Bi-fuzziness, Incompleteness, Inconsistency, Truth and Falsity Based on Saturation and Ignorance Functions. A New Approach of Penta-Valued Knowledge Representation

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Abstract

This paper presents a new five-valued knowledge representation of bipolar information. This representation is related to a five-valued logic that uses two logical values of truth (true, false) and three logical values of uncertainty (incomplete, inconsistent and fuzzy). The new approach is based on the concept of saturation function and ignorance function. In the framework of five-valued representation new formulae for union and intersection are constructed. Also, the paper presents a short application related to fuzzy preference modeling and decision making.

Introduction

Let X be a set of objects. We consider a property A, an object $x \in X$ and the following sentence $P_A(x)$: x has the property A. We want to know if the sentence $P_A(x)$ is true or false. After an evaluation, the information about logical value of sentence $P_A(x)$ is described by a scalar $T_A(x) \in [0,1]$. For the considered sentence, $T_A(x)$ represents its truth degree. In the same time, the function $T_A : X \to [0,1]$ defines a Zadeh fuzzy set associated to the property A (Zadeh 1965). Then, we compute the degree of falsity:

$$F_A(x) = 1 - T_A(x) \tag{1}$$

Using the scalar $T_A(x)$, we have obtained the following representation of information about sentence $P_A(x)$.

$$W_A(x) = (T_A(x), F_A(x)) \tag{2}$$

This information is normalized because the components of vector $W_A(x)$ verify the condition of partition of unity:

$$T_A(x) + F_A(x) = 1 \tag{3}$$

The representation (3) is related to a bi-valued logic based on true and false. The next step was done by Atanassov (Atanassov 1986). He considered that after evaluation, the information about logical value of sentence $P_A(x)$ is described by a vector with two components

$$V_A(x) = \left(T_A(x), F_A(x)\right) \tag{4}$$

and supplementary these two components verify the inequality:

$$T_A(x) + F_A(x) \le 1 \tag{5}$$

The information represented by vector $V_A(x)$ is not normalized but, Atanassov has introduced the intuitionistic index:

$$U_A(x) = 1 - T_A(x) - F_A(x) \tag{6}$$

Using the vector $V_A(x)$, we have obtained an intuitionistic representation of information about sentence $P_A(x)$.

$$W_A(x) = (T_A(x), U_A(x), F_A(x)) \tag{7}$$

This information is normalized because the components of vector $W_A(x)$ verify the condition of partition of unity:

$$T_A(x) + U_A(x) + F_A(x) = 1 \tag{8}$$

The representation (8) is related to a three-valued logic based on true, neutral and false.

In this paper we will consider the bipolar representation (Benferhat et al. 2006; Cornelis et al. 2003; Dubois et al. 2004) without having the condition (5). In this case, we cannot obtain immediately a normalized variant like (8). In the following, we present a method for obtaining a normalized representation of bipolar information.

The paper has the following structure: section two presents the concepts of saturation, ignorance and bi-fuzziness. Section three presents the construction method of five-valued representation. Section four presents a five-valued logic based on true, false, incomplete, inconsistent and fuzzy. Section five presents some operators for the five-valued structure. Section six presents the using of five-valued knowledge representation for fuzzy modeling of pairwise comparisons. Finally we present some conclusions.

Saturation, Ignorance and Bi-fuzziness Functions

In this section, firstly, we introduce the concepts of saturation function and ignorance function. These two functions are complementary. Both functions are essentially characterized by symmetry, boundary and monotonicity properties. Secondly, we introduce the concept of bi-fuzziness related to the index of indeterminacy (Patrascu 2008).

Definition 1: A saturation function is a mapping $S:[0,1]^2 \to [0,1]$ such that:

i) $S(x,y) = S(y,x)$
ii) $S(x,y) = 0$ if and only if $(x,y) = (0,0)$
iii) $S(x,y) = 1$ if and only if $(x,y) = (1,1)$
iv) $S(x,y)$ increases with respect to x and y

The property *a*) describes the commutativity and the property *d*) describes the monotonicity. From property *b)* it results that the saturation value is low if and only if both arguments have low value and from property *c)* it results that the saturation value is high if and only if both arguments have high value.

Example 1:

$$S(x,y) = \frac{x+y}{2}.$$

Example 2:

$$S(x,y) = \frac{\max(x,y)}{1+|x-y|}.$$

Example 3: For any *t-conorm* $\oplus$

$$S(x,y) = \frac{x \oplus y}{1+(1-x)\oplus(1-y)}.$$

Example 4: For any *t-conorm* $\oplus$

$$S(x,y) = \frac{x \oplus y}{x \oplus y + (1-x)\oplus(1-y)}.$$

Example 4:

$$S(x,y) = \frac{x+y}{2} + \frac{1-x-y}{2}|x-y|.$$

Notice that these particular saturation functions are not associative.

Definition 2: A ignorance function is a mapping $U:[0,1]^2 \to [0,1]$ such that:

i) $U(x,y) = U(y,x)$
ii) $U(x,y) = 0$ if and only if $(x,y) = (1,1)$
iii) $U(x,y) = 1$ if and only if $(x,y) = (0,0)$
iv) $U(x,y)$ decreases with respect to x and y

Example 1:

$$U(x,y) = 1 - \frac{x+y}{2}.$$

Example 2:

$$U(x,y) = \frac{1-\min(x,y)}{1+|x-y|}.$$

The following proposition shows the relation between saturation functions and ignorance functions and some supplementary properties.

Proposition 1: Let S be a saturation function. Then

$$U(x,y) = S(1-x,1-y) \tag{9}$$

is an ignorance function.

Proof: It is evident because the properties of both functions are complementary.

Proposition 2: Let U be an ignorance function. Then

$$S(x,y) = 1 - U(x,y) \tag{10}$$

is a saturation function.

Proof: It is evident because the properties of both functions are complementary.

Proposition 3: Let S be a saturation function let $\lambda \in (0,1)$. Then

$$P(x,y) = \frac{\lambda \cdot S(x,y)}{\lambda \cdot S(x,y) + (1-\lambda)\cdot(1-S(x,y))} \tag{11}$$

is a saturation function.

Proof: It is evident because in the new saturation function construction it was used the scalar addition based on the uninorm function.

Proposition 4: Let S be a saturation function let $\alpha \in (0,\infty)$. Then

$$Q(x,y) = \frac{S^{\alpha}(x,y)}{S^{\alpha}(x,y) + (1-S(x,y))^{\alpha}} \tag{12}$$

is a saturation function.

Proof: It is evident because in the new saturation function construction it was used the scalar multiplication based on the uninorm function.

Proposition 5: Let S be a saturation function. Then

$$R(x,y)=\frac{S(x,y)}{S(x,y)+S(1-x,1-y)} \quad (13)$$

is a saturation function.
Proof: It is results immediately that the new saturation function verifies the properties i), ii), iii) and iv).

Definition 3: A bi-fuzziness function is a mapping $I:[0,1]^2 \to [0,1]$ such that:

i) $I(x,y)=I(y,x)$
ii) $I(x,y)=I(1-x,y)$
iii) $I(x,y)=I(x,1-y)$
iv) $I(x,y)=0$ if and only if $x,y\in\{0,1\}$
v) $I(x,y)=1$ if and only if $(x,y)=(0.5,0.5)$
vi) $I(x,y)$ increases with x if $x\le 0,5$ and $I(x,y)$ decreases with x if $x\ge 0,5$
vii) $I(x,y)$ increases with y if $y\le 0,5$ and $I(x,y)$ decreases with y if $y\ge 0,5$

The bi-fuzziness function represents a measure of similarity between the point $(x,y)\in[0,1]^2$ and the center of unit square, the point $(0.5,0.5)$. The index of bi-fuzziness verifies, for each argument x and y, the properties considered by De Luca and Termini for fuzzy entropy definition (De Luca and Termini 1972).
If we replace y with the negation of x, namely $y=\bar{x}=1-x$, one obtains a fuzzy entropy function.

Proposition 6: Let S be a saturation function. Then

$$I(x,y)=(1-|S(x,\bar{y})-S(\bar{x},y)|)\cdot(1-|S(x,y)-S(\bar{x},\bar{y})|)$$

is a bi-fuzziness function.

Proposition 7: Let S be a saturation function. Then

$$I(x,y)=1-S(|2x-1|,|2y-1|)$$

is a bi-fuzziness function.

Example 1:

$$I(x,y)=1-|x-0.5|-|y-0.5|.$$

Example 2:

$$I(x,y)=\frac{(1-|x-y|)\cdot(1-|x+y-1|)|}{1-|x-y|\cdot|x+y-1|}.$$

Example 3:

$$I(x,y)=(1-|x-y|)(1-|x+y-1|).$$

Five-Valued Representation of Bipolar Information

Let S be a saturation function. For any pair (T,F), we define *the net truth* τ and *the definedness* δ by:

$$\tau(T,F)=S(T,\bar{F})-S(\bar{T},F)$$

$$\delta(T,F)=S(T,F)-S(\bar{T},\bar{F})$$

The uncertainty or the entropy (Kaufmann 1975; Patrascu 2010) is defined by:

$$h=1-|\tau| \quad (14)$$

and the certainty will be its negation:

$$g=|\tau|$$

The two functions define a partition with two fuzzy sets X_G and X_H: one related to the certainty and the other to the uncertainty.

The non-fuzziness id defined by:

$$z=|\delta| \quad (15)$$

The index of bi-fuzziness will be computed by difference between uncertainty and non-fuzziness:

$$i=1-|\tau|-|\delta| \quad (16)$$

The non-fuzziness and bi-fuzziness define two subsets of X_H, namely: X_Z and X_I. We compute the incompleteness (undefinedness) and inconsistency (contradiction) using the non-fuzziness:

$$u=\delta_- \quad (17)$$

$$c=\delta_+ \quad (18)$$

where $x_-=\max(0,-x)$ and $x_+=\max(0,x)$.

The incompleteness and inconsistency define two subsets of X_Z, namely: X_U and X_C. Notice that because $c\cdot u=0$ it results:

$$X_U\cap X_C=\Phi \quad (19)$$

Next we compute the index of truth and falsity using the net truth function τ:

$$t=\tau_+ \quad (20)$$

$$f=\tau_- \quad (21)$$

The index of truth and index of falsity define two subsets of X_G, namely: X_T and X_F. Notice that because $t\cdot f=0$ it results:

$$X_T \cap X_F = \Phi \tag{22}$$

The index of truth (20), the index of falsity (21), the index of bi-fuzziness (16), the index of incompleteness (17) and the index of inconsistency (18) define a partition of unity:

$$t + f + i + u + c = 1$$

In the construction method presented above, it was used the schema shown in figure 1.

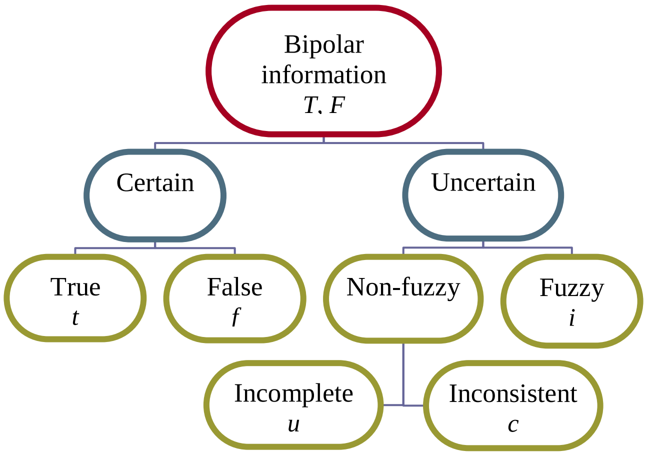

Figure 1. The construction schema for five-valued representation of bipolar information.

Five Valued Logic Based on Truth, Falsity, Inconsistency, Incompleteness and Bi-fuzziness

This five-valued logic is a new one, but is related to our previous work presented in (Patrascu 2008). In the framework of this logic we will consider the following five values: *true* t, *false* f, *incomplete* (*undefined*) u, *inconsistent* (*contradictory*) c, and *fuzzy (indeterminate)* i. We have obtained these five logical values, adding to the so called Belnap values (Belnap 1977) the fifth: *fuzzy* (*indeterminate*). Tables 1, 2, 3, 4, 5, 6 and 7 show the basic operators in this logic.

Table 1. The union.

$\cup$	t	c	i	u	f
t	t	t	t	t	t
c	t	c	i	i	c
i	t	i	i	i	i
u	t	i	i	u	u
f	t	c	i	u	f

Table 2. The intersection.

$\cap$	t	c	i	u	f
t	t	c	i	u	f
c	c	c	i	i	f
i	i	i	i	i	f
u	u	i	i	u	f
f	f	f	f	f	f

The main differences between the proposed logic and the Belnap logic are related to the logical values u and c. We have defined $c \cap u = i$ and $c \cup u = i$ while in the Belnap logic there were defined $c \cap u = f$ and $c \cup u = t$.

Table 3. The complement.

	$\neg$
t	f
c	c
i	i
u	u
f	t

Table 4. The negation.

	$-$
t	f
c	u
i	i
u	c
f	t

Table 5. The dual.

	$\approx$
t	t
c	u
i	i
u	c
f	f

The complement, the negation and the dual are interrelated and there exists the following equalities:

$$\approx x = -\neg x \tag{23}$$

$$\neg x = - \approx x \tag{24}$$

$$-x = \neg \approx x \tag{25}$$

Table 6. The S-implication

→	t	c	i	u	f
t	t	c	i	u	f
c	t	c	i	i	c
i	t	i	i	i	i
u	t	i	i	u	u
f	t	t	t	t	t

The *S-implication* is calculated by:

$$x \to y = \neg x \bigcup y \qquad (26)$$

Table 7. The equivalence

↔	t	c	i	u	f
t	t	c	i	u	f
c	c	c	i	i	c
i	i	i	i	i	i
u	u	i	i	u	u
f	f	c	i	u	t

The *equivalence* is calculated by:

$$x \leftrightarrow y = (\neg x \bigcup y) \bigcap (x \bigcup \neg y) \qquad (27)$$

New Operators Defined on Five-Valued Structure

There be $x = (t, c, i, u, f) \in [0,1]^5$, For this kind of vectors, one defines the union, the intersection, the complement, the negation and the dual operators. The operators are related to those define in (Patrascu 2007a; Patrascu 2007b).

The Union: For two vectors $a, b \in [0,1]^5$ where $a = (t_a, c_a, i_a, u_a, f_a)$, $b = (t_b, c_b, i_b, u_b, f_b)$, one defines the union (disjunction) $d = a \cup b$ by the formula:

$$\begin{aligned} t_d &= t_a \vee t_b \\ c_d &= (c_a + f_a) \wedge (c_b + f_b) - f_a \wedge f_b \\ u_d &= (u_a + f_a) \wedge (u_b + f_b) - f_a \wedge f_b \\ f_d &= f_a \wedge f_b \\ i_d &= 1 - (t_d + c_d + u_d + f_d) \end{aligned} \qquad (28)$$

The Intersection: For two vectors $a, b \in [0,1]^5$ one defines the intersection (conjunction) $c = a \cap b$ by the formula:

$$\begin{aligned} t_c &= t_a \wedge t_b \\ c_c &= (c_a + t_a) \wedge (c_b + t_b) - t_a \wedge t_b \\ u_c &= (u_a + t_a) \wedge (u_b + t_b) - t_a \wedge t_b \\ f_c &= f_a \vee f_b \\ i_c &= 1 - (t_c + c_c + u_c + f_c) \end{aligned} \qquad (29)$$

In formulae (28) and (29), the symbols "$\vee$" and "$\wedge$" represent the maximum and the minimum, namely:
$\forall x, y \in [0,1]$,

$$x \vee y = \max(x, y)$$
$$x \wedge y = \min(x, y)$$

The union "$\cup$" and intersection "$\cap$" operators preserve de properties $t + c + u + f \le 1$, $t \cdot f = 0$ and $u \cdot c = 0$, namely:

$$t_{a\cup b} + c_{a\cup b} + u_{a\cup b} + f_{a\cup b} \le 1$$
$$t_{a\cup b} \cdot f_{a\cup b} = 0$$
$$c_{a\cup b} \cdot u_{a\cup b} = 0$$

$$t_{a\cap b} + c_{a\cap b} + u_{a\cap b} + f_{a\cap b} \le 1$$
$$t_{a\cap b} \cdot f_{a\cap b} = 0$$
$$c_{a\cap b} \cdot u_{a\cap b} = 0$$

We remark that after union or intersection the certainty increases and uncertainty decreases.

The Complement: For $x = (t, c, i, u, f) \in [0,1]^5$ one defines the complement x^c by formula:

$$x^c = (f, c, i, u, t) \qquad (30)$$

The Negation: For $x = (t, c, i, u, f) \in [0,1]^5$ one defines the negation x^n by formula:

$$x^n = (f, u, i, c, t) \qquad (31)$$

The Dual: For $x = (t, c, i, u, f) \in [0,1]^5$ one defines the dual x^d by formula:

$$x^d = (t, u, i, c, f) \qquad (32)$$

In the set $\{0,1\}^5$ there are five vectors having the form $x = (t, c, i, u, f)$, which verify the condition $t + f + c + i + u = 1$: $T = (1,0,0,0,0)$ (*True*), $F = (0,0,0,0,1)$ (*False*), $C = (0,1,0,0,0)$ (*Inconsistent*), $U = (0,0,0,1,0)$ (*Incomplete*) and $I = (0,0,1,0,0)$ (*Fuzzy*).
Using the operators defined by (28), (29), (30), (31) and (32), the same truth table results as seen in Tables 1, 2, 3, 4, 5, 6 and 7.

Fuzzy Preference Relation in The Framework of Five-Valued Representation

A fuzzy preference relation A on a set of alternatives $X=\{x_1,x_2,...,x\}$ is a fuzzy set on the product set $X\times X$, that is characterized by a membership function $\mu_P : X\times X\to[0,1]$ (see Chiclana et al. 1998; Fodor et al. 1994; Tanino 1988). When cardinality of X is small, the preference relation may be represented by the $n\times n$ matrix $A=\{a_{ij}\}$ being $a_{ij}=\mu_A(x_i,x_j)$ $\forall i,j\in\{1,2,...,n\}$. a_{ij} is interpreted as the preference degree of the alternative x_i over x_j. From a preference relation A, Fodor and Roubens (Fodor 1994) derive the following three relations:

Strict preference: $p_{ij}=P(x_i,x_j)$ indicating that x_i preferred to x_j but x_j is not preferred to x_i.

Indifference: $i_{ij}=I(x_i,x_j)$ indicating that x_i and x_j are considered equal in the sense that x_i is as good as x_j.

Incomparability: $j_{ij}=J(x_i,x_j)$ which occurs if neither a_{ij} nor a_{ji}.

Taking into account the five-valued representation of bipolar information, we define five relations that characterize the following five fundamental attitudes:

Strict preference $t_{ij}=T(x_i,x_j)$ is a measure of strict preference of x_i over x_j, indicating that x_i preferred to x_j but x_j is not preferred to x_i.

Indifference: $c_{ij}=C(x_i,x_j)$ is a measure of the simultaneous fulfillment of a_{ij} and a_{ji}.

Incomparability: $u_{ij}=U(x_i,x_j)$ is a measure of the incomparability of x_i and x_j, which occurs if neither a_{ij} nor a_{ji}.

Strict aversion: $f_{ij}=F(x_i,x_j)$ that is a measure of strict preference of x_j over x_i, indicating that x_i is not preferred to x_j.

Undecidability: $i_{ij}=I(x_i,x_j)$ is a measure of undecidability between x_i and x_j which occurs when $a_{ij}\approx 0.5$ and $a_{ji}\approx 0.5$.

Next, we consider a decision making problem where, an expert supply the preferences over a set of n alternatives: $X=\{x_1,x_2,...x_n\}$. The preferences are represented by the following fuzzy relation:

$$A=\begin{bmatrix} 0 & a_{12} & ... & a_{1n} \\ a_{21} & 0 & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & ... & ... & 0 \end{bmatrix} \tag{38}$$

where $a_{ij}\in[0,1]$.

The algorithm that we propose to obtain the best alternative is the next:

Step 0: Initialize the matrix A and define the saturation function S.

Step 1: Compute the function t_{ij}, c_{ij}, u_{ij}, f_{ij} and i_{ij} using formulae (20), (21), (16), (17) and (18).

Step 2: Compute the relative score function by:

$$r_{ij}=\frac{t_{ij}+c_{ij}+0.5\cdot i_{ij}}{t_{ij}+2\cdot c_{ij}+1.5\cdot i_{ij}+u_{ij}+3\cdot f_{ij}} \tag{39}$$

Step 3: Compute the total score function by:

$$R_i=\sum_{\substack{j=1\\ j\neq i}}^{n} r_{ij} \tag{40}$$

Step 4: Choose

$$x_{optim}=\underset{k\in\{1,2,\ldots,n\}}{\arg\max}\{R_k\} \tag{41}$$

In the presented algorithm, the next five items hold:

If $a_{ij}=1$ and $a_{ji}=0$, then $r_{ij}=1$.

If $a_{ij}=1$ and $a_{ji}=1$, then $r_{ij}=0.5$.

If $a_{ij}=0.5$ and $a_{ji}=0.5$, then $r_{ij}=0.33$.

If $a_{ij}=0$, and $a_{ji}=0$, then $r_{ij}=0$.

If $a_{ij}=0$, and $a_{ji}=1$, then $r_{ij}=0$.

Numerical example: Let $X=\{x_1,x_2,x_3,x_4,x_5\}$ be a set of alternatives. Consider the fuzzy preference relation:

$$X=\begin{bmatrix} 0 & 0.06 & 0.42 & 0.84 & 0.50 \\ 0.20 & 0 & 0.85 & 0.02 & 0.70 \\ 0.60 & 0.45 & 0 & 0.68 & 0.43 \\ 0.27 & 0.93 & 0.20 & 0 & 0.30 \\ 0.20 & 0.47 & 0.67 & 0.83 & 0 \end{bmatrix} \tag{42}$$

If the saturation function is defined by

$$S(x,y)=\frac{x+y}{2},$$

it results:

$$t_{ij}=\left(a_{ij}-a_{ji}\right)_{+} \quad (43)$$

$$c_{ij}=\left(a_{ij}+a_{ji}-1\right)_{+} \quad (44)$$

$$u_{ij}=\left(1-a_{ij}-a_{ji}\right)_{+} \quad (45)$$

$$f_{ij}=\left(a_{ji}-a_{ij}\right)_{+} \quad (46)$$

$$i_{ij}=1-|a_{ij}-a_{ji}|-|a_{ij}+a_{ji}-1| \quad (47)$$

Using the presented algorithm one obtains:

$R_1=1.36$, $R_2=1.26$, $R_3=1.41$, $R_4=1.24$, $R_5=1.46$

It results $x_{optim}=x_5$.

Conclusions

In this paper, we propose a different functional approach to model the bipolar information. The new approach is based on two new information concepts: saturation function and ignorance function. Saturation function can be seen as way of generalizing t-conorms dropping out associativity. We must underline that the associativity is not crucial for the construction of five-valued representation. More than that, in our framework, the saturation function has only two arguments: the degree of truth and degree of falsity. Finally, we are dealing with a class of functions different from that of the t-conorms.

The saturation function measures the excess of information, while, the ignorance function measures the lack of information that an estimator suffers when trying to determine if a given sentence is true or false.

The third concept, bi-fuzziness function can be understood as an extension from fuzzy sets to bipolar fuzzy sets of the concept of fuzziness defined by Zadeh. In addition, the index of bi-fuzziness can be understood as a measure of partial uncertainty of bipolar information. Both saturation function and ignorance function are related. Each of them can be recovered in a functional way from the other.

If suitable saturation or ignorance functions are known that fit well for a given problem, they can be used to build a five-valued knowledge representation. In this way, we are able to provide a theoretical framework which is different from the usual one to represent truth, falsity, incompleteness, inconsistency and bi-fuzziness. In this framework, a new five-valued logic was presented based on five logical values: true, false, incomplete, inconsistent and fuzzy. It was identified two components for certainty and three components for uncertainty. Based on this logic, new union and intersection operators were defined for the existing five-valued structure of information.

We also propose an application in preferences under a novel score function. The using of the proposed five fundamental attitudes provides a new perspective in decision making and it offers a simple way to produce a comprehensive judgment.

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