Synthesis of General Petri Nets with Localities

Maciej Koutny and Marta Pietkiewicz-Koutny

School of Computing Science
Newcastle University
Newcastle upon Tyne, NE1 7RU
United Kingdom
{maciej.koutny,marta.koutny}@newcastle.ac.uk

Abstract. There is a growing need to introduce and develop computational models capable of faithfully modelling systems whose behaviour combines synchrony with asynchrony in a variety of complicated ways. Examples of such real-life systems can be found from VLSI hardware GALS systems to systems of cells within which biochemical reactions happen in synchronised pulses. One way of capturing the resulting intricate behaviours is to use Petri nets with localities where transitions are partitioned into disjoint groups within which execution is synchronous and maximally concurrent. In this paper, we generalise this type of nets by allowing each transition to belong to several localities. Moreover, we define this extension in a generic way for all classes of nets defined by net-types. We show that Petri nets with overlapping localities are an instance of the general model of nets with policies. Thanks to this fact, it is possible to automatically construct nets with localities from behavioural specifications given in terms of finite step transition systems. After that we outline our initial ideas concerning net synthesis when the association of transition to localities is not given and has to be determined by the synthesis algorithm.
Keywords: theory of concurrency, Petri nets, localities, analysis and synthesis, step sequence semantics, conflict, theory of regions, transition systems.

1 Introduction

In the formal modelling of computational systems there is a growing need to faithfully capture real-life systems exhibiting behaviour which can be described as 'globally asynchronous locally (maximally) synchronous' (GALS). Examples can be found in hardware design, where a VLSI chip may contain multiple clocks responsible for synchronising different subsets of gates [6], and in biologically inspired membrane systems representing cells within which biochemical reactions happen in synchronised pulses [15]. To capture such systems in a formal manner, [9] introduced *Place/Transition-nets with localities* (PTL-nets), where each locality identifies a distinct set of transitions which must be executed synchronously, i.e., in a maximally concurrent manner (akin to *local maximal concurrency*). The expressiveness of PTL-nets (even after enhancing them with

Recent Advances in Petri Nets and Concurrency, S. Donatelli, J. Kleijn, R.J. Machado, J.M. Fernandes (eds.), CEUR Workshop Proceedings, ISSN 1613-0073, Jan/2012, pp. 161–174.

inhibitor and activator arcs in [8]) was constrained by the fact that each transition belonged to a unique locality, and so the localities were all *non-overlapping*. In this paper, we drop this restriction aiming at a net model which we believe should provide a greater scope for faithful (or direct) modelling features implied by the complex nature of, for example, modern VLSI systems or biological systems.

> To explain the basic idea behind nets with *overlapping localities*, let us consider an array of n transitions t_i $(0 \leq i \leq n-1)$ which are arranged in a circular manner, i.e., t_i is adjacent to $t_{(i+n-1)\,\mathsf{mod}\,n}$ and $t_{(i+1)\,\mathsf{mod}\,n}$ which form its 'neighbourhood'. Each of the transitions belongs to some subsystem which is left unspecified. What is important from our point of view is that to be executed, t_i needs, in addition to being enabled by its subsystem, to receive an external stimulus (e.g., an electric charge when transitions represent biological cells) which then spreads to its neighbourhood forcing the execution of transition $t_{(i+n-1)\,\mathsf{mod}\,n}$ and $t_{(i+1)\,\mathsf{mod}\,n}$ provided they are enabled by their subsystems. Thus, stimulating a transition amounts to stimulating its neighbourhood, and neighbourhoods can overlap which means that a given transition can be triggered in possibly many ways. To model such a scenario in a direct way we can use a Petri net augmented with a locality mapping ℓ such that $\ell(t_i) = \{(i+n-1)\,\mathsf{mod}\,n, i, (i+1)\,\mathsf{mod}\,n\}$, where each integer represents a distinct locality, and assuming that a transition may be executed only if it belongs to some stimulated neighbourhood. For example, if all the transitions t_i are enabled by their subsystems, then the following are examples of legal steps of the Petri net:
>
> | | |
> |---|---|
> | $\{t_2, t_3, t_4\}$ | t_3 stimulated |
> | $\{t_2, t_3, t_4, t_5\}$ | t_3 and t_4 stimulated |
> | $\{t_2, t_3, t_4, t_5, t_8, t_9, t_{10}\}$ | t_3, t_4 and t_9 stimulated |
>
> and two examples of illegal steps are $\{t_2, t_3\}$ and $\{t_2, t_3, t_4, t_6\}$.
> In the abstract capture of the underlying mechanisms like that above, we will demand that an executed transition belongs to at least one *saturated* locality, i.e., it is not possible to additionally execute any more transitions associated with that locality.

Rather than introducing nets with overlapping localities for PT-nets or their extensions, we will move straight to the general case of τ-nets [2] which encapsulate a majority of Petri net classes for which the synthesis problem has been investigated. In fact, the task of defining τ-nets with (potentially) overlapping localities is straightforward, as the resulting model of *τ-nets with localities* turns out to be an instance of the general framework of τ-nets with *policies* introduced in [4].

After introducing the new model of nets, we turn our attention to their automatic synthesis from behavioural specifications given in terms of step transition systems. Since τ-nets with localities are an instance of a more general scheme treated in [4], we directly import synthesis results presented there which are based on the regions of a transition system studied in other contexts, in particular, in [1–3, 7, 13, 14, 16, 10, 11].

The results in [4] assume that policies are given which, in our case, means that we know exactly the localities associated with all the net transitions. This may be difficult to guarantee in practice, and so in the second part of the paper we outline our initial ideas concerning net synthesis when this is not the case, extending our previous work on non-overlapping localities reported in [12].

2 Preliminaries

In this section, we recall some basic notions concerning τ-nets, policies and the synthesis problem as presented in [4].

An *abelian monoid* is a set S with a commutative and associative binary (composition) operation $+$ on S, and a neutral element $\mathbf{0}$. The monoid element resulting from composing n copies of $s \in S$ will be denoted by $n \cdot s$, and so $\mathbf{0} = 0 \cdot s$ and $s = 1 \cdot s$.

A specific abelian monoid, $\langle T \rangle$, is the free abelian monoid generated by a set (of transitions) T. It can be seen as the set of all the multisets over T. We will use $\alpha, \beta, \gamma, \ldots$ to range over the elements of $\langle T \rangle$. Moreover, for all $t \in T$ and $\alpha \in \langle T \rangle$, we will use $\alpha(t)$ to denote the multiplicity of t in α. We will write $t \in \alpha$ whenever $\alpha(t) > 0$, and denote by $supp(\alpha)$ the set of all $t \in \alpha$. The size of α is given by $|\alpha| = \sum_{t \in T} \alpha(t)$.

We denote $\alpha \leq \beta$ whenever $\alpha(t) \leq \beta(t)$ for all $t \in T$ (and $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$). For $X \subseteq \langle T \rangle$, we denote by $\max_{\leq}(X)$ the set of all $\leq$-maximal elements of X, and by $\min_{\leq}(X)$ the set of all non-empty $\leq$-minimal elements of X.

If $T' \subseteq T$ then $\alpha|_{T'}$ is a multiset α' such that $\alpha'(t) = \alpha(t)$ if $t \in T'$ and otherwise $\alpha'(t) = 0$. The sum of two multisets, α and β, will be denoted by $\alpha + \beta$, and a singleton multiset $\{t\}$ simply by t.

A *transition system* over an abelian monoid S is a triple (Q, S, δ) such that Q is a set of *states*, and $\delta : Q \times S \to Q$ a partial *transition function*[1] satisfying $\delta(q, \mathbf{0}) = q$ for all $q \in Q$. An *initialised* transition system $\mathcal{T} \stackrel{\text{df}}{=} (Q, S, \delta, q_0)$ has in addition an *initial* state $q_0 \in Q$ from which every other state is *reachable*. For every state q of a (non-initialised or initialised) transition system TS, $enbld_{TS}(q) \stackrel{\text{df}}{=} \{s \in S \mid \delta(q, s) \text{ is defined}\}$.

Initialised transition systems $\mathcal{T}$ over free abelian monoids — called *step transition systems* — will represent concurrent behaviours of Petri nets. Non-initialised transition systems τ over arbitrary abelian monoids — called *net-types* — will provide ways to define various classes of nets. Throughout the paper, we will assume that:

- T is a fixed finite set (of net transitions);
- *Loc* is a fixed finite set (of net transitions' localities);

[1] Transition functions and net transitions are unrelated notions.

- $\mathcal{T} = (Q, S, \delta, q_0)$ is a fixed step transition system over $S = \langle T \rangle$.
- $\tau = (\mathbb{Q}, \mathbb{S}, \Delta)$ is a fixed net-type over an abelian monoid $\mathbb{S}$. In this paper, we will assume that τ is *substep closed* which means that, for every state $q \in Q$, if $\alpha + \beta \in enbld_\tau(q)$ then also $\alpha \in enbld_\tau(q)$. This will imply that substeps of resource enabled steps are also resource enabled which is a condition usually satisfied in practice.

The net-type defines a class of nets, by specifying the values (markings) that can be stored in net places ($\mathbb{Q}$), the operations and tests (inscriptions on the arcs) that a net transition may perform on these values ($\mathbb{S}$), and the enabling condition and the newly generated values for steps of transitions (Δ).

Definition 1 (τ-net). *A* τ-net system *is a tuple* $\mathcal{N} \stackrel{\text{df}}{=} (P, T, F, M_0)$*, where* P *and* T *are disjoint sets of places and transitions, respectively;* $F : (P \times T) \to \mathbb{S}$ *is a flow mapping; and* $M_0 : P \to \mathbb{Q}$ *is an* initial marking.

In general, any mapping $M : P \to \mathbb{Q}$ is a marking. For each place $p \in P$ and step $\alpha \in \langle T \rangle$, $F(p, \alpha) \stackrel{\text{df}}{=} \sum_{t \in T} \alpha(t) \cdot F(p, t)$.

Definition 2 (step semantics). *Given a* τ*-net system* $\mathcal{N} = (P, T, F, M_0)$*, a step* $\alpha \in \langle T \rangle$ *is* (resource) enabled *at a marking* M *if, for every place* $p \in P$*:*

$$F(p, \alpha) \in enbld_\tau(M(p)) .$$

We denote this by $\alpha \in enbld_{\mathcal{N}}(M)$*. The* firing *of such a step produces the marking* M' *such that, for every* $p \in P$*:*

$$M'(p) \stackrel{\text{df}}{=} \Delta(M(p), F(p, \alpha)) .$$

Step firing policies are means of controlling and constraining the huge number of execution paths resulting from the concurrent nature of a majority of computing systems.

Let $\mathcal{X}_\tau$ be the family of all sets of steps enabled at some reachable marking M of some τ-net $\mathcal{N}$ with the set of transitions T.

Definition 3 (bounded step firing policy). *A* bounded step firing policy *for* τ*-nets over* $\langle T \rangle$ *is given by a control disabled steps mapping* $cds : 2^{\langle T \rangle} \to 2^{\langle T \rangle \setminus \{\mathbf{0}\}}$ *such that, for all* $X \subseteq \langle T \rangle$*, the following hold:*

1. *If* X *is infinite then* $cds(X) = \varnothing$*.*
2. *If* X *is finite then, for every* $Y \subseteq X$*:*
   (a) $cds(X) \subseteq X$*;*
   (b) $cds(Y) \subseteq cds(X)$*; and*
   (c) $X \in \mathcal{X}_\tau$ *and* $X \setminus cds(X) \subseteq Y$ *imply* $cds(X) \cap Y \subseteq cds(Y)$*.*

We will now discuss further step firing policies and their effect on net behaviour.

Definition 4 (τ-net with policy). *Let cds be a bounded step firing policy for τ-nets over $\langle T \rangle$. A tuple $\mathcal{NP} \stackrel{\text{df}}{=} (P, T, F, M_0, cds)$ is a τ-*net system with policy *if $\mathcal{N} = (P, T, F, M_0)$ is a τ-net and the (control) enabled steps of $\mathcal{NP}$ at a marking M are:*

$$Enbld_{\mathcal{NP}}(M) \stackrel{\text{df}}{=} enbld_{\mathcal{N}}(M) \setminus cds(enbld_{\mathcal{N}}(M)).$$

Moreover, let $enbld_{\mathcal{NP}}(M) \stackrel{\text{df}}{=} enbld_{\mathcal{N}}(M)$ be the set of resource enabled steps of $\mathcal{NP}$ at marking M. The effect of executions of enabled steps in $\mathcal{NP}$ is the same as in $\mathcal{N}$.

We will denote by $CRG(\mathcal{NP})$ the step transition system with the initial state M_0 formed by firing inductively from M_0 all possible control enabled steps of $\mathcal{NP}$, and call it concurrent reachability graph *of $\mathcal{NP}$.*

In this paper our concern will be to find a general solution to the synthesis problem for τ-nets with localities. Since they are special kinds of τ-nets with policies we will be able to use the theory developed for those nets in [4]. By solving a synthesis problem we mean finding a procedure for building a net of a certain class with the desired behaviour (in our case, concurrent reachability graph). In our case the problem can be defined as follows.

SYNTHESIS PROBLEM

Let $\mathcal{T}$ be a given finite step transition system. Provide necessary and sufficient conditions for $\mathcal{T}$ to be *realised* by some τ-net system with policy $\mathcal{NP}$ (i.e., $\mathcal{T} \cong CRG(\mathcal{NP})$ where $\cong$ is transition system isomorphism preserving the initial states and transition labels).

The solution of the synthesis problem is based on the idea of a region of a transition system.

Definition 5 (τ-region). *A* τ-region *of $\mathcal{T}$ is a pair of mappings*

$$(\sigma : Q \to \mathbb{Q} \ , \ \eta : \langle T \rangle \to \mathbb{S})$$

such that η is a morphism of monoids and, for all $q \in Q$ and $\alpha \in enbld_{\mathcal{T}}(q)$:

$$\eta(\alpha) \in enbld_{\tau}(\sigma(q)) \quad \text{and} \quad \Delta(\sigma(q), \eta(\alpha)) = \sigma(\delta(q, \alpha)) \ .$$

For every state q of Q, we denote by $enbld_{\mathcal{T},\tau}(q)$ the set of all steps α such that $\eta(\alpha) \in enbld_{\tau}(\sigma(q))$, for all τ-regions (σ, η) of $\mathcal{T}$.

We then have the following general result from [4].

Theorem 1. *$\mathcal{T}$ can be realised by a τ-net system with a (bounded step firing) policy cds* iff *the following two regional axioms are satisfied:*

AXIOM I: STATE SEPARATION

For any pair of states $q \neq r$ of $\mathcal{T}$, there is a τ-region (σ, η) of $\mathcal{T}$ such that $\sigma(q) \neq \sigma(r)$.

AXIOM II: FORWARD CLOSURE WITH POLICIES

For every state q of $\mathcal{T}$, $enbld_{\mathcal{T}}(q) = enbld_{\mathcal{T},\tau}(q) \setminus cds(enbld_{\mathcal{T},\tau}(q))$. $\square$

A solution to the synthesis problem is obtained if one can compute a finite set $\mathcal{WR}$ of τ-regions of $\mathcal{T}$ *witnessing* the satisfaction of all instances of AXIOMS I and II [5]. A suitable τ-net system with policy cds, $\mathcal{NP}_{\mathcal{WR}} = (P, T, F, M_0, cds)$, can be then constructed with $P = \mathcal{WR}$ and, for any place $p = (\sigma, \eta)$ in P and every $t \in T$, $F(p,t) = \eta(t)$ and $M_0(p) = \sigma(q_0)$ (recall that q_0 is the initial state of $\mathcal{T}$, and $T \subseteq \langle T \rangle$).

3 τ-nets with localities

We will now introduce a general class of Petri nets with localities, based on a specific class of control disabled steps mappings.

A locality mapping for the transition set T is any $\ell : T \to 2^{Loc}$ such that $\ell(t) \neq \varnothing$ for all $t \in T$. (Below we will denote $l \in \ell(\alpha)$, for every step α and a locality $l \in Loc$, whenever there is a transition $t \in \alpha$ such that $l \in \ell(t)$.) Then the induced control disabled steps mapping is

$$cds_\ell : 2^{\langle T \rangle} \to 2^{\langle T \rangle \setminus \{\mathbf{0}\}}$$

such that, for all $X \subseteq \langle T \rangle$:

$$cds_\ell(X) \stackrel{\text{df}}{=} \begin{cases} \{\alpha \in X \mid \exists t \in \alpha \; \forall l \in \ell(t) \; \exists \alpha + \beta \in X : \; l \in \ell(\beta)\} & \text{if } X \text{ is finite} \\ \varnothing & \text{otherwise .} \end{cases}$$

Proposition 1. *cds_ℓ is a bounded step firing policy.*

Proof. All we need to prove is that if $X \in \mathcal{X}_\tau$ is finite and $Y \subseteq X$ and $X \setminus cds_\ell(X) \subseteq Y$ and $\alpha \in cds_\ell(X) \cap Y$, then $\alpha \in cds_\ell(Y)$.

We first observe that $\max_\leq(X) \cap cds_\ell(X) = \varnothing$ and so $\max_\leq(X) \subseteq X \setminus cds_\ell(X) \subseteq Y$. Then we observe that since X is finite and $\alpha \in cds_\ell(X)$, there is $t \in \alpha$ such that for all $l \in \ell(t)$ there exists $\alpha + \beta \in \max_\leq(X) \subseteq Y$ satisfying $l \in \ell(\beta)$. This and the fact that Y is finite (as $Y \subseteq X$) means that $\alpha \in cds_\ell(Y)$. □

We will call a τ-net system with a policy cds_ℓ a *τ/ℓ-net system* (or τ-net with localities). Moreover, we will call $\mathcal{T}$ a *τ/ℓ-transition system* if AXIOM I and AXIOM II are satisfied for $\mathcal{T}$ with policy $cds = cds_\ell$.

Proposition 2. *Let M be a marking of a τ/ℓ-net system $\mathcal{NP}$ such that the set $enbld_{\mathcal{NP}}(M)$ is finite. A step $\alpha \in enbld_{\mathcal{NP}}(M)$ belongs to $Enbld_{\mathcal{NP}}(M)$* iff *for every $t \in \alpha$ there is $l \in \ell(t)$ such that:*

$$l \in \ell(t') \implies \alpha + t' \notin enbld_{\mathcal{NP}}(M) \, ,$$

for every transition t'.

Proof. Follows from $\max_\leq(enbld_{\mathcal{NP}}(M)) \subseteq Enbld_{\mathcal{NP}}(M)$, and the fact that $\gamma \leq \delta$ and $\delta \in enbld_{\mathcal{NP}}(M)$ together imply $\gamma \in enbld_{\mathcal{NP}}(M)$. □

We obtain an immediate solution to the synthesis problem for τ/ℓ-nets.

Theorem 2. *A finite step transition system $\mathcal{T}$ can be realised by a τ/ℓ-net system* iff *$\mathcal{T}$ is a τ/ℓ-transition system.*

Proof. Follows from Theorem 1 and Proposition 1. □

The synthesis problem for PT-nets and EN-systems with localities (and with or without inhibitor and read arcs) have been investigated in [10–12]. For such nets, the locality mapping ℓ has the property that $|\ell(t)| = 1$, for all $t \in T$. Such an ℓ defines localities which are mutually disjoint or *non-overlapping*. In this paper, we allow fully general, i.e., possibly overlapping localities.

As to the effective construction of synthesised net, it has been demonstrated in [10–12] that this can be easily done for net classes with non-overlapping localities mentioned above. Similar argument can be applied also in the general setting of overlapping localities and τ-nets corresponding to PT-nets and EN-systems with localities. We omit details.

Finally, it is interesting to observe that in the (previously considered) case of non-overlapping localities, cds_ℓ can be defined through a pre-order on steps. This is no longer the case for the general locality mappings.

4 Towards synthesis with unknown localities

The synthesis result presented in the previous section was obtained assuming that the locality mapping was given. However, in practice such a mapping might be unknown (or partially known), and part of the outcome of a successful synthesis procedure would be a suitable (or *good*) locality mapping. Clearly, what really matters in a locality mapping is the identification of (possibly overlapping) clusters of transitions, each cluster containing all transitions sharing a locality. Since there are only finitely many clusters, there are also finitely many non-equivalent locality mappings, and the synthesis procedure could simply enumerate them and then check one-by-one using Theorem 2. This, however, would be highly impractical as the number of clusters is exponential in the number of transitions. We will now present some initial ideas and results aimed at reducing the number of checks.

From now on we will assume that $\mathcal{T}$ is *finite*. We will also assume that we have checked that, for every state q of $\mathcal{T}$, the set of steps $enbld_{\mathcal{T},\tau}(q)$ is finite; otherwise $\mathcal{T}$ could not be isomorphic to the concurrent reachability graph of any τ-net with localities (see AXIOM II and Theorem 2).

In the rest of this section, for every state q of $\mathcal{T}$ and locality mappings ℓ, ℓ':

- $allSteps_q$ is the set of all steps labelling arcs outgoing from q.
- $minSteps_q$ is the set of all non-empty steps $\alpha \in allSteps_q$ for which there is no non-empty $\beta \in allSteps_q$ such that $\beta < \alpha$.
- T_q is the set of all net transitions occurring in the steps of $allSteps_q$.
- $clusters_q^\ell$ is the set of all sets $\{t \in T_q \mid l \in \ell(t)\}$, for every $l \in \ell(T_q)$.

– ℓ and ℓ' are node-consistent if $clusters_r^{\ell} = clusters_r^{\ell'}$, for every state r of $\mathcal{T}$.

A general result concerning locality mappings is that they are equally suitable for being good locality mapping whenever they induce the same clusters of co-located transitions in each individual node of the step transition system.

Proposition 3. *Let ℓ and ℓ' be two node-consistent locality mappings. Then $\mathcal{T}$ is τ/ℓ-transition system* iff *$\mathcal{T}$ is τ/ℓ'-transition system.*

Proof. Suppose that $\mathcal{T}$ is τ/ℓ-transition system. First we notice that AXIOM I does not depend on the locality mapping. For AXIOM II and ℓ' it suffices to show that, for each state q of $\mathcal{T}$:

$$cds_{\ell}(enbld_{\mathcal{T},\tau}(q)) = cds_{\ell'}(enbld_{\mathcal{T},\tau}(q)) \ . \tag{1}$$

We observe that the steps from $enbld_{\mathcal{T},\tau}(q)$ have transitions belonging to T_q (as the maximal steps in $enbld_{\mathcal{T},\tau}(q)$ never belong to $cds_{\ell}(enbld_{\mathcal{T},\tau}(q))$ and AXIOM II holds for ℓ), and thus according to the definition of $cds_{\ell}(X)$ the influence of each locality $l \in \ell(T_q)$ and $l' \in \ell'(T_q)$ can be accurately represented by the clusters $\{t \in T_q \mid l \in \ell(t)\}$ and $\{t' \in T_q \mid l' \in \ell'(t')\}$, respectively. Hence, since ℓ and ℓ' are node-consistent, (1) holds. $\square$

As a consequence, a good locality mapping can be arbitrarily modified to yield another good locality mapping as long as the two mappings are node-consistent (there is no need to re-check the two axioms involved in Theorem 2). This should allow one to search for an optimal good locality mapping starting from some initial choice (for example, one might prefer to have as few localities per transition as possible).

The construction of a good locality mapping could be seen as modular process, in the following way. First, separately for each state q, we produce a list of possible cluster-sets of transitions in T_q induced by hypothetical good locality mappings. Each such *cluster-set* $clSet \stackrel{\text{df}}{=} \{C_1, \ldots, C_k\}$ is composed of non-empty subsets of T_q so that $C_1 \cup \ldots \cup C_k = T_q$ and:

$$enbld_{\mathcal{T}}(q) = enbld_{\mathcal{T},\tau}(q) \setminus cds_{clSet}(enbld_{\mathcal{T},\tau}(q))$$

where

$$cds_{clSet}(X) \stackrel{\text{df}}{=} \{\alpha \in X \mid \exists t \in \alpha \ \forall i \le k : \ (t \in C_i \ \Rightarrow \ \exists t' \in C_i : \ \alpha + t' \in X)\} \ .$$

Similarly, one may produce, for each state q, a characterisation of inadmissible clustering of transitions. We can then select different cluster-sets (one per each state of the step transition system) and check whether combining them together yields a good locality mapping. Such a procedure was used in [12] to construct 'canonical' locality mappings for the case of non-overlapping localities (and the combining of cluster-sets was based on the operation of transitive closure).

The search for a good locality mapping outlined above can be improved if one looks for solutions in a specific class of nets, or if the locality mapping is partially known or constrained (for example, that two specific transitions cannot share a locality).

4.1 Localised conflicts

Intuitively, localities and conflicts may have opposite effects on step enabledness. Whereas joining two localities may reduce the number of control enabled steps, adding a conflict between transitions with shared localities may turn a non-enabled step into a control enabled one. It is therefore interesting what simplifications, if any, one might obtain if conflicts were constrained to exist between transitions sharing localities.

In the paper [12] we looked at this issue in the context of PTL-nets and ENL-nets, coming up with the notion of nets with *localised conflicts*. For the synthesis problem for such nets, it turned out that for each state q there was at most one cluster-set to be considered, providing particularly pleasant simplification of the original problem. In the rest of this section, we provide some initial results towards extending this to the case of nets with overlapping localities. Below, for a step α and locality l we denote $\alpha|_l \stackrel{\text{df}}{=} \alpha|_{\{t\in T \mid l\in\ell(t)\}}$ assuming that ℓ is given. Moreover, $\mathit{Enbld}^{min}_{\mathcal{NP}}(M) \stackrel{\text{df}}{=} \min_{\leq}\{\alpha \in \mathit{Enbld}_{\mathcal{NP}}(M) \mid \alpha \neq \varnothing\}$.

To start with, the set of *saturated localities* of a step α which is resource enabled at some marking M of a τ/ℓ-net $\mathcal{NP}$ is defined as:

$$\mathit{satlocalities}_M(\alpha) \stackrel{\text{df}}{=} \{l \in \ell(\alpha) \mid \neg\,\exists\, \alpha+\beta \in \mathit{enbld}_{\mathcal{NP}}(M) :\ l \in \ell(\beta)\}\ .$$

Intuitively, saturated localities are those which have been 'active' during the execution of a step α. It is immediate to see that if $\alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)$ then, for all $t \in \alpha$:

$$\mathit{satlocalities}_M(\alpha) \cap \ell(t) \neq \varnothing\ .$$

Moreover, the above intersection may contain more than one active localities which are 'responsible' for the execution of transition t. At the level of potential clusters of a step transition system (i.e., groups of transitions which share a locality), we can define *saturated clusters* in a state q as:

$$\mathit{satclusters}_q(\alpha) \stackrel{\text{df}}{=} \{C \subseteq T_q \mid \alpha|_C \neq \varnothing \ \wedge\ \forall\, \alpha+\beta \in \mathit{allSteps}_q :\ \beta|_C = \varnothing\}\ .$$

Note that if $\mathcal{T}$ is the concurrent reachability graph of a τ/ℓ-net $\mathcal{NP}$, and M is a reachable marking of $\mathcal{NP}$, then:

$$\{\ \{t \in T_M \mid l \in \ell(t)\}\ \mid\ l \in \mathit{satlocalities}_M(\alpha)\ \} \subseteq \mathit{satclusters}_M(\alpha)\ .$$

The following definition is our first attempt to generalise the notion of nets with localised conflicts investigated in [12].

Definition 6 (localised conflicts). *A τ/ℓ-net system $\mathcal{NP}$ has* partially localised conflicts *if for all reachable markings M and non-empty steps α belonging to $\mathit{enbld}_{\mathcal{NP}}(M)$,*

$$t \in \mathit{enbld}_{\mathcal{NP}}(M) \quad \textit{and} \quad \alpha + t \notin \mathit{enbld}_{\mathcal{NP}}(M)$$

implies $\mathit{satlocalities}_M(\alpha) \neq \varnothing$ and

$$\forall\, l \in \ell(t) \cap \mathit{satlocalities}_M(\alpha)\ :\ \alpha|_l + t \notin \mathit{enbld}_{\mathcal{NP}}(M)\ .$$

Intuitively, if there is a (global) conflict between a transition and a step, then this conflict can also be witnessed locally. We will now be concerned with the synthesis problem aimed at constructing τ/ℓ-net systems with partially localised conflicts.

Proposition 4. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking.*
If $\alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)$ then $\alpha|_l \in \mathit{Enbld}_{\mathcal{NP}}(M)$, for all $l \in \mathit{satlocalities}_M(\alpha)$.

Proof. Suppose that $\widetilde{l} \in \mathit{satlocalities}_M(\alpha)$ and $\alpha|_{\widetilde{l}} \notin \mathit{Enbld}_{\mathcal{NP}}(M)$.

Since $\alpha|_{\widetilde{l}} \leq \alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)$, we have $\alpha|_{\widetilde{l}} \in \mathit{enbld}_{\mathcal{NP}}(M)$. Hence there is $\widetilde{t} \in \alpha|_{\widetilde{l}}$ such that, for all $l \in \ell(\widetilde{t})$, there is $\alpha|_{\widetilde{l}} + t' \in \mathit{enbld}_{\mathcal{NP}}(M)$ with $l \in \ell(t')$. In particular, since $\widetilde{l} \in \ell(\widetilde{t})$, there is $\alpha|_{\widetilde{l}} + \widehat{t} \in \mathit{enbld}_{\mathcal{NP}}(M)$ such that $\widetilde{l} \in \ell(\widehat{t})$. Hence $\widehat{t} \in \mathit{enbld}_{\mathcal{NP}}(M)$, and so we can use Definition 6 to infer that $\alpha + \widehat{t} \in \mathit{enbld}_{\mathcal{NP}}(M)$, producing a contradiction with $\widetilde{l} \in \mathit{satlocalities}_M(\alpha)$. $\square$

Thus in terms of selecting clusters in the construction outlined in the previous section, if C has been selected at a state q then, for every α such that $C \in \mathit{satclusters}_q(\alpha)$, it must be the case that $\alpha|_C \in \mathit{allSteps}_q$.

Proposition 5. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking.*
If $\alpha \in \mathit{Enbld}^{min}_{\mathcal{NP}}(M)$ then $\alpha = \alpha|_l$, for all $l \in \mathit{satlocalities}_M(\alpha)$.

Proof. By Proposition 4, we have $\alpha|_l \in \mathit{Enbld}_{\mathcal{NP}}(M)$, and by definition of $\alpha|_l$, we have that $\alpha|_l \leq \alpha$. Moreover, $\alpha|_l \leq \alpha$ and $\alpha|_l \neq \varnothing$ (as $l \in \ell(\alpha)$). Hence, as α is a minimal non-empty step in $\mathit{Enbld}_{\mathcal{NP}}(M)$, we have $\alpha = \alpha|_l$. $\square$

Thus in terms of selecting clusters in the construction outlined in the previous section, if C has been selected at a state q then, for every $\alpha \in \mathit{minSteps}_q$ such that $C \in \mathit{satclusters}_q(\alpha)$, it must be the case that $\alpha|_C = \alpha$.

We will now present a series of results which can all be useful in the selection of clusters in the construction outlined in the previous section.

Proposition 6. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. Then, for all $\alpha \in \mathit{Enbld}^{min}_{\mathcal{NP}}(M)$:*

$$\mathit{satlocalities}_M(\alpha) \subseteq \bigcap\{\ell(t) \mid t \in \alpha\} .$$

Proof. Let $l \in \mathit{satlocalities}_M(\alpha)$. By Proposition 5, we have $\alpha = \alpha|_l$. Hence $l \in \ell(t)$, for all $t \in \alpha$. $\square$

Corollary 1. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. Then, for all $\alpha \in \mathit{Enbld}^{min}_{\mathcal{NP}}(M)$:*

$$\bigcap\{\ell(t) \mid t \in \alpha\} \neq \varnothing .$$

Proof. By $\alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)$, $\mathit{satlocalities}_M(\alpha) \cap \ell(t) \neq \varnothing$, for every $t \in \alpha$. Hence $\mathit{satlocalities}_M(\alpha) \neq \varnothing$ and the result follows from Proposition 6. $\square$

We will now need the following auxiliary fact.

Proposition 7. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. If $\alpha, \beta \in \mathit{Enbld}_{\mathcal{NP}}(M)$ and $\alpha \leq \beta$ then $\mathit{satlocalities}_M(\alpha) \subseteq \mathit{satlocalities}_M(\beta)$.*

Proof. Suppose $l \in \mathit{satlocalities}_M(\alpha) \setminus \mathit{satlocalities}_M(\beta)$.
From $l \in \mathit{satlocalities}_M(\alpha)$ we have that $l \in \ell(\alpha)$ and:

$$\forall t: \; l \in \ell(t) \implies \alpha + t \notin \mathit{enbld}_{\mathcal{NP}}(M) \; . \tag{2}$$

From $l \notin \mathit{satlocalities}_M(\beta)$ we have that either $l \notin \ell(\beta)$, or $l \in \ell(\beta)$ and there is $\widetilde{t}$ such that:

$$l \in \ell(\widetilde{t}) \quad \wedge \quad \beta + \widetilde{t} \in \mathit{enbld}_{\mathcal{NP}}(M) \; . \tag{3}$$

Only the latter is possible, because $l \in \ell(\alpha)$ and $\alpha \leq \beta$. From (2) and (3) we have that $\alpha + \widetilde{t} \notin \mathit{enbld}_{\mathcal{NP}}(M)$ and $\beta + \widetilde{t} \in \mathit{enbld}_{\mathcal{NP}}(M)$, which produces a contradiction with $\alpha + \widetilde{t} \leq \beta + \widetilde{t}$. $\square$

Proposition 8. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. Moreover, let $\alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)$ and $\widetilde{l} \in \mathit{satlocalities}_M(\alpha)$ be such that $\alpha|_l \not< \alpha|_{\widetilde{l}}$, for all $l \in \mathit{satlocalities}_M(\alpha) \setminus \{\widetilde{l}\}$.*
Then $\alpha|_{\widetilde{l}} \in \mathit{Enbld}^{min}_{\mathcal{NP}}(M)$.

Proof. From Proposition 4 we have that $\alpha|_{\widetilde{l}} \in \mathit{Enbld}_{\mathcal{NP}}(M)$. Suppose there is a non-empty step $\beta \in \mathit{Enbld}_{\mathcal{NP}}(M)$ such that $\beta < \alpha|_{\widetilde{l}}$. From Proposition 7 and $\alpha|_{\widetilde{l}} \leq \alpha$, it follows that

$$\mathit{satlocalities}_M(\beta) \subseteq \mathit{satlocalities}_M(\alpha|_{\widetilde{l}}) \subseteq \mathit{satlocalities}_M(\alpha) \; .$$

As $\beta \neq \varnothing$ and $\beta \in \mathit{Enbld}_{\mathcal{NP}}(M)$ we obtain

$$\mathit{satlocalities}_M(\beta) \neq \varnothing \; .$$

We have that $\widetilde{l} \notin \mathit{satlocalities}_M(\beta)$ as $\beta < \alpha|_{\widetilde{l}}$. Let $\widehat{l} \in \mathit{satlocalities}_M(\beta)$. Since $\mathit{satlocalities}_M(\beta) \subseteq \mathit{satlocalities}_M(\alpha)$, we have $\widehat{l} \in \mathit{satlocalities}_M(\alpha)$. Hence, by Proposition 4, $\alpha|_{\widehat{l}} \in \mathit{Enbld}_{\mathcal{NP}}(M)$. By the assumption we made, $\alpha|_{\widehat{l}} \not< \alpha|_{\widetilde{l}}$ as $\widehat{l} \neq \widetilde{l}$. So, there is $\widehat{t}$ such that $\alpha|_{\widehat{l}}(\widehat{t}) \geq \alpha|_{\widetilde{l}}(\widehat{t}) > \beta(\widehat{t})$, producing a contradiction with $\widehat{l} \in \mathit{satlocalities}_M(\beta)$. $\square$

Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. Then:

$$\mathit{maxind}^M_t \stackrel{\mathrm{df}}{=} \max\{\alpha(t) \mid \alpha \in \mathit{Enbld}_{\mathcal{NP}}(M)\} \; ,$$

for every net transition t which is resource enabled at M.

Proposition 9. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. Moreover, let t and u be distinct transitions which are resource enabled at M and share a locality $\widetilde{l}$. Then exactly one of the following holds:*

- *There is no step $\alpha \in Enbld_{\mathcal{NP}}(M)$ such that $\widetilde{l} \in satlocalities_M(\alpha)$ and*

$$maxind_t^M + maxind_u^M = \alpha(t) + \alpha(u) \tag{4}$$

and, for all $l \in satlocalities_M(\alpha) \setminus \{\widetilde{l}\}$, we have $\alpha|_l \not< \alpha|_{\widetilde{l}}$.
- *There is $\alpha \in Enbld_{\mathcal{NP}}^{min}(M)$ such that $t, u \in \alpha$.*

Proof. We have $\widetilde{l} \in \ell(t) \cap \ell(u)$. Suppose that there is $\alpha \in Enbld_{\mathcal{NP}}(M)$ such that (4) holds and $\widetilde{l} \in satlocalities_M(\alpha)$ and for all $l \in satlocalities_M(\alpha) \setminus \{\widetilde{l}\}$, $\alpha|_l \not< \alpha|_{\widetilde{l}}$. Since t and u are resource enabled at M and (4) holds, we have $\alpha(t) \geq 1$ and $\alpha(u) \geq 1$. On the other hand, $\widetilde{l} \in \ell(t) \cap \ell(u)$. Then, $t, u \in \alpha|_{\widetilde{l}}$. We can see that all the conditions of Proposition 8 are satisfied for α and $\widetilde{l}$, and so we obtain that $\alpha|_{\widetilde{l}} \in Enbld_{\mathcal{NP}}^{min}(M)$. □

Unique and minimal step covers It is not possible to reverse the inclusion in Proposition 6, i.e., there can be $\alpha \in Enbld_{\mathcal{NP}}^{min}(M)$ such that:

$$\bigcap\{\ell(t) \mid t \in \alpha\} \subseteq satlocalities_M(\alpha)$$

does no hold. As an example, we can take a PT-net with two concurrent transitions, a and b, each having one pre-place marked with a single token and no post-places, satisfying $\ell(a) = \{l, l'\}$ and $\ell(b) = \{l\}$. Then the step $\alpha = \{a\}$ belongs to $Enbld_{\mathcal{NP}}^{min}(M_0)$ yet:

$$\bigcap\{\ell(t) \mid t \in \alpha\} = \{l, l'\} \not\subseteq \{l'\} = satlocalities_{M_0}(\alpha) \ .$$

Looking at the last example, one can make a comment about the advantages of allowing a single transition to have more than one locality. In such a situation, different localities can define different modes of engagement/co-operation. For transition a, the locality l could be interpreted as defining a 'co-operative mode', while l' a 'self-sufficient' mode. In this way, some localities may force big sets of transitions to work in synchrony, while other localities may allow smaller sets to be synchronised, or even single transitions to be executed alone. Intuitively, we can model different 'circles of co-operations' for net transitions.

If we take again the last two transitions, and this time consider the step $\{a, b\}$, then one may observe that there is certain ambiguity as to which localities have been active during its execution, as both $L = \{l\}$ and $L' = \{l, l'\}$ could be taken. We will now investigate the role of such sets of localities.

Definition 7 (step covers). *A* (locality) cover *of a step α is a set of localities L such that:*

$$supp(\alpha) = \bigcup\{supp(\alpha|_l) \mid l \in L\} \ ,$$

and it is minimal *if no proper subset of L is a locality cover of α. Moreover, a minimal locality cover L is* unique *if there is no other minimal locality cover L' for α such that $\{\alpha|_l \mid l \in L\} = \{\alpha|_{l'} \mid l' \in L'\}$.*

Example 1. Let $\ell(a) = \{l, l'\}$ and $\ell(b) = \{l\}$. Then $L = \{l, l'\}$ is not a minimal cover for $\alpha = \{a\}$ as $L' = \{l'\}$ is also a cover.

Example 2. Let $\ell(a) = \ell(b) = \{l, l'\}$. Then there are two minimal covers for $\alpha = \{a, b\}$, $L = \{l\}$ and $L' = \{l'\}$, which are not unique.

Example 3. Let $\ell(a) = \{l, l'\}$, $\ell(b) = \{l\}$ and $\ell(c) = \{l'\}$. Then $L = \{l, l'\}$ is a unique cover for $\alpha = \{a, b, c\}$.

Example 4. Let $\ell(a) = \{l_1, l_3\}$, $\ell(b) = \{l_1, l_4\}$, $\ell(c) = \{l_2, l_3\}$ and $\ell(d) = \{l_2, l_4\}$. Then $\alpha = \{a, b, c, d\}$ has two unique minimal covers: $L = \{l_1, l_2\}$ and $L' = \{l_3, l_4\}$.

Equipped with the concept of a unique minimal cover, we can reverse the inclusion in Proposition 6.

Proposition 10. *Let $\mathcal{NP}$ be a τ/ℓ-net system with partially localised conflicts and M be its reachable marking. If $\alpha \in Enbld^{min}_{\mathcal{NP}}(M)$ and all its minimal covers are unique, then:*

$$satlocalities_M(\alpha) = \bigcap\{\ell(t) \mid t \in \alpha\} .$$

Proof. We need to show the ($\supseteq$) inclusion as the opposite one follows from Proposition 6. Suppose that $\widetilde{l} \in \bigcap\{\ell(t) \mid t \in \alpha\} \setminus satlocalities_M(\alpha)$.

From $\widetilde{l} \in \bigcap\{\ell(t) \mid t \in \alpha\}$ we have $\alpha = \alpha|_{\widetilde{l}}$. Since $\alpha \in Enbld^{min}_{\mathcal{NP}}(M)$, we have $satlocalities_M(\alpha) \cap \ell(t) \neq \varnothing$, for all $t \in \alpha$. Suppose $\widehat{l} \in satlocalities_M(\alpha) \cap \ell(\widetilde{t})$, for some $\widetilde{t} \in \alpha$. Notice that $\widehat{l} \neq \widetilde{l}$ as $\widetilde{l} \notin satlocalities_M(\alpha)$. From Proposition 4, $\alpha|_{\widehat{l}} \in Enbld_{\mathcal{NP}}(M)$.

We have $\alpha|_{\widehat{l}} \leq \alpha = \alpha|_{\widetilde{l}}$. We cannot have $\alpha|_{\widehat{l}} = \alpha|_{\widetilde{l}}$, because then both $\{\widetilde{l}\}$ and $\{\widehat{l}\}$ would be two different minimal covers for α, and so neither of them be unique. Hence $\alpha|_{\widehat{l}} < \alpha$, producing a contradiction with the minimality of α. □

Thus in terms of selecting clusters in the construction outlined earlier on, if C has been selected at a state q then, for every $\alpha \in minSteps_q$, we must have $C \in satclusters_q(\alpha)$ iff $\alpha|_C = \alpha$.

5 Concluding remarks

In this paper, we only initiated the investigation of intricate relationships between localities, conflicts and step covers. In the future research we plan to develop stronger results on this topic, aiming at an efficient synthesis procedure of τ-nets with localities with unknown locality mappings.

Acknowledgement We would like to thank Piotr Chrzastowski-Wachtel for his suggestion to investigate nets with overlapping localities. We are also grateful to the reviewers for their detailed and helpful comments. This research was supported by the RAE&EPSRC DAVAC VERDAD projects, and NSFC Grants 60910004 and 2010CB328102.

References

1. Badouel, E., Bernardinello, L., Darondeau, Ph.: The Synthesis Problem for Elementary Net Systems is NP-complete. Theoretical Computer Science **186** (1997) 107–134
2. Badouel, E., Darondeau, Ph.: Theory of Regions. In: Reisig, W., Rozenberg, G. (eds.): Lectures on Petri Nets I: Basic Models, Advances in Petri Nets. Lecture Notes in Computer Science **1491**. Springer-Verlag, Berlin Heidelberg New York (1998) 529–586
3. Bernardinello, L.: Synthesis of Net Systems In: Marsan, M.A. (ed.): Application and Theory of Petri Nets 1993. Lecture Notes in Computer Science **691**. Springer-Verlag, Berlin Heidelberg New York (1993) 89–105
4. Darondeau, P., Koutny, M., Pietkiewicz-Koutny, M., Yakovlev, A.: Synthesis of Nets with Step Firing Policies. Fundamenta Informaticae **94** (2009) 275–303
5. Desel, J., Reisig, W.: The Synthesis Problem of Petri Nets. Acta Informatica **33** (1996) 297–315
6. Dasgupta, S., Potop-Butucaru, D., Caillaud, B., Yakovlev, A.: Moving from Weakly Endochronous Systems to Delay-Insensitive Circuits. Electronic Notes in Theoretical Computer Science **146** (2006) 81–103
7. Desel, J., Reisig, W.: The Synthesis Problem of Petri Nets. Acta Informatica **33** (1996) 297–315
8. Kleijn, J., Koutny, M.: Processes of Membrane systems with Promoters and Inhibitors. Theoretical Computer Science **404** (2008) 112–126
9. Kleijn, H.C.M., Koutny, M., Rozenberg, G.: Towards a Petri Net Semantics for Membrane Systems. In: Freund, R., Paun, G., Rozenberg, G., Salomaa, A. (eds.): WMC 2005. Lecture Notes in Computer Science **3850**. Springer-Verlag, Berlin Heidelberg New York (2006) 292–309
10. Koutny, M., Pietkiewicz-Koutny, M.: Transition Systems of Elementary Net Systems with Localities. In: Baier, C., Hermanns, H. (eds.): CONCUR 2006. Lecture Notes in Computer Science **4137**. Springer-Verlag, Berlin Heidelberg New York (2006) 173–187
11. Koutny, M., Pietkiewicz-Koutny, M.: Synthesis of Elementary Net Systems with Context Arcs and Localities. Fundamenta Informaticae **88** (2008) 307–328
12. Koutny, M., Pietkiewicz-Koutny, M.: Synthesis of Petri Nets with Localities. Scientific Annals of Computer Science **19** (2009) 1–23
13. Mukund, M.: Petri Nets and Step Transition Systems. International Journal of Foundations of Computer Science **3** (1992) 443–478
14. Nielsen, M., Rozenberg, G., Thiagarajan, P.S.: Elementary transition systems. Theoretical Compututer Science **96** (1992) 3–33
15. Păun, G.: Membrane Computing, An Introduction. Springer-Verlag, Berlin Heidelberg New York (2002)
16. Pietkiewicz-Koutny, M.: The Synthesis Problem for Elementary Net Systems with Inhibitor Arcs. Fundamenta Informaticae **40** (1999) 251–283