Proc. 23rd Int. Workshop on Description Logics (DL2010), CEUR-WS 573, Waterloo, Canada, 2010.

Decidability of Description Logics with Transitive Closure of Roles in Concept and Role Inclusion Axioms

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Abstract. This paper investigates Description Logics which allow transitive closure of roles to occur not only in concept inclusion axioms but also in role inclusion axioms. First, we propose a decision procedure for the description logic $\mathcal{SHIO}_+$, which is obtained from $\mathcal{SHIO}$ by adding transitive closure of roles. Next, we show that $\mathcal{SHIO}_+$ has the *finite model property* by providing a upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts with respect to sets of concept and role inclusion axioms. Additionally, we prove that if we add number restrictions to $\mathcal{SHI}_+$ then the satisfiability problem is undecidable.

Introduction

The ontology language OWL-DL is widely used to formalize resources on the Semantic Web. This language is mainly based on the description logic $\mathcal{SHOIN}$ which is known to be decidable [1]. Although $\mathcal{SHOIN}$ is expressive and provides *transitive roles* to model transitivity of relations, we can find several applications in which *the transitive closure of roles*, that is more expressive than transitive roles, is necessary. An example in [2] describes two categories of devices as follows: (1) Devices have as their direct part a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$, (2) Devices have at some level of decomposition a battery: $\mathsf{Device} \sqcap \exists \mathsf{hasPart}^+.\mathsf{Battery}$. However, if we now define $\mathsf{hasPart}$ as a *transitive role*, the concept $\mathsf{Device} \sqcap \exists \mathsf{hasPart}.\mathsf{Battery}$ does not represent the devices as described above since it does not allow one to describe these categories of devices as two different sets of devices. We now consider another example in which we need to use the transitive closure of roles in role inclusion axioms.

Example 1. A process accepts a set $\mathbf{S}$ of possible states where $\mathsf{start} \in \mathbf{S}$ is an initial state. The process can reach two disjoint phases $\mathsf{A}, \mathsf{B} \subseteq \mathbf{S}$, considered as two sets of states. To go from a state to another one, the process has to perform an action next. Sometimes, it can execute a jump that implies a sequence of actions next.

To specify the behavior of the process as described, we might need a role name next to express the fact that a state follows another one, a nominal o for start, a role name jump for jumps, concept names A, B for the phases and the following axioms:
(1) $\mathsf{o} \sqsubseteq \neg\mathsf{A} \sqcap \neg\mathsf{B}$, $\mathsf{A} \sqcap \mathsf{B} \sqsubseteq \bot$, $\mathsf{o} \sqsubseteq \forall \mathsf{next}^-.\bot$
(2) $\top \sqsubseteq \exists \mathsf{next}.\top$, $\mathsf{jump} \sqsubseteq \mathsf{next}^+$
Since jumps are arbitrarily executed over $\mathbf{S}$ and they form (non-directed) cycles with next instances, we cannot use concept axioms to express them. In addition, if a transitive

role is used instead of transitive closure, we cannot express the property : an execution of jump implies a sequence of actions next. Therefore, the axiom $\mathsf{jump} \sqsubseteq \mathsf{next}^+$ is necessary.

Such examples motivate the study of Description Logics (DL) that allow the transitive closure of roles to occur in both concept and role inclusion axioms. We introduce in this work a DL that can express the process as described in Ex.1 and propose a decision procedure for concept satisfiability problem in this DL. To the best of our knowledge, the decidability of $\mathcal{SHIO}_+$, which is obtained from $\mathcal{SHIO}$ by adding transitive closure of roles, is unknown. [3] has established a decision procedure for concept satisfiability in $\mathcal{SHI}_+$ by using neighborhoods to build completion graphs. In the literature, many decidability results in DLs can be obtained from their counterparts in modal logics [4], [5]. However, these counterparts do not take into account expressive role inclusion axioms. In particular, [5] has shown the decidability of a very expressive DL, so-called $\mathcal{CATS}$, including $\mathcal{SHIQ}$ with the transitive closure of roles but not allowing it to occur in role inclusion axioms. [5] has pointed out that the complexity of concept subsumption in $\mathcal{CATS}$ is EXPTIME-complete by translating $\mathcal{CATS}$ into the logic Converse PDL in which inference problems are well studied.

Recently, there have been some works (e.g. in [6]) which have attempted to augment the expressiveness of role inclusion axioms. A decidable logic, namely $\mathcal{SROIQ}$, resulting from these efforts allows for new role constructors such as composition, disjointness and negation. In addition, [7] has introduced a DL, so-called $\mathcal{ALCQI}b^+_{reg}$, which can capture $\mathcal{SRIQ}$, and obtained the worst-case complexity (EXPTIME-complete) of the satisfiability problem by using automata-based technique. $\mathcal{ALCQI}b^+_{reg}$ allows for a rich set of operators on roles by which one can simulate role inclusion axioms. However, transitive closures in role inclusion axioms are expressible neither in $\mathcal{SROIQ}$ nor in $\mathcal{ALCQI}b^+_{reg}$.

Tableaux-based algorithms for expressive DLs like $\mathcal{SHIQ}$ [8] and $\mathcal{SROIQ}$ [6] result in efficient implementations. This kind of algorithms relies on two structures, the so-called *tableau* and *completion graph*. Roughly speaking, a tableau for a concept represents a model for the concept and it is possibly infinite. A tableau translates satisfiability of all given concept and role inclusion axioms into the satisfiability of semantic constraints imposed *locally* on each individual of the tableau. This feature of tableaux will be called *local satisfiability property*. The algorithm in [9] for satisfiability in $\mathcal{ALC}_{reg}$ (including the transitive closure of roles and other role operators) introduced a method to deal with loops which can hide unsatisfiable nodes.

To check satisfiability of a concept, tableaux-based algorithms try to build a completion graph whose finiteness is ensured by a technique, the so-called *blocking technique*. It provides a termination condition and guarantees soundness and completeness. The underlying idea of the blocking mechanism is to detect “loops” which are repeated pieces of a completion graph.

The contribution of the present paper consists of (i) proving that $\mathcal{SHIO}_+$ is decidable and it has the *finite model property* by providing a upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts with respect to (w.r.t.) sets of concept and role inclusion axioms, (ii) establishing a reduction of the domino problem to the concept satisfiability problem in the logic $\mathcal{SHIN}_+$ that is obtained from $\mathcal{SHI}_+$ by adding number restrictions on *simple* roles. This reduction shows that $\mathcal{SHIN}_+$ is undecidable.

The Description Logic $\mathcal{SHIO}_+$

The logic $\mathcal{SHIO}_+$ is an extension of $\mathcal{SHIO}$ by allowing for transitive closure of roles. In this section, we present the syntax and semantics of $\mathcal{SHIO}_+$. The definitions reuse notation introduced in [8].

Definition 1. *Let* $\mathbf{R}$ *be a non-empty set of* role names. *We denote* $\mathbf{R}_\mathsf{I} = \{P^- \mid P \in \mathbf{R}\}$, $\mathbf{R}_+ = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. *The set of* $\mathcal{SHIO}_+$*-roles is* $\mathbf{R} \cup \mathbf{R}_\mathsf{I} \cup \mathbf{R}_+$. *A* role inclusion axiom *is of the form* $R \sqsubseteq S$ *for two* $\mathcal{SHIO}_+$*-roles* R *and* S. *A* role hierarchy $\mathcal{R}$ *is a finite set of role inclusion axioms.*
$*$ *An interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty set* $\Delta^\mathcal{I}$ *(*domain*) and a function* $\cdot^\mathcal{I}$ *which maps each role name to a subset of* $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ *such that, for* $R \in \mathbf{R}_\mathsf{I}$, $Q^+ \in \mathbf{R}_+$,
$R^{-\mathcal{I}} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \langle y, x\rangle \in R^\mathcal{I}\}$, *and* $Q^{+\mathcal{I}} = \bigcup_{n>0} (Q^n)^\mathcal{I}$ *with* $(Q^1)^\mathcal{I} = Q^\mathcal{I}$,
$(Q^n)^\mathcal{I} = \{\langle x, y\rangle \in (\Delta^\mathcal{I})^2 \mid \exists z \in \Delta^\mathcal{I}, \langle x, z\rangle \in (Q^{n-1})^\mathcal{I}, \langle z, y\rangle \in Q^\mathcal{I}\}$.
An interpretation $\mathcal{I}$ *satisfies a role hierarchy* $\mathcal{R}$ *if* $R^\mathcal{I} \subseteq S^\mathcal{I}$ *for each* $R \sqsubseteq S \in \mathcal{R}$. *Such an interpretation is called a* model *of* $\mathcal{R}$, *denoted by* $\mathcal{I} \models \mathcal{R}$.
$*$ *Function* Inv *returns the inverse of a role as follows:*

$$\mathsf{Inv}(R) := \begin{cases} R^- & \text{if } R \in \mathbf{R}, \\ S & \text{if } R = S^- \text{ where } S \in \mathbf{R}, \\ (Q^-)^+ & \text{if } R = Q^+ \text{ where } Q \in \mathbf{R}, \\ Q^+ & \text{if } R = (Q^-)^+ \text{ where } Q \in \mathbf{R} \end{cases}$$

$*$ *A relation* $\underline{\circledast}$ *is defined as the transitive-reflexive closure of* $\sqsubseteq$ *on* $\mathcal{R} \cup \{\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_\mathsf{I}\}$. *We denote* $S \equiv R$ *iff* $R \underline{\circledast} S$ *and* $S \underline{\circledast} R$. *We may abuse the notation by saying* $R \underline{\circledast} S \in \mathcal{R}$.

Notice that we introduce into role hierarchies axioms $Q \sqsubseteq Q^+$ which allows us (i) to propagate $(\forall Q^+.A)$ correctly, and (ii) to take into account the fact that $R \sqsubseteq S$ implies $R^+ \sqsubseteq S^+$.

Definition 2. *Let* $\mathbf{C}' = \mathbf{C} \cup \mathbf{C}_o$ *be a non-empty set of* concept names *where* $\mathbf{C}$ *is a set of normal concept names and* $\mathbf{C}_o$ *is a set of* nominals.
$*$ *The set of* $\mathcal{SHIO}_+$*-concepts is inductively defined as the smallest set containing all* C *in* $\mathbf{C}'$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$ *where* C *and* D *are* $\mathcal{SHIO}_+$*-concepts,* R *is an* $\mathcal{SHIO}_+$*-role,* S *is a simple role and* $n \in \mathbb{N}$. *We denote* $\bot$ *for* $\neg\top$.
$*$ *An interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *consists of a non-empty set* $\Delta^\mathcal{I}$ *(*domain*) and a function* $\cdot^\mathcal{I}$ *which maps each concept to a subset of* $\Delta^\mathcal{I}$ *such that* $\mathsf{card}\{o^\mathcal{I}\} = 1$ *for all* $o \in \mathbf{C}_o$ *where* $\mathsf{card}\{\cdot\}$ *is denoted for the cardinality of a set* $\{\cdot\}$,
$\top^\mathcal{I} = \Delta^\mathcal{I}$, $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$, $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \backslash C^\mathcal{I}$,
$(\exists R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I}, \langle x, y\rangle \in R^\mathcal{I} \wedge y \in C^\mathcal{I}\}$,

$(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}},\, \langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$
$*$ *$C \sqsubseteq D$ is called a general concept inclusion (GCI) where C, D are $\mathcal{SHIO}_+$-concepts (possibly complex), and a finite set of GCIs is called a terminology $\mathcal{T}$. An interpretation $\mathcal{I}$ satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $\mathcal{I}$ satisfies a terminology $\mathcal{T}$ if $\mathcal{I}$ satisfies each GCI in $\mathcal{T}$. Such an interpretation is called a* model *of $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$.*
$*$ *A concept C is called* satisfiable *w.r.t. a role hierarchy $\mathcal{R}$ and a terminology $\mathcal{T}$ iff there is some interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a* model *of C w.r.t. $\mathcal{R}$ and $\mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called an $\mathcal{SHIO}_+$* ontology *and said to be consistent if there is a model of $(\mathcal{T}, \mathcal{R})$. A concept D* subsumes *a concept C w.r.t. $\mathcal{R}$ and $\mathcal{T}$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{R})$.*

Notice that a transitive role S can be expressed by using a role axiom $S^+ \sqsubseteq S$. Since negation is allowed in the logic $\mathcal{SHIO}_+$, unsatisfiability and subsumption w.r.t. $(\mathcal{T}, \mathcal{R})$ can be reduced each other: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable. In addition, we can reduce ontology consistency to concept satisfiability w.r.t. an ontology: $(\mathcal{T}, \mathcal{R})$ is consistent if $A \sqcup \neg A$ is satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$ for some concept name A.

For the ease of construction, we assume all concepts to be in *negation normal form* (NNF) i.e. negation occurs only in front of concept names. Any $\mathcal{SHIO}_+$-concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [8]. For a concept C, we denote the nnf of C by $\mathsf{nnf}(C)$ and the nnf of $\neg C$ by $\dot\neg C$.
Let D be an $\mathcal{SHIO}_+$-concept in NNF. We define $\mathsf{sub}(D)$ to be the smallest set that contains all sub-concepts of D including D. For an ontology $(\mathcal{T}, \mathcal{R})$, we define the set of all sub-concepts $\mathsf{sub}(\mathcal{T}, \mathcal{R})$ as follows:

$$\mathsf{sub}(\mathcal{T}, \mathcal{R}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(\mathsf{nnf}(\neg C \sqcup D), \mathcal{R})$$

$$\mathsf{sub}(E, \mathcal{R}) := \mathsf{sub}(E) \cup \{\dot\neg C \mid \neg C \in \mathsf{sub}(E)\} \cup \{\forall S.C \mid (\forall R.C \in \mathsf{sub}(E), S \sqsubseteq^{*} R) \vee (\dot\neg\forall R.C \in \mathsf{sub}(E), S \sqsubseteq^{*} R) \text{ and } S \text{ occurs in } \mathcal{T} \text{ or } \mathcal{R}\}$$

For the sake of simplicity, for each concept D w.r.t. $(\mathcal{T}, \mathcal{R})$ we denote $\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)$ for $\mathsf{sub}(\mathcal{T}, \mathcal{R}) \cup \mathsf{sub}(D)$, and $\mathbf{R}_{(\mathcal{T}, \mathcal{R}, D)}$ for the set of roles R occurring in $\mathcal{T}, \mathcal{R}, D$ with the inverse and transitive closure of each R. If it is clear from the context we will use $\mathbf{R}$ instead of $\mathbf{R}_{(\mathcal{T}, \mathcal{R}, D)}$.

A decision procedure for $\mathcal{SHIO}_+$

In our approach, we define a sub-structure of graphs, called *neighborhood*, which consists of a node together with its neighbors. Such a neighborhood captures all semantic constraints imposed by the logic constructors of $\mathcal{SHIO}$. A graph obtained by "tiling" neighborhoods together allows us to represent in some way a model for a concept in $\mathcal{SHIO}_+$. In fact, we embed in this graph another structure, called *cyclic path*, to express transitive closure of roles. Since all expansion rules for $\mathcal{SHIO}$ can be translated into construction of neighborhoods, the algorithm presented in this paper focuses on defining cyclic paths over such a graph. In this way, the non-determinism resulting from satisfying the transitive closure of roles can be translated into the search in a space of all possible graphs obtained from tiling neighborhoods.

Neighborhood for $\mathcal{SHIO}_+$

Tableau-based algorithms, as presented in [8], use expansion rules representing tableau properties to build a completion graph. Applying expansion rules makes all nodes of a completion graph satisfy semantic constraints imposed by concept definitions in the label associated with each node. This means that *local* satisfiability in such completion graphs is sufficient to ensure *global* satisfiability. The notion of *neighborhood* introduced in Def. 3 expresses exactly the expansion rules for $\mathcal{SHIO}$, consequently, guarantees local satisfiability. Therefore, a completion graph built by a tableau-based algorithm can be considered as set of neighborhoods which are tiled together. In other terms, building a completion tree by applying expansion rules is equivalent to the search of a tiling of neighborhoods.

Definition 3 (Neighborhood). *Let D be an $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. We denote $\mathbf{R}$ for the set of roles R occurring in D and $\mathcal{T},\mathcal{R}$ with the inverse of each R. A* neighborhood*, denoted (v_B, N_B, l), for D w.r.t. $(\mathcal{T},\mathcal{R})$ is formed from a* core *node v_B, a set of* neighbor *nodes N_B, edges $\langle v_B, v\rangle$ with $v \in N_B$ and a labelling function l such that $l(u) \in 2^{\mathsf{sub}(\mathcal{T},\mathcal{R},D)}$ with $u \in \{v_B\} \cup N_B$ and $l\langle v_B, v\rangle \in 2^{\mathbf{R}}$ with $v \in N_B$.*

1. *A node $v \in \{v_B\} \cup N_B$ is* nominal *if there is $o \in \mathbf{C}_o$ such that $o \in l(v)$. Otherwise, v is a* non-nominal *node;*
2. *A node $v \in \{v_B\} \cup N_B$ is* valid *w.r.t. D and $(\mathcal{T},\mathcal{R})$ iff*
   - *(a) If $C \sqsubseteq D \in \mathcal{T}$ then $\mathsf{nnf}(\neg C \sqcup D) \in l(v)$, and*
   - *(b) $\{A, \neg A\} \not\subseteq l(v)$ with each concept name A, and*
   - *(c) If $C_1 \sqcap C_2 \in l(v)$ then $\{C_1, C_2\} \subseteq l(v)$, and*
   - *(d) If $C_1 \sqcup C_2 \in l(v)$ then $\{C_1, C_2\} \cap l(v) \neq \emptyset$.*
3. *A neighborhood $B = (v_B, N_B, l)$ is* valid *iff all nodes $\{v_B\} \cup N_B$ are valid and the following conditions are satisfied:*
   - *(a) If $\exists R.C \in l(v_B)$ then there is a neighbor $v \in N_B$ such that $C \in l_B(v)$ and $R \in l\langle v_B, v\rangle$;*
   - *(b) For each $v \in N_B$, if $R \in l\langle v_B, v\rangle$ and $R \underline{\circledast} S$ then $S \in l\langle v_B, v\rangle$;*
   - *(c) For each $v \in N_B$, if $R \in l\langle v_B, v\rangle$ (resp. $R \in \mathsf{Inv}(l\langle v_B, v\rangle)$) and $\forall R.C \in l(v_B)$ (resp. $\forall R.C \in l(v)$) then $C \in l(v)$ (resp. $C \in l(v_B)$);*
   - *(d) For each $v \in N_B$, if $Q^+ \in l\langle v_B, v\rangle$ (resp. $Q^+ \in \mathsf{Inv}(l\langle v_B, v\rangle)$), $Q^+ \underline{\circledast} R \in \mathcal{R}$ and $\forall R.D \in l(v_B)$ (resp. $\forall \mathsf{Inv}(R).D \in l(v)$) then $\forall Q^+.D \in l(v)$ (resp. $\forall \mathsf{Inv}(Q^+).D \in l(v_B)$);*
   - *(e) For each $o \in \mathbf{C}_o$, if $o \in l(u) \cap l(v)$ with $\{u, v\} \subseteq \{v_B\} \cup N_B$ then $l(u) = l(v)$;*
   - *(f) There is at most one node $v \in N_B$ such that $l(v) = \mathscr{C}$ and $l\langle v_B, v\rangle = \mathscr{R}$ for each $\mathscr{C} \in 2^{\mathsf{sub}(\mathcal{T},\mathcal{R},D)}$, $\mathscr{R} \in 2^{\mathbf{R}}$.*

We denote $\mathbb{B}_{(\mathcal{T},\mathcal{R},D)}$ for a set of all valid neighborhoods for D w.r.t. $(\mathcal{T},\mathcal{R})$. When it is clear from the context we will use $\mathbb{B}$ instead of $\mathbb{B}_{(\mathcal{T},\mathcal{R},D)}$.

The condition 3f in Def. 3 ensures that any neighborhood has a finite number of neighbors. As mentioned, a valid neighborhood as presented in Def. 3 satisfies all concept definitions in the label associated with the core node. For this reason, neighborhoods can be still used to tile a completion tree for $\mathcal{SHIO}_+$ without taking care of expansion rules for $\mathcal{SHIO}$. In other terms, the neighborhood notion expresses the local satisfiability property in a sufficient way for being used in a global context.

Lemma 1. *Let D be an $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $(v_B, N_B, l), (v_{B'}, N_{B'}, l)$ be two valid neighborhoods with $l(v_B) = l(v_{B'})$. If there is $v \in N_B$ such that there does not exist any $v' \in N_{B'}$ satisfying $l(v') = l(v)$ and $l\langle v_B, v\rangle = l\langle v_{B'}, v'\rangle$ then the neighborhood $(v_{B'}, N_{B'} \cup \{u\}, l)$ is valid where $l(u) = l(v)$ and $l\langle v_{B'}, u\rangle = l\langle v_B, v\rangle$.*

This lemma holds due to the facts that (i) a valid neighbor in a valid neighborhood B is also a valid neighbor in another valid neighborhood B' if the labels of two core nodes of B and B' are identical, (ii) since $\mathcal{SHIO}_+$ does not allow for number restrictions hence Def. 3 has no restriction on the number of neighbors of a core node.

Completion Tree with Cyclic Paths

As discussed in works related to tableau-based technique, the blocking technique fails in treating DLs with the transitive closure of roles. It works correctly only if the satisfiability of a node in completion tree can be decided from its neighbors and itself i.e. *local* satisfiability must be sufficient for such completion trees. However, the presence of the transitive closure of roles makes satisfiability of a node depend on further nodes which can be arbitrarily far from it. The problem becomes harder when we add the transitive closure of roles to role hierarchies. For instance, if $P \sqsubseteq Q^+, Q \sqsubseteq S^+$ are axioms in a role hierarchy then each Q-edge generated for satisfying Q^+ may lead to generate an arbitrary number of S-edges for satisfying S^+.

More precisely, satisfying the transitive closure P^+ in an edge $\langle x, y\rangle$ (i.e. $P^+ \in L\langle x, y\rangle$) is related to a set of nodes on a path rather than a node with its neighbors i.e. it imposes a semantic constraint on a set of nodes $x, x_1, \cdots, x_n, y$ such that they are connected together by P-edges. In general, satisfying the transitive closure is quite nondeterministic since the semantic constraint can lead to be applied to an *arbitrary* number of nodes. In addition, the presence of transitive closure of roles in a role hierarchy makes this difficulty worse. For instance, if $P \sqsubseteq Q^+, Q \sqsubseteq S^+$ are axioms in a role hierarchy then each Q-edge generated for satisfying Q^+ may lead to generate an arbitrary number of S-edges for satisfying S^+.

The most common way for dealing with a new logic constructor is to add a new expansion rule for satisfying the semantic constraint imposed by the new constructor. Such an expansion rule for the transitive closure of roles must: (i) find or create a set of P-edges forming a path for each occurrence of P^+ in the label of edges; (ii) deal with non-deterministic behaviours of the expansion rule resulting from the semantics of the transitive closure of roles; and (iii) enable to control the expansion of completion trees by a new blocking technique which has to take into account the fact that satisfying the transitive closure of a role may add an arbitrary number of new transitive closures to be satisfied. To avoid these difficulties, our approach does not aim to directly extend the construction of completion trees by using a new expansion rule, but to translate this construction into selecting a "good" completion tree, namely *completion tree with cyclic*

paths, from a finite set of trees without taking into account the semantic constraint imposed by the transitive closure of roles. The process of selecting a "good" completion tree is guided by finding in a completion tree (which is well built in advance) a *cyclic path* for each occurrence of the transitive closure of a role.
Summing up, a completion tree with cyclic paths will be built in two stages. The first one which yields a tree consists of tiling valid neighborhoods together such that two neighborhoods are tiled if they have *compatible* neighbors. The second stage deals with the transitive closure of roles by defining cyclic paths over the tree obtained from the first stage. Both of them are presented in Def. 4.

Definition 4 (Completion Tree with Cyclic Paths). *Let D be a $\mathcal{SHIO}_+$ concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $\mathbb{B}$ be the set of all valid neighborhoods for D w.r.t. $(\mathcal{T},\mathcal{R})$. A tree $\mathbf{T} = (V, E, L)$ for D w.r.t. $(\mathcal{T},\mathcal{R})$ is defined from $\mathbb{B}$ as follows.*

1. *If there is a valid neighborhood $(v_0, N_0, l) \in \mathbb{B}$ with $D \in l(v_0)$ then a root node x_0 and successors x of x_0 are added to V such that $L(x_0) = l(v_0)$, and $L(x) = l(v)$, $L\langle x_0, x\rangle = l\langle v_0, v\rangle$ for each $v \in N_0$.*
2. *For each node $x \in V$ with its predecessor x',*
   (a) *If there is an ancestor y of x such that $L(y) = L(x)$ then x is* blocked *by y. In this case, x is a leaf node;*
   (b) *Otherwise, if we find a valid neighborhood (v_B, N_B, l) from $\mathbb{B}$ such that*
      i. *$l(v_B) = L(x), l(v) = L(x')$, $\mathsf{Inv}(\ l\langle v_B, v\rangle\ ) = L\langle x', x\rangle$ for some $v \in N_B$, and*
      ii. *if there is some nominal $o \in \mathbf{C}_o$ such that $o \in l(u) \cap L(w)$ with $u \in N_B \setminus \{v\}$, $w \in V$ then $l(u) = L(w)$*

      then we add a successor y of x for each $u \in N_B \setminus \{v\}$ such that $L(y) = l(u)$ and $L(\langle x, y\rangle) = l(\langle v_B, u\rangle)$.

We say a node x is an R-successor of $x' \in V$ if $R \in L\langle x', x\rangle$. A node x is called an R-neighbor of x' if x is an R-successor of x' or x' is a $\mathsf{Inv}(R)$-successor of x. In addition, a node x is called an R-block of x' if x blocks an R-successor of x' or x' blocks a $\mathsf{Inv}(R)$-successor of x.
$\mathbf{T} = (V, E, L)$ is called a completion tree with cyclic paths *if for each $\langle u, v\rangle \in E$ such that $Q^+ \in L\langle u, v\rangle$ and $Q \notin L\langle u, v\rangle$ there exists a* cyclic path *$\varphi = \langle x_0, \cdots, x_n\rangle$ which is formed from nodes $v_i \in V$ and satisfies the following conditions:*

- *$x_0 = u$ and x_i is not blocked for all $i \in \{0, \cdots, n\}$;*
- *There do not exist $i, j \in \{1, \cdots, n-1\}$ with $j > i$ such that $L(x_i) = L(x_j)$;*
- *$L(x_n) = L(v)$ and x_i is a Q-neighbor or Q-block of x_{i+1} for all $0 \le i \le n-1$.*

In this case, φ is called a cyclic path and denoted by $\varphi_{\langle u,v\rangle}$.

At this point we have gathered all necessary elements to introduce a decision procedure for the concept satisfiability in $\mathcal{SHIO}_+$. However, in order to provide a upper bound on the size of models of satisfiable $\mathcal{SHIO}_+$-concepts we need an extra structure, namely *reduced tableau*.

Definition 5 (Reduced Tableau). *Let $\mathbf{T} = (V, E, L)$ be a completion tree with cyclic paths for a $\mathcal{SHIO}_+$-concept D w.r.t. $(\mathcal{T}, \mathcal{R})$. An equivalence relation $\sim$ over V is defined as follows: $x \sim y$ iff $L(x) = L(y)$.*
Let $V/\sim := \{[x] \mid x \in V\}$ be the set of all equivalence classes of V by $\sim$. A graph $G = (V/\sim, E', L)$ is called reduced tableau *for D w.r.t. $(\mathcal{T}, \mathcal{R})$ if:*

- $L([x]) = L(x')$ *for any* $x' \in [x]$*;*
- $\langle [x], [y] \rangle \in E'$ *iff there are* $x' \in [x], y' \in [y]$ *such that* $\langle x', y' \rangle \in E$*;*
- $L(\langle [x], [y] \rangle) = \bigcup_{x' \in [x], y' \in [y], \langle x', y' \rangle \in E} L(\langle x', y' \rangle) \cup \bigcup_{x' \in [x], y' \in [y], \langle y', x' \rangle \in E} \mathsf{Inv}(L\langle y', x' \rangle)$
  where $\mathsf{Inv}(L\langle x, y \rangle) = \{\mathsf{Inv}(R) \mid R \in L\langle y, x \rangle\}$

A reduced tableau as defined in Def. 5 identifies nodes whose labels are the same. The following lemma states an important property of reduced tableaux.

Lemma 2. *Let D be a $\mathcal{SHIO}_+$-concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. Let $G = (V/\sim, E', L)$ be a reduced tableau for D w.r.t. $(\mathcal{T}, \mathcal{R})$. We define $\Delta^{\mathcal{I}} = V/\sim$ and a function $\cdot^{\mathcal{I}}$ that maps:*

- *each concept name A occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $A^{\mathcal{I}} \subseteq V/\sim$ such that* $A^{\mathcal{I}} = \{[x] \mid A \in L([x])\}$*;*
- *each role name R occurring in $D, \mathcal{T}$ and $\mathcal{R}$ to $R^{\mathcal{I}} \subseteq (V/\sim)^2$ such that* $R^{\mathcal{I}} = \{\langle [x], [y] \rangle \mid R \in L\langle [x], [y] \rangle\} \cup \{\langle [y], [x] \rangle \mid \mathsf{Inv}(R) \in L\langle [x], [y] \rangle\}$

If D has a reduced tableau G then $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a model of D w.r.t. $(\mathcal{T}, \mathcal{R})$.

Lem. 2 affirms that a reduced tableau of a concept D can represent a model of D. The construction of reduced tableaux as presented in Def. 5 preserves not only the validity of neighborhoods but also cyclic paths. The following result is an immediate consequence of Lem. 2.

Proposition 1. *Let D be a $\mathcal{SHIO}_+$-concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. If there is a completion tree with cyclic paths $\mathbf{T}$ for D w.r.t. $(\mathcal{T}, \mathcal{R})$ then D has a finite model whose size is bounded by an exponential function in the size of $D, \mathcal{T}, \mathcal{R}$.*

Indeed, by the construction of the reduced tableau $G = (V/\sim, E', L)$, the number of nodes of G is bounded by 2^K where $K = \mathsf{card}\{\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)\}$ and K is a polynomial function in the size of $D, \mathcal{T}$ and $\mathcal{R}$.

Lemma 3. *Let D be a $\mathcal{SHIO}_+$-concept with a terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$. If D has a model w.r.t. $(\mathcal{T}, \mathcal{R})$ then there exists a completion tree with cyclic paths.*

A proof of Lem. 3 can be performed in three steps. First, we define directly valid neighborhoods from individuals of a model. Next, a completion tree can be built by tiling valid neighborhoods with help of role relationships between individuals of the model. Finally, cyclic paths are embedded into the obtained tree by devising paths from finite cycles for the transitive closure of roles in the model. Lem. 1 makes possible adding a new node to a given neighborhood as neighbor if the new node is a neighbor of a node whose label equals to that of the core node of the neighborhood.

From the construction of completion trees with cyclic paths according to Def. 4 and Lem. 2 and 3, we can devise immediately Algorithm 1 for the concept satisfiability in $\mathcal{SHIO}_+$.

Algorithm 1: Decision procedure for concept satisfiability in $\mathcal{SHIO}_+$

Input : Concept D, terminology $\mathcal{T}$ and role hierarchy $\mathcal{R}$
Output: IsSatisfiable(D)

foreach *Tree* $\mathbf{T} = (V, E, L)$ *obtained from tiling valid neighborhoods* **do**
  if *For each* $\langle x, y\rangle \in E$ *with* $Q^+ \in L\langle x, y\rangle, Q \notin L\langle x, y\rangle$, $\mathbf{T}$ *has a* $\varphi_{\langle x,y\rangle}$ **then**
    return `true`;

return `false`;

Lemma 4 (Termination). *Alg. 1 for $\mathcal{SHIO}_+$ terminates and the size of completion trees is bounded by a double exponential function in the size of inputs.*

Termination of Alg. 1 is a consequence of the following facts: (i) the number of valid neighborhoods is bounded, (ii) the size of completion trees which are tiled from valid neighborhoods is bounded by $(2^{m\times n})^{2^{n\times(m+1)}}$ where $m = \mathsf{card}\{\mathsf{sub}(\mathcal{T}, \mathcal{R}, D)\}, n = \mathsf{card}\{\mathbf{R}\}$.
Alg. 1 is highly complex since it is not a goal-directed procedure. Such an exhaustive behavior is very different from that of tableau-based algorithms in which the construction of a completion tree is inherited from step to step. In Alg. 1, when a tree obtained from tiling neighborhoods cannot satisfy an occurrence of the transitive closure of a role (after satisfying others), the construction of tree has to restart. The following theorem is a direct consequence of Lem.3 and 4.

Theorem 1. *Alg. 1 is a decision procedure for the satisfiability of $\mathcal{SHIO}_+$-concepts w.r.t. a terminology and role hierarchy, and it runs in deterministic 3-*EXPTIME *and nondeterministic 2-*EXPTIME*.*

Thm. 1 is a consequence of the following facts: (i) the size of completion trees is bounded by a double exponential function in the size of inputs , and (ii) the number of of completion trees is bounded by a triple exponential function in the size of inputs.

Remark 1. From the construction of reduced tableaux in Def. 5, we can devise an algorithm for deciding the satisfiability in $\mathcal{SHIO}_+$ which runs in nondeterministic EXPTIME. In fact, such an algorithm can check the validity of neighborhoods and cycles for transitive closures in a graph whose size is bounded by an exponential function in the size of inputs.

Adding number restrictions to $\mathcal{SHI}_+$

The logic $\mathcal{SHIN}_+$ is obtained from $\mathcal{SHI}_+$ ($\mathcal{SHIO}_+$ without nominals) by allowing, additionally, for number restrictions as follows:

Definition 6. *Let* $\mathbf{R}, \mathbf{C}$ *be sets of role and concept names. The set of $\mathcal{SHIN}_+$-roles, role hierarchy $\mathcal{R}$ and model $\mathcal{I}$ of $\mathcal{R}$ are defined similarly to those in Def. 1.*
$*$ *A role R is called* simple *w.r.t. $\mathcal{R}$ iff $(Q^+ \sqsubseteq\!\!\!\!\sqsubseteq R) \notin \mathcal{R}$ for any $Q^+ \in \mathbf{R}_+$.*
$*$ *The set of $\mathcal{SHIN}_+$-concepts is inductively defined as the smallest set containing all*

$C \in \mathbf{C}$, $\top$, $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n\, S)$ and $(\geq n\, S)$ where C and D are $\mathcal{SHIN}_+$-concepts, R is a $\mathcal{SHIN}_+$-role and S is a simple role. We denote $\bot$ for $\neg\top$.
*$*$ An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non empty set $\Delta^{\mathcal{I}}$ (*domain*) and a function $\cdot^{\mathcal{I}}$ which maps each concept name to a subset of $\Delta^{\mathcal{I}}$. In addition, the function $\cdot^{\mathcal{I}}$ satisfies the conditions for the logic constructors in $\mathcal{SHI}_+$ (as introduced in Def. 2 without nominal), and*
$(\geq n\, R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in \Delta^{\mathcal{I}} \mid \langle x, y\rangle \in R^{\mathcal{I}}\} \geq n\}$,
$(\leq n\, R)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \mathsf{card}\{y \in \Delta^{\mathcal{I}} \mid \langle x, y\rangle \in R^{\mathcal{I}})\} \leq n\}$
$*$ Satisfiability of a $\mathcal{SHIN}_+$-concept C w.r.t. a role hierarchy $\mathcal{R}$ and a terminology $\mathcal{T}$ is defined similarly to that in Def. 2.

A definition for neighborhoods in $\mathcal{SHIN}_+$ would be provided if we adopt that there may be two neighborhoods such that the labels of their core nodes are identical but they cannot be merged together i.e. a property being similar to Lem. 1 no longer holds for $\mathcal{SHIN}_+$. In such a situation, the local information related to the labels of the ending nodes of a path would be not sufficient to form a cycle. This prevents us from embedding cyclic paths to a completion tree in guaranteeing the soundness and completeness. Note that for the logics $\mathcal{SHI}_+$ and $\mathcal{SHIO}_+$ we can transform a reduced tableau to a tableau (e.g. as described in [8]) such that if any two nodes x, y having the same label then there is an isomorphism between the two neighborhoods (x, N_x, l) and (y, N_y, l). This means that if we know the label of a node in such a tableau it is possible to determine all nodes which are arbitrarily far from this node. This property does not hold for $\mathcal{SHIN}_+$ tableaux.

In the sequel, we show that the difficulty mentioned is insurmountable i.e. the concept satisfiability problem in $\mathcal{SHIN}_+$ is undecidable. The undecidability proof uses a reduction of the domino problem [10]. The following definition, which is taken from [8], reformulates the problem in a more precise way.

Definition 7. *A domino system $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ consists of a non-empty set of domino types $\mathcal{D} = \{D_1, \cdots, D_l\}$ and of sets of horizontally and vertically matching pairs $\mathcal{H} \subseteq \mathcal{D} \times \mathcal{D}$ and $\mathcal{V} \subseteq \mathcal{D} \times \mathcal{D}$. The problem is to determine if, for a given $\mathbf{D}$, there exists a tiling of an $\mathbb{N} \times \mathbb{N}$ grid such that each point of the grid is covered with a domino type in $\mathcal{D}$ and all horizontally and vertically adjacent pairs of domino types are in $\mathcal{H}$ and $\mathcal{V}$ respectively, i.e., a mapping $t : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ such that for all $m, n \in \mathbb{N}$, $\langle t(m, n), t(m + 1, n)\rangle \in \mathcal{H}$ and $\langle t(m, n), t(m, n + 1)\rangle \in \mathcal{V}$.*

The reduction of the domino problem to the satisfiability of $\mathcal{SHIN}_+$-concepts will be carried out by (i) constructing a concept, namely A, and two sets of concept and role inclusion axioms, namely $\mathcal{T}_D$ and $\mathcal{R}_D$, and (ii) showing that the domino problem is equivalent to the satisfiability of A w.r.t. $\mathcal{T}_D$ and $\mathcal{R}_D$. Axioms in Def. 8 specify a grid that represents such a domino system. Globally, given a domino set $\mathcal{D} = \{D_1, \cdots, D_l\}$, we need axioms that impose that each point of the plane is covered by exactly one $D_i^{\mathcal{I}}$ (axiom 8 in Def. 8) and ensure that each D_i is compatibly placed in the horizontal and vertical lines (axiom 9). Locally, the key idea is to use $\mathcal{SHIN}_+$ axioms 3, 5, 10, 11, 12 and 13 in Def. 8 for describing the grid as illustrated in Fig. 1.

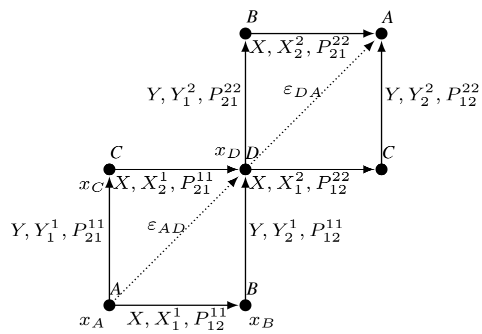

Fig. 1. How each square can be formed from a diagonal represented by an ε

Definition 8. *Let* $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$ *be a domino system with* $\mathcal{D} = \{D_1, \cdots, D_l\}$. *Let* N_C *and* N_R *be sets of concept and role names such that* $N_C = \{A, B, C, D\} \cup \mathcal{D}$, $N_R = \{X_j^i \mid i, j \in \{1,2\}\} \cup \{X, Y\} \cup \{P_{rs}^{ij} \mid i, j, r, s \in \{1,2\}, r \neq s\} \cup \{\varepsilon_{AD}, \varepsilon_{DA}, \varepsilon_{BC}, \varepsilon_{CB}\}$.

Role hierarchy:

1. $X_r^i \sqsubseteq P_{rs}^{ij}, Y_s^j \sqsubseteq P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
2. $X_r^i \sqsubseteq X, Y_r^i \sqsubseteq Y$ *for all* $i, r \in \{1,2\}$,
3. $\varepsilon_{AD} \sqsubseteq P_{12}^{11+}, \varepsilon_{AD} \sqsubseteq P_{21}^{11+}, \varepsilon_{DA} \sqsubseteq P_{12}^{22+}, \varepsilon_{DA} \sqsubseteq P_{21}^{22+}$,
4. $\varepsilon_{BC} \sqsubseteq P_{21}^{21+}, \varepsilon_{BC} \sqsubseteq P_{12}^{21+}, \varepsilon_{CB} \sqsubseteq P_{21}^{12+}, \varepsilon_{CB} \sqsubseteq P_{12}^{12+}$,

Concept inclusion axioms:

5. $\top \sqsubseteq \leq 1 P_{rs}^{ij}$ *for all* $i, j, r, s \in \{1,2\}, r \neq s$,
6. $\top \sqsubseteq \ \leq 1X, \top \sqsubseteq \ \leq 1Y$,
7. $\top \sqsubseteq \leq 1\varepsilon_{AD}, \top \sqsubseteq \leq 1\varepsilon_{DA}, \top \sqsubseteq \leq 1\varepsilon_{BC}, \top \sqsubseteq \leq 1\varepsilon_{CB}$,
8. $\top \sqsubseteq \bigsqcup_{1 \leq i \leq l} (D_i \sqcap (\bigsqcap_{1 \leq j \leq l, j \neq i} \neg D_j))$,
9. $D_i \sqsubseteq \forall X. \bigsqcup_{(D_i, D_j) \in \mathcal{H}} D_j \sqcap \forall Y. \bigsqcup_{(D_i, D_k) \in \mathcal{V}} D_k$ *for each* $D_i \in \mathcal{D}$,
10. $A \sqsubseteq \neg B \sqcap \neg C \sqcap \neg D \sqcap \exists X_1^1.B \sqcap \exists Y_1^1.C \sqcap \exists \varepsilon_{AD}.D \sqcap \forall P_{12}^{22}.\bot \sqcap \forall P_{21}^{22}.\bot$,
11. $B \sqsubseteq \neg A \sqcap \neg C \sqcap \neg D \sqcap \exists X_2^2.A \sqcap \exists Y_2^1.D \sqcap \exists \varepsilon_{BC}.C \sqcap \forall P_{21}^{12}.\bot \sqcap \forall P_{12}^{12}.\bot$,
12. $C \sqsubseteq \neg A \sqcap \neg B \sqcap \neg D \sqcap \exists X_2^1.D \sqcap \exists Y_2^2.A \sqcap \exists \varepsilon_{CB}.B \sqcap \forall P_{21}^{21}.\bot \sqcap \forall P_{12}^{21}.\bot$,
13. $D \sqsubseteq \neg A \sqcap \neg B \sqcap \neg C \sqcap \exists X_1^2.C \sqcap \exists Y_1^2.B \sqcap \exists \varepsilon_{DA}.A \sqcap \forall P_{12}^{11}.\bot \sqcap \forall P_{21}^{11}.\bot$.

Theorem 2 (Undecidability of $\mathcal{SHIN}_+$). *The concept* A *is satisfiable w.r.t. concept and role inclusion axioms in Def. 8 iff there is a compatible tiling* t *of the first quadrant* $\mathbb{N} \times \mathbb{N}$ *for a given domino system* $\mathbf{D} = (\mathcal{D}, \mathcal{H}, \mathcal{V})$.

Complete proofs of the obtained results in this work can be found in [11].

Conclusion and Discussion

We have presented in this paper a decision procedure for the logic $\mathcal{SHIO}_+$ and shown the finite model property for this logic. To do this we have introduced the neighborhood notion which is an abstraction of the local satisfiability property of tableaux enables us to encapsulate all semantic constraints imposed by the logic constructors in $\mathcal{SHIO}$, and thus to deal with transitive closure of roles independently from the other constructors. According to Rem. 1, we can devise a decision procedure for deciding the concept satisfiability in $\mathcal{SHIO}_+$ so that it runs in nondeterministic exponential time (NEXPTIME). This result with the proof of Lem. 4 implies that this procedure runs in a deterministic doubly exponential. However, the worst-case complexity of the problem remains an open question.

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