential *z*, which results in $Vn_{\alpha j} = \{z\}$. In each of these η_k, we apply the $\exists_F$-form for each $y \in Vn_{\alpha k}$.

The forms Tr≡ are applied every time a pair of formulas of the form ${}_{\alpha|-}\equiv_{xy}$ and ${}_{\alpha|-}\equiv_{yz}$ appears, where at least one of them is *new* to the sequent. The forms of the type ≡rp (substitution of equals) are applied each time a pair of formulas appears, one of which is of the form ${}_{\alpha|-}\equiv_{xy}$, and the other is one of the forms ${}_{\alpha|-}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(p)$, ${}_{\alpha-|}R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(p)$, ${}_{\alpha|-}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(p)$, ${}_{\alpha-|}\neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(p)$, where at least one of them is new to the sequent.

Next, we apply ${}_{-|}\Box$-forms, and finally, at the end of the stage, we apply $-{}_{|-}\Box$-forms.

If the construction of the sequent tree is completed successfully, then we obtain a closed tree.

If the construction does not complete, then we have an infinite, unclosed tree. In such a tree, there is an unclosed path $\wp$ (by König's lemma, see [15]), all of whose vertices are unclosed sequents. Each of the formulas from the initial sequent Σ will appear on $\wp$ and become accessible.

For the proposed sequent calculi for TML, the soundness and completeness theorems hold. For these calculi, the theorems are formulated in a similar manner, with the relations $|=_{IR}$, $|=_T$, $|=_F$, $|=_{TF}$ corresponding to the calculi $MC^{Q\text{-}IR}$, $MC^{Q\text{-}T}$, $MC^{Q\text{-}F}$, $MC^{Q\text{-}TF}$.

Theorem 8 (soundness). Let the sequent ${}_{|-}\Gamma_{-|}\Delta$ be derivable in the calculus C; then $\Gamma \,|=_* \Delta$.

Let ${}_{|-}\Gamma_{-|}\Delta$ be derivable in the calculus C. Then, a closed sequent tree has been constructed for it. All its leaves are closed sequents, so for each such leaf ${}_{|-}X_{-|}Z$, we have $X \,|=_* Z$. The movement from the leaves of the tree to its root is accomplished using sequent forms. By Theorem 7, the relation $|=^*$ is preserved when moving from the premises of the forms to the conclusions. Therefore, $\Lambda |=^* K$ for each vertex ${}_{|-}\Lambda_{-|}K$ of the sequent tree. In particular, for the root ${}_{|-}\Gamma_{-|}\Delta$ we also have $\Gamma \,|=_* \Delta$.

The proof of completeness for the sequent calculi of TML relies on the theorem about constructing a counter-model using an unclosed path in the sequent tree built in the corresponding calculus. The proof of the counter-model theorems is based on the method of model (Hintikka) sets (see [11]).

Theorem 9 (on counter-models for the calculus MCQ-^{TF}). Let $\wp$ be an unclosed path in a sequent tree constructed for the sequent ${}_{|-}\Gamma_{-|}\Delta$ in the calculus $MC^{Q\text{-}TF}$, and let S be the set of names of states of the world in the specified formulas along the path $\wp$. Then there exist GMS $M^T = (St, R, A, Im_T)$, $M^F = (St, R, A, Im_F)$, and $\delta \in {}^V\!A$ such that for all $\alpha \in S$:

$${}_{\alpha|-}\Phi \in H_\alpha \Rightarrow \Phi^T{}_\alpha(\delta) = T; \quad {}_{\alpha-|}\Phi \in H_\alpha \Rightarrow \Phi^T{}_\alpha(\delta) \neq T.$$
$${}_{\alpha|-}\Phi \in H_\alpha \Rightarrow \Phi^F{}_\alpha(\delta) \neq F; \quad {}_{\alpha-|}\Phi \in H_\alpha \Rightarrow \Phi^F{}_\alpha(\delta) = F.$$

Here, $\Phi^T{}_\alpha$ and $\Phi^F{}_\alpha$ denote $Im_T(\Phi, \alpha)$ and $Im_F(\Phi, \alpha)$, respectively.

Such GMS M^T and M^F are called T-conter-model and F-counter-model for ${}_{|-}\Gamma_{-|}\Delta$.

Let M be the union of all world model schemes of the sequents on the path $\wp$, then S is the set of state names from M. Let H_α be the set of all specified formulas of the state α on the path $\wp$ $W_\alpha = nm(H_\alpha)\backslash unv(H_\alpha)$, $W = \bigcup_{\alpha \in S} W_\alpha$, $H_M = (\{H_\alpha \mid \alpha \in S\}, M)$.

Such H_M is called a model system (i.e. a set of model sets).

Equality predicates induce equivalence relations on the sets W_α:

$$x \sim_\alpha y \;\Leftrightarrow\; {}_{\alpha|-}\equiv_{xy},\ {}_{\alpha|-}Ex,\ {}_{\alpha|-}Ey \in H_\alpha.$$

Let us denote $\langle v\rangle_\alpha = \{u \mid v \sim_\alpha u\}$. Now, we define $\langle v\rangle = \{u \mid v \sim_\alpha u$ for some $\alpha \in S\}$.

This definition is correct. It is based on the interpretation of the equality of basic data as an identity: for the same data d, it is impossible for $d(x)\!\downarrow = d(y)\!\downarrow$ on one state and $d(x)\!\downarrow \neq d(y)\!\downarrow$ on another state.

We denote $A_\alpha = \{\langle v\rangle \mid v \in W_\alpha]$. Then $A = \bigcup_{\alpha \in S} A_\alpha = \{\langle v\rangle \mid v \in W\}$.

Let us specify $\delta = [v \mapsto \langle v\rangle \mid v \in W]$ and $\delta_\alpha = [v \mapsto \langle v\rangle \mid v \in W_\alpha]$.

For predicates-indicators and equality predicates in GMS M^T and M^F, we have:

- ${}_{\alpha|-}Ex \in H_\alpha$ implies $x \in W$, so $Ex^T{}_\alpha(\delta) = T$ and $Ex^F{}_\alpha(\delta) = T$, therefore $Ex^F{}_\alpha(\delta) \neq F$;
- ${}_{\alpha-|}Ex \in H_\alpha$ implies $x \notin W$, so $Ex^T{}_\alpha(\delta) = F$, therefore $Ex^T{}_\alpha(\delta) \neq T$, and $Ex^F{}_\alpha(\delta) = F$;
- ${}_{\alpha|-}\equiv_{xy} \in H_\alpha \Rightarrow (\equiv_{xy})_\alpha(\delta) = T$ and $(\equiv_{xy})^F{}_\alpha(\delta) = T$, therefore $(\equiv_{xy})^F{}_\alpha(\delta) \neq F$;
- ${}_{\alpha-|}\equiv_{xy} \in H_\alpha \Rightarrow (\equiv_{xy})^T{}_\alpha(\delta) = F$, therefore $(\equiv_{xy})^T{}_\alpha(\delta) \neq T$, and $(\equiv_{xy})^F{}_\alpha(\delta) = F$.

Let us specify the values of the predicates represented by predicate symbols and their negations and by primitive *Un*-formulas and their negations on δ in GMS M^T and M^F:

- $_{\alpha}\vdash p\in H_{\alpha} \Rightarrow p^{T}_{\alpha}(\delta)=T$ and $p^{F}_{\alpha}(\delta)\neq F$
- $_{\alpha}\dashv p\in H_{\alpha} \Rightarrow p^{T}_{\alpha}(\delta)\neq T$ and $p^{F}_{\alpha}(\delta)=F$;
- $_{\alpha}\vdash \neg p\in H_{\alpha} \Rightarrow \neg p^{T}_{\alpha}(\delta)=T$ and $\neg p^{F}_{\alpha}(\delta)\neq F$
- $_{\alpha}\dashv \neg p\in H_{\alpha} \Rightarrow \neg p^{T}_{\alpha}(\delta)\neq T$ and $\neg p^{F}_{\alpha}(\delta)=F$;
- $_{\alpha}\vdash R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow p^{T}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=T$ and $p^{F}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))\neq F$;
- $_{\alpha}\dashv R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow p^{T}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))\neq T$ and $p^{F}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=F$;
- $_{\alpha}\vdash \neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow \neg p^{T}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=T$ and $\neg p^{F}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))\neq F$;
- $_{\alpha}\dashv \neg R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow \neg p^{T}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))\neq T$ and $\neg p^{F}_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=F$.

Next, we prove by induction on the formula structure.

Similarly, the theorems on counter-models for the calculi MCQ≡T and MCQ≡F can be formulated.

Theorem 10 (on a counter-model for the calculus $MC^{Q\equiv T}$). Let ℘ be an unclosed path in a sequent tree constructed for the sequent $\vdash\Gamma\dashv\Delta$ in the calculus MCQ≡T, and let S be the set of names of states of the world in the specified formulas along the path ℘. Then there exist GMS M = (*St*, *R*, *A*, *Im*) and $\delta\in{}^{V}A$ such that for all α∈S:

$_{\alpha}\vdash\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)=T$; $_{\alpha}\dashv\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)\neq T$.

Such GMS M will be called a *T*-counter-model for $\vdash\Gamma\dashv\Delta$.

Theorem 11 (on a counter-model for the calculus $MC^{Q\equiv F}$). Let ℘ be an unclosed path in a sequent tree constructed for the sequent $\vdash\Gamma\dashv\Delta$ in the calculus $MC^{Q\equiv F}$, and let S be the set of names of states of the world in the specified formulas along the path ℘. Then there exist GMS M = (St, R, A, Iт) and δ∈VA such that for all α∈S:

$_{\alpha}\vdash\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)\neq F$; $_{\alpha}\dashv\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)=F$.

Such GMS M will be called an *F*-counter-model for $\vdash\Gamma\dashv\Delta$.

Let us examine in detail the theorem on the counter-model for the calculus $MC^{Q\equiv IR}$.

Theorem 12 (on a counter-model for the calculus $MC^{Q\equiv IR}$). Let ℘ be an unclosed path in a sequent tree constructed for the sequent $\vdash\Gamma\dashv\Delta$ in the calculus $MC^{Q\equiv IR}$, and let S be the set of names of states of the world in the specified formulas along the path ℘. Then there exist GMS M = (St, R, A, Iт) and δ∈VA such that for all α∈S:

$_{\alpha}\vdash\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)=T$; $_{\alpha}\dashv\Phi\in H_{\alpha} \Rightarrow \Phi_{\alpha}(\delta)=F$.

Such GMS M will be called an *IR*-conter-model for $\vdash\Gamma\dashv\Delta$.

We specify $\delta = [v\mapsto\langle v\rangle \mid v\in W]$ and $\delta_{\alpha} = [v\mapsto\langle v\rangle \mid v\in W_{\alpha}]$ as described in Theorem 9.

For predicates-indicators and equality predicates in GMS M, we have:

- $_{\alpha}\vdash Ex\in H_{\alpha}$ implies $x\in W$, therefore $Ex_{\alpha}(\delta) = T$;
- $_{\alpha}\dashv Ex\in H_{\alpha}$ implies $x\notin W$, therefore $Ex_{\alpha}(\delta) = F$;
- $_{\alpha}\vdash \equiv_{xy}\in H_{\alpha} \Rightarrow (\equiv_{xy})_{\alpha}(\delta) = T$;
- $_{\alpha}\dashv \equiv_{xy}\in H_{\alpha} \Rightarrow (\equiv_{xy})_{\alpha}(\delta) = F$.

Let us specify the values of the predicates represented by predicate symbols and primitive Un-formulas on δ in GMS M:

- $_{\alpha}\vdash p\in H_{\alpha} \Rightarrow p_{\alpha}(\delta)=T$;
- $_{\alpha}\dashv p\in H_{\alpha} \Rightarrow p_{\alpha}(\delta)=F$;
- $_{\alpha}\vdash R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow p_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=T$;
- $_{\alpha}\dashv R^{\bar{v},\bar{u}}_{\bar{x},\perp}(p)\in H_{\alpha} \Rightarrow p_{\alpha}(\mathrm{r}^{\bar{v},\bar{u}}_{\bar{x},\perp}(\delta))=F$.

Next, we prove by induction on the formula structure.

From the theorems on constructing counter-models, we obtain the completeness theorems.

Theorem 13 (completeness of $MC^{Q\equiv IR}$). Let $\Gamma \models_{IR} \Delta$; then the sequent $\vdash\Gamma\dashv\Delta$ is derivable in the calculus $MC^{Q_{=}, IR}$.

Let us assume the opposite: suppose $\Gamma \models_{IR} \Delta$, i.e. $\Gamma\ {}_M\!\models_{IR} \Delta$ holds for every consistent GMS M, but the sequent $\Sigma = {}_{|-}\Gamma_{-|}\Delta$ is not derivable. Then there exists an unclosed path in the tree for Σ. By Theorem 12, there exist GMS $M = (S, R, A, Jm)$ and $\delta \in {}^V\!A$: ${}_{\alpha|-}\Phi \in H_\alpha \Rightarrow \Phi_\alpha(\delta) = T$ and ${}_{\alpha-|}\Phi \in H_\alpha \Rightarrow \Phi_\alpha(\delta) = F$. In particular, this holds for the formulas of the sequent ${}_{|-}\Gamma_{-|}\Delta$. Therefore, $\Phi_\alpha(\delta) = T$ for all $\Phi^\alpha \in \Gamma$ and $\Psi_\beta(\delta) = F$ for all $\Psi^\beta \in \Delta$. This contradicts $\Gamma\ {}_M\!\models_{IR} \Delta$, hence $\Gamma \not\models_{IR} \Delta$. We have reached a contradiction. Thus, the assumption that ${}_{|-}\Gamma_{-|}\Delta$ is not derivable is incorrect, which proves the theorem.

Theorem 14 (completeness of $MC^{Q_{=}, TF}$). Let $\Gamma \models_{TF} \Delta$; then the sequent ${}_{|-}\Gamma_{-|}\Delta$ is derivable in the calculus $MC^{Q_{=}, TF}$.

Let us assume the opposite: suppose $\Gamma \models_{TF} \Delta$, i.e. $\Gamma\ {}_M\!\models_{TF} \Delta$ holds for every consistent GMS M, but the sequent $\Sigma = {}_{|-}\Gamma_{-|}\Delta$ is not derivable. Then there exists an unclosed path in the tree for Σ. By Theorem 9, there exist $M^T = (St, R, A, Im_T)$, $M^F = (St, R, A, Im_F)$, and $\delta \in {}^V\!A$ such that:

$${}_{\alpha|-}\Phi \in H_\alpha \Rightarrow \Phi^T{}_\alpha(\delta) = T; \quad {}_{\alpha-|}\Phi \in H_\alpha \Rightarrow \Phi^T{}_\alpha(\delta) \neq T;$$

$${}_{\alpha|-}\Phi \in H_\alpha \Rightarrow \Phi^F{}_\alpha(\delta) \neq F; \quad {}_{\alpha-|}\Phi \in H_\alpha \Rightarrow \Phi^F{}_\alpha(\delta) = F.$$

For a T-counter-model, according to ${}_{|-}\Gamma_{-|}\Delta \subseteq H$, for all $\Phi^\alpha \in \Gamma$ we have $\Phi^T{}_\alpha(\delta) = T$, and for all $\Psi^\beta \in \Delta$ we have $\Psi^T{}_\beta(\delta) \neq T$. This contradicts $\Gamma\ {}_M\!\models_T \Delta$, therefore, $\Gamma \not\models_{TF} \Delta$.

For an F-counter-model, according to ${}_{|-}\Gamma_{-|}\Delta \subseteq H$, for all $\Phi^\alpha \in \Gamma$ we have $\Phi^T{}_\alpha(\delta) \neq F$, and for all $\Psi^\beta \in \Delta$ we have $\Psi^T{}_\beta(\delta) = F$. This contradicts $\Gamma\ {}_M\!\models_F \Delta$, therefore, $\Gamma \not\models_{TF} \Delta$.

Thus, the assumption that $|-\Gamma-|\Delta$ is not derivable is incorrect, which proves the theorem.

The completeness theorem for the calculi $MC^{Q_{=}, T}$ and $MC^{Q_{=}, F}$ can be proved in the similar manner.

6. Conclusion

The work investigates program-oriented logical formalisms of the modal type – pure first-order modal logics of partial non-monotonic quasiary predicates. Variants of such logics with strong equality predicates and weak equality predicates are proposed. The semantic models and languages of these logics are described, with a focus on properties related to equality predicates, specifically the characteristics of the substitution of equals. A number of logical consequence relations for sets of formulas specified with states are defined, and their main properties are outlined. Based on this semantic foundation, the corresponding sequent type calculi for the studied logics are proposed. The varieties of these calculi for different logical consequence relations are described, along with the basic sequent forms and the conditions for the closedness of sequents. The construction of derivations (sequent trees) in the proposed calculi is explained, and the soundness and completeness theorems for the calculi are proved.

References

[1] S. Abramsky, D.M. Gabbay and T.S.E. Maibaum (eds), Handbook of Logic in Computer Science, Vol. 1-5, Oxford University Press, 1993–2000.

[2] D. Bjorner, M.C. Henson (eds), Logics of Specification Languages, EATCS Series, Monograph in Theoretical Computer Science, Heidelberg: Springer, 2008.

[3] P. Blackburn, J. van Benthem and F. Wolter (eds), Handbook of Modal Logics, Vol. 3, Studies in Logic and Practical Reasoning, Elsevier, 2006.

[4] M. Huth and M. Ryan, Logic in Computer Science, Second Edition, Cambridge University Press, 2004.

[5] V. Goranko, Temporal Logics, Cambridge University Press.

[6] M. Fisher, An Introduction to Practical Formal Methods Using Temporal Logic, Somerset, NJ: Wiley, 2011.

[7] F. Kröger, S. Merz, Temporal Logic and State Systems, Berlin-Heidelberg: Springer-Verlag, 2008.

[8] M. Nikitchenko, O. Shkilniak, and S. Shkilniak, Pure First-Order Logics of Quasiary Predicates, No 2–3 of Problems in Programming, Kyiv, 2016, pp. 73–86 (in Ukrainian).
[9] M. Fitting, R.L. Mendelsohn, First-Order Modal Logic, 2nd edition, Springer, 2023.
[10] S. Shkilniak, First-Order Composition Nominative Logics with Predicates of Weak Equality and of Strong Equality, No 3 of Problems in Programming, Kyiv, 2019, pp. 28–44 (in Ukrainian).
[11] O. Shkilniak, Transitional Modal Logics of Non-monotone Partial Predicates, Bulletin of Taras Shevchenko National University of Kyiv, Series: Physics & Mathematics. 3, 2015, pp. 141–147 (in Ukrainian).
[12] O. Shkilniak, Transitional Modal Logics of Non-monotone Quasiary Predicates, Computer Mathematics, Issue 2, 2014, pp. 99–110 (in Ukrainian).
[13] O. Shkilniak, S. Shkilniak, First-Order Sequent Calculi of Logics of Quasiary Predicates with Extended Renominations and Equality, UkrPROG'2022, CEUR Workshop Proceedings (CEUR-WS.org), 2023, pp. 3–18.
[14] Mykola Nikitchenko, Oksana Shkilniak, Stepan Shkilniak. Sequent Calculi of First-order Logics of Partial Predicates with Extended Renominations and Composition of Predicate Complement, UkrPROG'2020, CEUR Workshop Proceedings (CEUR-WS.org), 2020, pp.182–197.
[15] S. C. Kleene, Mathematical Logic, Dower Publications, 2013.