ACTIVATING STUDENTS IN INTRODUCTORY MATHEMATICS TUTORIALS (EuroPLOP 2008)

Pattern for introductory mathematics tutorials following a constructivist approach

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Abstract. *This paper describes a pedagogical pattern for mathematics tutorials with two different solutions depending on the underlying philosophy of learning or teaching. The aim of tutorials for introductory mathematics courses is for students to practice and apply what they have learned during the lecture. Adopting the traditional approach tutors show how to solve the given problems. Students observe the tutor solving problems on the chalkboard, copy the solution, and usually assume they are able to solve similar problems by themselves next time. Following a constructivist philosophy of learning we present a 'parallel' – different – solution to the same problem where learners do actively mathematics while learning how to solve mathematical problems.*

Preliminary remarks

This pattern describes design decisions which 'designers' of mathematics courses have to take. There is a long tradition of teaching mathematics to freshman at universities. In the pedagogical discussion after the TIMSS and PISA studies researchers in mathematics education recommend teaching and learning scenarios in schools where learners actively **do** mathematics, solve complex problems, reason and communicate mathematically, and make connections between different fields and topics. This concept differs widely from traditional tutorials at the universities.

In this paper we state the context and problem and then, present two different solutions based on different philosophies of teaching and learning. One solution follows the traditional approach, the other is based on the constructivist philosophy of learners actively building their own knowledge.

In presenting the patterns, we merge the traditional pattern format and a formalism developed by Wippermann (2008) especially for e-learning scenarios. In our experience, this formalism communicates the main ideas of an educational setting better than patterns designed for technological areas.

Context

Tutorials for introductory mathematics usually support mathematics lectures. In Germany there are often several hundred students in 'Introductory Math' courses at universities. The presentation of the lecture is usually given by a professor or lecturer. The tutorials differ widely in the number of attending students but usually there are smaller numbers of students (20 to 40 students in each group).

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Tutorials normally don't introduce new topics but concern themselves with practicing the previous lectures' contents by solving mathematical problems given in worksheets.

The discussed setting in this paper is an introductory mathematics course for students who want to become math teachers for primary and lower secondary schools. This is one of the main reasons we follow a non-traditional teaching philosophy. As future school teachers these students have to gain enough experience learning in constructivist learning scenarios to be able to teach in these scenarios as well.

The pattern for constructivist mathematics learning can be easily adapted to other school forms (i.e. high school) or even subjects with similar processes and competencies like theoretical physics or computer science.

Problem / Challenge / Motivation[1]

Freshmen usually have to get used to the difference between mathematics at school and at universities. At universities, performing mathematics means much more than just solving a predefined set of mathematical problems in a given thematic context (e.g., arithmetic or geometry). It means applying solution strategies and problem solving heuristics such as finding examples and counter-examples, making conjectures, and drawing graphs. In addition, performing 'real' mathematics often means solving problems with no pre-defined single solution. Therefore, students have to decide which information is relevant for the solution, how to process this information, and how to present the results. In addition, they have to choose appropriate tools like spreadsheet calculators or dynamic geometry systems.

Especially future teachers should experience this kind of performing mathematics very early in their studies. They do not have to focus on the product (the solution of a problem), but on the mathematical processes necessary for solving the problem. The latter is the 'real' objective of learning mathematics. This change in the view on mathematics education is clearly stated, e.g., in the NCTM 'Principles and Standards for School Mathematics' (2000) where the process standards *problem solving*, *reasoning and proof*, *communication*, *connections*, and *representation* are considered as important as the content standards *number and operation*, *algebra*, *geometry*, *measurement, data analysis and probability*.

How do beginners learn these strategies efficiently?

Forces

Here is the point where the pattern follows two different paths to reach answers to the above stated challenge. Depending on the fundamental philosophy of learning and teaching two different solutions arise. The descriptions are given parallel in the following table to simplify the comparison.

The learning/teaching philosophies are deeply intertwined with theoretical pedagogical approaches which will also be described in the table (s. 'Rationale').

[1] In a pedagogical context the word 'problem' is not really adequate here. Students learning geometry is not a problem which can be solved once and for all – it's more a 'challenge'.

Traditional teaching philosophy	Learner activating teaching philosophy
"Mathematics is learned by being exposed to definitions, theorems, proofs, techniques and examples, through which one is exposed to formalization, proof, modeling, techniques etc. The teacher's job is to lay out the material clearly and logically. Students must rehearse many examples in order to develop facility and through facility, gain understanding of the concepts, the techniques and why the techniques work. Hard work is valued, work consisting largely of working through notes and problems to try to understand them." (Holton, 2001, 73) This is also the kind of learning most of the professors at German universities experienced when learning mathematics themselves.	"Mathematics is learned by reconstructing for oneself what others have thought and tried to expound clearly and logically. Reconstruction is carried out through constructing special and illustrative cases, trying to see generality through the particular, guided by theorems and other exposition. Exposition and practice on exercises is useful, but only as means to reconstruction. Facility and understanding grow together, as each contributes to the other and neither necessarily precedes the other." (Holton, 2001, 73) The basic idea of this concept is to motivate students actively doing mathematics. The students work on their own choice of problems, organized in small groups and aided by tutors. There are several forms of feedback during the tutorials as well as virtually in learning platforms.

Solution

Students **work on a set of given problems** and follow **the demonstrations of solved problems** and proofs of theorems during the tutorial. ***Planning and preparation:*** The **person responsible (lecturer or advanced tutor)** creates problem sheets containing a number of problems all of which are supposed to be solved by all students.	Students **pick from 5-6 weekly problem suggestions** and work in groups during the tutorial on the chosen problems **guided by the tutor** who doesn't give the correct solution away. ***Planning and preparation:*** The **person responsible (lecturer or advanced tutor)** creates problem sheets containing problems with the same mathematical topics but in different contexts for the students to choose from. The wording focuses on the solution process rather than the correct solution.

A typical problem:

Show that for any x, y, z $\in \mathbb{R}$, | x - z| ≤ | x - y| + | y - z|.

An expected typical solution to the problem:

$|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$ *(triangle inequality)*[2]

The problems are related closely to the content of the lecture. A sample solution also has to be created. The problem sheets are delivered to students one week or more before the tutorial session. Tutors get the problem sheets at the same time plus a sample solution to prepare for the demonstrations.

A typical problem:

Make conjectures of several unit fractions concerning their decimal representation. What kind of decimal do you get?

If it is not a terminating decimal: How long are the periods and the delays of the periods? Make conjectures on the base of your data. Which properties determine the kind of decimal? Which properties determinate the length of the period and the delay? Test your hypotheses with other unit fractions.

Hints / techniques:

- *You can use the Excel spreadsheets available in Moodle.*
- *Which of the unit fractions are good indicators for your conjectures?*

Expected activities:

The students are expected to try to understand the properties of decimal numbers using spreadsheets and to find the significance of denominators only containing powers of 2 and 5 compared to denominators without 2 and 5 or 'mixed' ones.[3]

The problem suggestions have to cover enough of the mathematical content so that students can not evade basic concepts such as reasoning and proof, finding examples, or special techniques like using group tables or important mathematical content. Some useful hints and references which don't give away too much have to be added.

Tutors need to be provided with ideas, hints, and strategies for exemplary solutions and problem solving. Also, they have to be

[2] Explanations for non-mathematicians: $|x - z| = |x - y + y - z|$ *because $-y + y = 0$ for any $y \in \mathbb{R}$ and the triangle inequality is* $|x + y| \leq |x| + |y|$ *which is true for all $x, y \in \mathbb{R}$*

[3] All the fractions with denominators which consists only of powers of 2 and 5 (e.g. $1/40 = 1/(5*2^3)=0.025$) are terminating. All the fractions with denominators which consist of numbers without the factors of 2 and/or 5 are periodic ($1/33=1/(3*11)=0.030303\ldots$) and the rest are delayed periodic ($1/12=1/(3*2^2)=0.083333\ldots$).

Tutors read through the given sample solution to be sure that they understand everything. **Students** work on the problems before the tutorial session. This is not monitored or supported by tutors.	briefed about aspects of insufficient solutions and problematic problem-solving strategies. Usually even tutors don't get the 'correct' solution from the lecturer. In fact, they have to think through the problems on their own and actively 'do mathematics'.
During the tutorials:	***During the tutorials:***
The **tutors** present the problem solutions on the chalkboard. **Students** watch the demonstration and compare the solutions with their own (if they have created one). Alternatively, a student may be asked to present her or his own solution. Students may ask questions and discuss different solutions.	**Groups of students** start work on the problems during the tutorial session. They choose the problems they want to work on, discuss ideas, find examples, and verify or disprove statements on the worksheets and of others. They can use every tool they want to support the problem solving process - laptops, calculators or whatever. If necessary they ask for help and explain their problems. Finally they have to decide whether something is a solution or not. If they can't find a solution, they can also ask for help on their problem in discussion forums in the online learning platform. Then other students, tutors, or the lecturer may assist. The **tutors** give feedback, ask helpful questions and confirm that a solution is ok but don't give the correct solution. They give hints or mirror back some ideas and questions brought already up by the students. Very often, they just 'sit around' and do nothing.
After the tutorials:	***After the tutorials:***
The sample solutions are given to the students. **Students** try to transfer the solutions to similar problems (reproduction). They practice under their own steam.	**All the participants** join in the online discussion. **Students** finish not yet solved problems and continue working on them based on their lecture notes, trying to connect them to problems they worked on during the tutorials.

Rationale

Theoretical Background	Theoretical Background
Basically this kind of teaching is experts demonstrating learners how mathematics “goes”. The social cognitive theory (Bandura, 1989) states that people may learn by observing others doing something. The process of demonstrating a specific behavior is called *modeling* (Bandura, 2001). An example for this is a person in a restaurant struggling to eat a lobster. Observing other guests, this person is able to perform the task by imitating other guests’ procedures. Vicarious experience is one source of self-efficacy expectations (Bandura, 1998). People who see others performing well may think that they could also master the task (Schunk, 1999; Margolis, 2005). In the cognitive load theory (Chandler & Sweller, 1991; Sweller, van Merriënboer, & Paas, 1998) instructional guidelines are developed which help to avoid irrelevant cognitive load during problem solving (Anderson, 1995). In many European universities mathematics lectures and tutorials are designed following this idea. Students follow an expert presenting the 'published' mathematics[4] in a very condensed way, avoiding wrong turns (Holton, 2001).	The basic idea of this concept is to motivate students to actively do mathematics. It is based on constructivist teaching philosophy insofar as by actively dealing with the mathematical problems themselves students gain experiences and insights and adjust their ideas and mathematical concepts which is all part of building new knowledge. Constructivist teachers[5] believe strongly in the idea that students construct knowledge for themselves and they will not truly learn something until they spend a good deal of time asking questions and actively thinking about the topic. The job as a teacher is to provide context, motivation and guidance. According to the cognitive apprenticeship model (Collins, Brown, & Newman, 1989) the latter has to fade with the learner’s increasing expertise. Motivation is a crucial factor for learning. Intrinsically motivated students show higher levels of cognitive engagement in tasks than students who are more extrinsically motivated (Pintrich & Schrauben, 1992). Ryan and Deci (2002) state that three factors promote intrinsic motivation: *perceived choice*, *perceived competence*, and *related-ness*. The perception of competence is also related to the construct of self-efficacy (Bandura, 1998). Mathematical self-efficacy is the belief of a person that she or he is able to solve a mathematical problem (Betz &

[4] ‘Published’ means here the way mathematics is displayed in books where e.g. the concise and elegant form of a proof is printed and not the easier to follow but longer approach which shows how the proof was found the first time.
[5] For some first ideas on constructivism in mathematics education see e.g. http://mathforum.org/library/ed_topics/constructivism/, last visited May 31st, 2009

	Hackett, 1983; Pajares & Miller, 1995). One major source of efficacy expectations is performance accomplishments. Repeated successes have a positive influence on self-efficacy. Only if students try to solve the problems on their own there is a chance to increase their self-efficacy based on performances.
Reflections This is a very efficient way – from the teacher's point of view – to teach large groups of students how to solve mathematical problems. If 'an expert' (tutor or good student) demonstrates his previously prepared solution all students have seen at least one correct way of solving that particular problem. This solution is often used in introductory mathematics courses at universities and since a lot of mathematics teachers went through this system they obviously were successful. But there are a lot of drawbacks in this solution. 1. Non preparation Often students come to the tutorial session without own solutions or without even having read the problem sheet. Therefore they just copy the solutions without any understanding. Students need a lot of time and effort just before the exams to 'catch up' with all the missing understanding and often there just isn't enough time. There are possibilities to deal with this non-preparation of students and force them to work on the problems before they get the solution presented: At the beginning of the session a list with all problems is passed around and students have to tick the problems they have prepared. The tutor then calls students according to the list to present the solution. Every student has to tick at least 50% (or more) of all	**Reflections** Students have to get used to this kind of tutorial. A lot of the students are not very confident that they can really recognize a correct or incorrect solution. They want always some authority to check their answers. For these students there are several support structures beside the weekly face-to-face tutorials and online discussions. For example, there is an 'open math room' three to four times during the week where tutors answer questions. Learning to be a good problem solver requires working on problems at the same time as reflecting on the problem solving processes. This is very easy in collaborative settings and by actively participating in the problem solving processes. Working in groups students automatically communicate, ask questions, represent mathematical ideas, etc. and therefore mathematical processes can be experienced, reflected and discussed. One issue is definitely the time needed by the tutors for preparation and feedback.

problems otherwise he or she is not allowed to take the exam. Also, each student has to present solutions a certain number of times during the semester.

Another approach is to state problems that should be done at home by the students. Written solutions will have to be handed to the tutor at the beginning of the session. These solutions are graded (or just a feedback is given) and also an average grade must be reached to take part in the exams.

2. Illusion of Understanding
Presenting solutions without eliciting deep processing often creates the 'illusion of understanding' (cf. Atkinson et al., 2000). Students assume they have understood the solution. Realization that they didn't often occurs during the final exam.
This problem can be mitigated if students *actively process* the demonstrations (cf. Mayer, 2004). Some guidelines for the design of worked examples have been developed to increase the student's cognitive activity, e.g., giving incomplete worked examples (*completion problems*; Sweller et al., 1998), emphasizing the structure of the solution (Catrambone & Holyoak, 1990), or prompting students to elicit self-explanations (Chi, de Leeuw, Chiu, & LaVancher, 1994).

3. Motivation
Predominantly, students are extrinsically motivated. They want to pass the final test. But intrinsic motivation is a crucial factor in learning: see 'Theoretical background' of the 'Learner activating teaching philosophy – solution'.

4. 'Modern' learning theories
See: 'Theoretical background' of the 'Learner activating teaching philosophy – solution'.

Examples

This kind of tutorials can still be found at many German universities.	This kind of tutorials were realized from October 2007 until July 2008 at the University of Education Ludwigsburg in the courses Introduction to Arithmetic for Secondary Teachers (Einführung in die Arithmetik für Lehramt Realschule) and Introduction to Geometry for Secondary Teachers (Einführung in die Geometrie für Lehramt Realschule). The whole setting of weekly lectures and tutorials were evaluated by different instruments: a questionnaire on mathematical self-efficacy and a questionnaire on learning motivation. The results are published in Bescherer and Spannagel (2008; in German). Since October 2008 these tutorials are developed further in the context of the research project SAiL-M (www.sail-m.de) funded by the German Federal Ministry of Education and Research.

Related Patterns

	PATTERNS FOR ACTIVE LEARNING by Eckstein, Bergin, Sharp (http://www.pedagogicalpatterns.org/current/activelearning.pdf) TECHNOLOGY ON DEMAND, HELP ON DEMAND, FEEDBACK ON DEMAND (all Bescherer & Spannagel, 2009) and HINT ON DEMAND (www.sail-m.de)

Summary

The above presented pattern gives two solutions to the same challenge (problem) based on two different teaching philosophies. Of course these solutions describe more or less the extreme variations of tutorials and all shades in between these are possible.

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