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Author for correspondence:
Yan Chen
e-mail: yanchenm@umich.edu

# Oscillations in hyperasymptotic series

Yan Chen[1], and Harvey Segur[2]

[1]School of Information, University of Michigan, Ann Arbor, Michigan, US

[2]Department of Applied Mathematics, University of Colorado, Boulder, Colorado, US

The method of hyperasymptotic series was invented by Berry & Howls to approximate solutions of Schrödinger-type ordinary differential equations (ODEs) more accurately. We present here a variation on their method to find the hyperasymptotic series for the Airy function, $Ai(z)$, which solves an ODE of this type. Berry and Howls applied their method to exactly this problem; we analyze the same problem as they did in order to make clear how the two methods differ. Our most surprising result is that the hyperasymptotic series for $Ai(z)$ for $z > 0$ exhibits small oscillations, even though $Ai(z)$ is not oscillatory for $z > 0$. We show that these oscillations arise as a natural consequence of the formulation of a hyperasymptotic series.

## 1. Introduction

Students of calculus learn about infinite series, almost always in terms of *convergent series*. Less well known are *asymptotic series*, which can provide useful approximations of functions that might not have a convergent series representation. Asymptotic series go at least as far back as 1730, when James Stirling showed how to approximate $log(n!)$ with a divergent series that becomes increasingly accurate as $n$ increases [13]. In their early days asymptotic series (called *divergent series* at the time) were ill-defined, so some people used them appropriately and others did not. In 1828, Neils Abel wrote "*The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever.*" [1]

In 1886, Poincaré [10] and Stieltjes [12] published papers that provided a solid mathematical basis for asymptotic series. Here is Poincaré's definition of a divergent asymptotic series. A formal power series has the form



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$$\sum_{m=0}^{\infty} a_m \left(x - x_0\right)^m \ , \tag{1.1}$$

where $x$ is a variable, $x_0$ is a fixed number, and $\{a_m\}$ is an infinite sequence of known numbers or functions of other variables. If $x_0$ is infinite, then a power series take the form

$$\sum_{m=0}^{\infty} b_m x^{-m} \ , \tag{1.2}$$

and again $\{b_m\}$ denotes an infinite sequence of known numbers or functions of other variables. Denote a partial sum of the series in (1.2) by $S_M(x) = \sum_{m=0}^{M} b_m x^{-m}$. In the definition of Poincaré, the series in (1.2) is *asymptotic* to a function, $P(x)$, as $x \to \infty$ if for every positive integer $M < \infty$ ,

$$x^M \left|P(x) - S_M(x)\right| \to 0 \qquad as\, x \to \infty,\ M\, fixed, \tag{1.3}$$

and if

$$x^M \left|P(x) - S_M(x)\right| \to \infty \qquad as\, M \to \infty,\ x\, fixed, \tag{1.4}$$

In words, (1.3) guarantees that the error obtained in truncating the series at $(M+1)$ terms always tends to zero faster than the last term retained in the series, as $x \to \infty$. Meanwhile, (1.4) guarantees that the series in (1.2) does **not** converge to $P(x)$ for $x$ fixed, positive and large enough; i.e., the series is divergent.

Stieltjes [12] addressed the issue of how to use a divergent series like that in (1.2) to obtain the best possible approximation for $P(x)$, at least for $x > 0$. For the series in (1.2) to be asymptotic in the sense of Poincaré, the coefficients $b_m$ must grow in magnitude faster than any power of $m$. A common pattern is that the error in approximating $P(x)$ by $S_M(x)$ at fixed $x$,

$$\{P(x) - S_M(x)\} = \sum_{m=M+1}^{\infty} b_m x^{-m},$$

decreases as $M$ increases for small values of $M$, but eventually the error starts to grow (as it must because the series in (1.2) is divergent). For larger values of $x$, one goes to higher values of $M$ before the error starts to grow, and the minimal error is smaller for a larger value of $x$. Stieltjes showed that the optimal stopping procedure at a fixed $x$ is to take terms from the series as long as the error at that $x$ decreases with each additional term kept; once the error starts to increase, then drop the last term retained. The series is asymptotic, so the first term *not* kept is a bound on the total error at this $x$. By choosing to truncate the series so that the smallest term in the series is the first not kept, one obtains an approximation for $P(x)$ with a bound on the error that is as small as possible by this method.

Berry [4] points out that Stokes [14] had observed that the most accurate approximation in an asymptotic series is obtained by stopping "at the least term", and that Stokes' observation preceded the work of Poincaré and Stieltjes by nearly 40 years. Berry notes that the actual error obtained by stopping in this way is often much smaller than $x^{-M}$ (so algebraically small) – it is often on the order of $e^{-x}$ (so exponentially small).

A convergent Taylor series and a divergent asymptotic series are similar in that each gives a sequence of increasingly accurate approximations of a given function, by truncating an infinite series with an increasing but finite number of terms. An important difference arises in terms of increasing the accuracy of the approximation.

- A Taylor series is convergent, so a more accurate approximation for $P(x)$ can be obtained simply by keeping more terms in the convergent series.

- If one truncates a divergent asymptotic series for $P(x)$ at the optimal stopping point for the series at a fixed $x$, then this approximation is the most accurate one available from the series for this $x$. The error cannot be reduced further without going outside the framework of the asymptotic series.

In the 1980s, one century after the work of Poincaré and Stieltjes, several research groups sought reliable methods to quantify the (usually exponentially small) term that follows the smallest algebraic term in a formally asymptotic expansion of a given function. This work goes by several names: *asymptotics beyond all orders*, *exponential asymptotics*, *hyperasymptotics* and others. See [6] or [11] for discussions of some of these methods, and for their application to a variety of physical problems. The objective of most of these methods is to obtain an accurate representation of the first term beyond the optimal approximation obtained from the standard asymptotic series. This term is usually exponentially small (in $x$, as $x \to \infty$), which explains the titles of *exponential asymptotics* and of *asymptotics beyond all orders*.

The objective of Berry & Howls [5] in developing a *hyperasymptotic series* is more profound. Given a function, $P(x)$, that they seek to approximate near some limit (e.g., as $x \to \infty$), they imagine starting with a standard asymptotic series, and truncating it at the optimal stopping point for a fixed, large $x$. Then they changes variables, using a transformation developed earlier by Dingle [9], and find a new problem to solve for the error remaining from the original problem,$\{P(x) - S_M(x)\}$. Then they solve the new problem asymptotically in a series involving the new variable of Dingle. The series so obtained is still divergent, and they use Borel summation of the new series to assign meaning to this new, divergent series. Then they iterate this process: they keep terms (in terms of their new variable) of their new series until they reach its optimal stopping point, then change variables again, obtain a third equation for the much smaller error remaining from the second problem, and repeat. According to Berry [4], the error cannot be driven to zero by this approach, but it can be made much smaller. Their prototypical example problem is to find very accurate approximations of $Ai(z)$, for large positive $z$. In this paper, we present a variation on the method of Berry & Howls [5], which avoids some limitations of their approach.

The remainder of this paper is as follows. The problem in question is to find the solution of the differential equation of Airy [3],

$$\frac{d^2y}{dz^2} = z \cdot y(z), \tag{1.5}$$

that is bounded as $z \to \infty$ and that satisfies a given normalization condition. We seek a representation of this solution that is valid for all $z > 0$. Eq'n (1.5) is a special case of a one-dimensional Schrödinger-type equation,

$$\frac{d^2y(z,\lambda)}{dz^2} = \lambda^2 Z(z) \cdot y(z;\lambda), \tag{1.6}$$

where $Z(z)$ is a known function and $\lambda$ is a constant. In §2, we outline both the method of Berry & Howls [5] and our method to solve (1.5) for $z > 0$. Both methods begin with a transformation due to Dingle [9], which rewrites (1.5) in "better" variables ($z \to F(z)$, and $z > 0 \Leftrightarrow F > 0$). The two methods diverge at the next step. We solve the transformed version of (1.5) for $F > 0$ by standard means: recast the new differential equation as an integral equation, and construct a solution of the integral equation in terms of a series that converges absolutely for $F > 0$. For large $F > 0$, the first term in this convergent series dominates all the other terms, so we can analyze the series one term at a time. The first term is defined in terms of an integral over known functions, and in §3 we construct a formal asymptotic expansion for this first term by integrating by parts repeatedly. The result is an expansion that can be carried to arbitrarily high order, but always with an explicit error term, so there is no loss of information is constructing this series. We view this as an important ingredient in our approach.

- We approximate terms in the exact solution of the equation, rather than approximates the differential equation itself.
- Our approximation of each term in the solution has an explicit remainder, so there is never any question about the size of the truncation error.

Proceeding in this way, we arrive at the biggest surprise (to us) of this analysis: the series that we construct in this way exhibits small but clearly measurable oscillations. In §4, we analyze these oscillations, which seem not to have been observed in earlier work on this problem, but which arise naturally as a consequence of the structure of a hyperasymptotic series. These oscillations arise in the analysis of the first term in the convergent series of the solution. In §5, we analyze some of the higher order terms in the convergent series, and show that they amplify the oscillations, so the oscillations are not restricted to the first term in the series. We expect to see such oscillations in any approximate solution of a differential equation in which one goes beyond the optimal truncation of a standard asymptotic expansion. Our conclusions are summarized in §6.

## 2. Recurrent series to solve the Airy equation

For $\lambda >> 1$, standard WKB methods provide the dominant terms of two solutions of (1.6):

$$y_{\pm}(z,\lambda) \sim \frac{exp\left(\pm\lambda\int_{z^*}^{z} Z^{\frac{1}{2}}(\zeta)d\zeta\right)}{Z^{\frac{1}{4}}(z)}, \tag{2.1}$$

where $z^*$ is an arbitrary reference point. This was the starting point for the work Dingle [9], which motivated later work by Berry & Howls [5]. Dingle viewed the divergence of the series that follow these dominant terms as a natural consequence of the fact that either solution in (2.1) makes use of only one of the two roots of $Z(z)$. Because of this, Dingle introduced a new independent variable,

$$F(z) := 2\lambda\int_{z^*}^{z} Z^{\frac{1}{2}}(\zeta)d\zeta, \tag{2.2}$$

which is the difference between the two exponents in (2.1). The problem of interest here is to find the solution of (1.6) that is exponentially small as $z\to+\infty$ and satisfies a normalization condition. Then the new dependent variable that represents this small solution is represented by $Y(F;\lambda)$, defined by

$$y(z,\lambda) = \left\{\frac{e^{-\frac{1}{2}F}}{Z^{\frac{1}{4}}(z)}\right\} Y(F;\lambda). \tag{2.3}$$

Note that the relation in (2.1) is approximate, but (2.2) and (2.3) are exact. Using (2.2) and (2.3) to rewrite (1.6) in terms of these new variables leads to the (exact) ODE that defines $Y(F;\lambda)$:

$$Y''(F;\lambda) - Y'(F;\lambda) - \Gamma(F;\lambda)\,Y(F;\lambda) = 0, \tag{2.4$a$}$$

where

$$\Gamma(F;\lambda) = \frac{\left(Z^{-\frac{1}{4}}(z)\right)_{zz}}{4\lambda^2 Z^{\frac{3}{4}}(z)}. \tag{2.4$b$}$$

In the special case of (1.5), $\lambda=1$, $Z(z)=z$, (2.4$a$) is unchanged, (2.2) becomes

$$F(z) = \frac{4}{3}z^{\frac{3}{2}}, \tag{2.4$c$}$$

and (2.4$b$) reduces to

$$G(F) := \Gamma(F;\lambda=1) = -\frac{5}{36F^2}. \tag{2.4d}$$

To this point, the methods of Dingle [8], Berry & Howls [5] and us are all essentially the same. Our approaches differ after this point because of how we each analyze (2.4). Dingle [9] and Berry & Howls [5] follow the logic of a WBK expansion of the solution of (2.4*a*),(2.4*b*) and seek a solution in the form

$$Y(F;\lambda) = \sum_{r=0}^{\infty} (-1)^r \lambda^{-r} Y_r(F), \tag{2.5a}$$

with

$$Y_0(F) = 1. \tag{2.5b}$$

Then the sequence of functions $\{Y_r(F)\}$ satisfy the kind of recurrence relation that arises naturally in WKB theory,

$$Y'_{r+1}(F) = -Y''_r(F) - G(F)Y_r(F), \tag{2.5c}$$

with

$$G(F) = \lambda^2 \Gamma(F;\lambda) = \frac{\left(Z^{-\frac{1}{4}}(z)\right)_{zz}}{4Z^{\frac{3}{4}}(z)}. \tag{2.5d}$$

(The formula given in [5] for $G(F)$ contains misprints.) If the series in (2.5*a*) converged, then the sum of (2.5*c*) over all non-negative values of $r$ would be exactly equivalent to (2.4*a*). However, Berry [4] observes that the series in (2.5*a*) is divergent, so summing (2.5*c*) over all positive values of $r$ need not be equivalent to (2.4*a*). In addition, Berry & Howls [5] use Borel summation to assign finite values to their divergent series. This procedure is systematic, but the finite value it produces may or may not be the "correct" one for a particular problem. For both of these reasons, we sought another way to construct extremely accurate approximations of the Airy function for $z>0$, which we discuss next.

The method of variation of parameters is a standard way to reformulate a differential equation like (2.4) as an integral equation. The integral equation then provides a one-parameter family of solutions of (2.4), each of which tends to zero as $z \to +\infty$. The free parameter in this family can then be chosen to satisfy the normalization condition, so the construction yields $Ai(z)$ without approximation. The integral equation corresponding to (2.4) is

$$Y(F) = A + \int_F^{\infty} \left(1 - e^{F-f}\right) G(f) Y(f)\, df, \tag{2.6}$$

where $G(f)$ is given in (2.4*d*), and $A$ is a free constant. One can verify by differentiating (2.6) twice that if (2.6) has a solution that is bounded for all $F>0$, then that solution also satisfies (2.4). The solution of (2.6) can be written in the form

$$Y(F) = \sum_{n=0}^{\infty} Y_n(F), \tag{2.7a}$$

$$Y_0(F) = A, \tag{2.7b}$$

and for $n \geq 0$,

$$Y_{n+1}(F) = -\frac{5}{36}\int_F^{\infty} \left(1 - e^{F-f}\right) f^{-2} Y_n(f)\, df. \tag{2.7c}$$

Eq'ns (2.7*a*),(2.7*b*) resemble (2.5*a*),(2.5*b*) with $\lambda = 1$, but we emphasize that (2.7*c*) does not agree with (2.5*c*). Proceeding order-by-order, one obtains the following bounds from (2.7*b*),(2.7*c*):

$$|Y_0(F)| = |A|, \tag{2.7d}$$

$$|Y_1(F)| \le |A| \left(\frac{5}{36}\right) F^{-1}; \tag{2.7e}$$

for $n \ge 1$, one shows by induction that

$$|Y_n(F)| \le |A| \left(\frac{5}{36}\right)^n \frac{F^{-n}}{n!}. \tag{2.7f}$$

Summing these, one finds that

$$|Y(F)| \le \sum_{n=0}^{\infty} |Y_n(F)| \le |A| \sum_{n=0}^{\infty} \frac{\left(\frac{5}{36F}\right)^n}{n!} = |A|\, e^{\left(\frac{5}{36F}\right)}. \tag{2.8}$$

Therefore the series in (2.7*a*),(2.7*b*) converges absolutely for $F > 0$, and (2.6) has a solution that is bounded for all $F > 0$. Formally differentiating (2.6) yields an explicit formula for $Y^{'}(F)$, and (2.8) guarantees that $Y^{'}(F)$ is also bounded for all $F > 0$. Differentiating (2.6) a second time confirms that the solution of (2.6) also satisfies (2.4).

Combining these results with (2.3) shows that for all $z > 0$

$$Ai(z) = \left\{ \frac{e^{-\frac{1}{2}F}}{z^{\frac{1}{4}}} \right\} Y(F), \tag{2.9}$$

where $Y(F)$ is the solution to (2.6), $F(z)$ is given by (2.4*c*), and the constant, $A$, is chosen to satisfy the normalization condition. In addition, $Y(F)$ is defined by the convergent infinite series in (2.7*a*),(2.7*b*), and the bounds in (2.7*d*),(2.7*e*),(2.7*f*) show that the terms in this series are naturally ordered, so that higher terms in the series decay faster for $F >> 1$. Therefore, the series in (2.7*a*),(2.7*b*) automatically satisfies (1.3), the first of Poincaré's requirements for an asymptotic series as $F \to \infty$ , but not (1.4), because the series is convergent, not divergent.

We summarize this section with three comments.

- No approximations have been made to this point.
- The series in (2.7) is different from that in (2.5), even though both are motivated by a desire to find more accurate approximations for $Ai(z)$ than one obtains by a standard asymptotic series.
- Because the terms in the series in (2.7) are naturally ordered as shown in (2.7*d*),(2.7*e*),(2.7*f*), we can obtain more detailed (i.e., hyperasymptotic) information by analyzing each term in the series in (2.7). That analysis is carried out in the next section.

## 3. Analysis of the $Y_1(F)$ term in (2.7)

From (2.7*b*),(2.7*c*), the $Y_1(F)$ term in the series in (2.7) is

$$Y_1(F;A) = -\frac{5}{36} A \int_F^{\infty} \left(1 - e^{F-f}\right) f^{-2} df. \tag{3.1}$$

Integrating the first term in the integrand provides the first term in the asymptotic expansion of $Y_1(F;A)$ for $F >> 1$. Then repeatedly integrating the second term in the integrand by parts yields subsequent terms in this asymptotic series, with a different remainder term appearing after each integration. The result after $N$ integrations is

$$Y_1(F;A) = \frac{5}{36} A \left[ \sum_{n=1}^{N} (-1)^n (n-1)! F^{-n} + (-1)^{N-1} N! Q_N(F) \right], \tag{3.2a}$$

where we define

$$N!\,Q_N(F) := \min_{N} \left[ N! \int_F^\infty e^{F-f} f^{-(N+1)} df \right]. \tag{3.2b}$$

We discover an interesting phenomenon for picking the optimal $N$ at which to truncate this asymptotic series. Eq'n (3.2$b$) gives an explicit formula for the remainder – the error incurred in approximating $Y_1(F;A)$ by only the finite sum in (3.2$a$), so for any $F > 0$, we want the value of $N$ that minimizes this remainder at that $F$. We find that as $F$ increases a little beyond the mid-value of two consecutive integers, the optimal stopping point $(N)$ changes from the integer just below $F$ to the integer just above it. Therefore we choose our optimal $N$ by directly computing the remainder itself, and we let $N!Q_N(F)$ denote the minimum of this remainder.

To control the remainder, $N!Q_N(F)$, we first find upper and lower bounds on it. As shown in Appendix A, if $|N - F| \le 1$ and $3 \le F < \infty$ , then $N!Q_N(F)$ has an upper bound,

$$N!\,Q_N(F) < 4\left(\frac{e^{-F}}{\sqrt{F}}\right). \tag{3.3}$$

The numerical coefficient in (3.3) can be decreased (from 4 to $\sqrt{2\pi}e^{\frac{1}{36}}$) if we evaluate $N!Q_N(F)$ only at integer values of $F$, and it must be increased if we consider values of $N$ beyond $|N - F| \le 1$. However, there seems to be no reason to consider this broader range, because numerical calculations of the quantity minimized in (3.2$b$) shows that the minimum remainder always occurs for an $N$ with $|N - F| < 1$.

A lower bound is obtained by similar means. As shown in Appendix A, for $|N - F| < 1$, $F \ge 3$,

$$N!\,Q_N(F) > 1.53\left(\frac{e^{-(F+1)}}{\sqrt{F+1}}\right). \tag{3.4}$$

These bounds establish part of the structure of the hyperasymptotic series. Note that both upper and lower bounds decay exponentially fast as $F \to +\infty$, as predicted by Stokes [14] and Stieltjes [12].

Note also that the upper and lower bounds decay at approximately the same rate $\left(\frac{e^{-F}}{\sqrt{F}}\right)$, which suggests that $N!Q_N(F)$ might decay at this rate as well. Figure 1 shows that this suggestion is correct, at least at leading order and at least in three widely separated intervals of $F > 0$.

Figure 1 also provides an estimate of the numerical coefficient in the leading order behavior of $N!Q_N(F)$ as $F \to +\infty$. From Figure 1, we estimate that for $F >> 1$,

$$N!\,Q_N(F) \sim \sqrt{\frac{\pi}{2}}\left(\frac{e^{-F}}{\sqrt{F}}\right). \tag{3.5}$$

This estimate of the numerical coefficient in (3.5) can be refined by subsequent analysis.

The main tools in our analysis, in (3.5) and in what follows, are direct numerical evaluation of the exact remainder, given in (3.2$b$), plus curve-fitting. Nothing more complicated is needed for this problem, because the remainder is known explicitly and is relatively simple.

The oscillations advertised in the title of this paper are not obvious in Figure 1, because their amplitude is too small to be seen with the resolution used in Figure 1. Even so, their existence is important in a hyperasymptotic series, which promises to provide detailed information to levels of accuracy much more precise than those available in a standard asymptotic series.

The next step in the analysis, therefore, is to examine the structure of $N!Q_N(F)$, as a function of $F > 0$ in more detail. This step is carried out in section 4.

## 4. Oscillations in the hyperasymptotic series

Figure 2 shows the graph of the left side of (3.5), divided by the right side, as a function of $F$. If (3.5) were an exact equation, the graph would be a horizontal line, at height 1. Instead, Figure 2

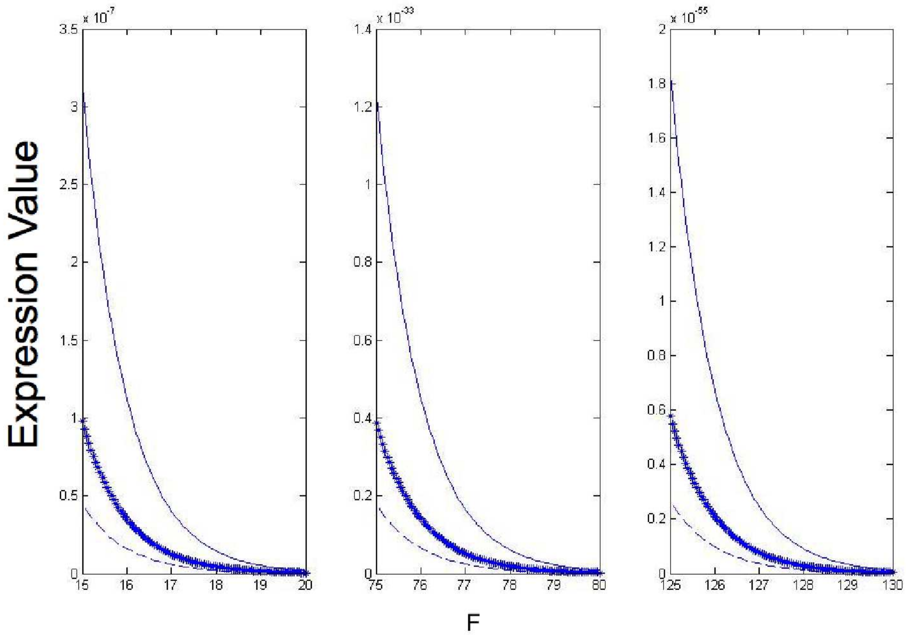


**Figure 1.** Graphs of the upper bound (——), lower bound (- - - -) and direct evaluation (*****) of $N!Q_N(F)$, in the intervals (15, 20), (75, 80), (125, 130).

shows a well defined oscillation, with a period of approximately 1, and an amplitude and mean that vary slowly. The vertical scale in Figure 2 is more refined than the first plot in Figure 1 by a factor of more than 30, which is why the oscillations shown clearly in Figure 2 are not evident in Figure 1.

Let $R(F)$ denote the envelope of the local maxima of $N!Q_N(F)$ in Figure 2 (i.e., a smooth continuous curve through the (*) points in Figure 2). After some searching around, we found an approximate linear relation between $ln(F)$ and $ln(ln(\frac{1}{R}))$, shown in Figure 3. (As shown in Figure 2, $0 < R(F) < 1$, so $ln(ln(\frac{1}{R}))$ is real-valued.)

A least-squares fit of a straight line through the data points in Figure 3 gives the result

$$ln\left(ln\left(\frac{1}{R}\right)\right) = -3.178 - ln\,(F)\,, \tag{4.1}$$

$$\Rightarrow \qquad ln\left(\frac{1}{R}\right) = e^{-3.178}F^{-1} = 0.04167\,F^{-1},$$

$$\Rightarrow \qquad R = e^{-0.04167\,F^{-1}}. \tag{4.2}$$

It remains to find another curve to fit the oscillations shown in Figure 3. We find that the observed oscillations are fit well with half-periods of a sine function with a half-period of 1. The final result is that as $F \to +\infty$,

$$N!\,Q_N(F) \sim \sqrt{\frac{\pi}{2}}\left(\frac{e^{-F}}{\sqrt{F}}\right)\left[e^{\left(-0.04167\,F^{-1}\right)} - \frac{0.12236}{F}\left|sin\left(\pi F - \frac{\pi}{2}\right)\right|\right]. \tag{4.3}$$

This is a curve-fitted representation of the oscillations shown in Figure 2, embedded within the decaying curve shown in Figure 1. The terms in the square brackets in (4.3) describe the small-scale oscillations, which are not visible in Figure 1 and not represented in (3.5). Comments:

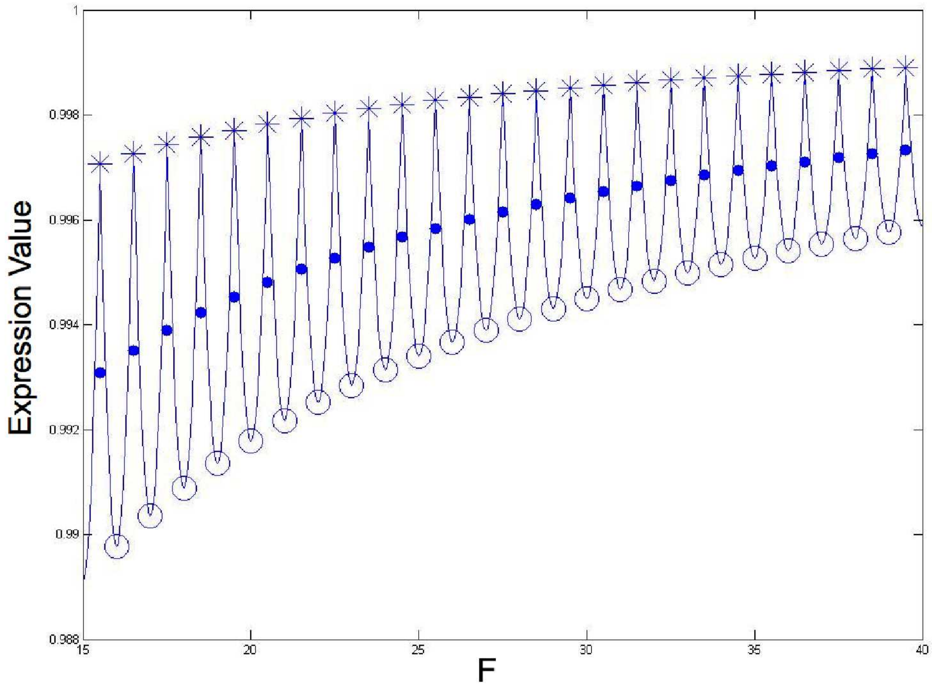


**Figure 2.** $\{N!\,Q_N(F)\}\cdot\left\{\sqrt{\frac{2}{\pi}}\,e^F\sqrt{F}\right\}$, as a function of $F$, for $15 < F < 40$. Within each oscillation, (*) marks the maximum, (o) marks the minimum and (●) marks the mean value.

- The first term in the square brackets in (4.3) was obtained by fitting a straight line through the local maxima of the oscillations, as shown in Figure 3. The oscillatory term in the square brackets in (4.3) vanishes at each local maximum, so any error in the first term is due to the failure of the straight line in Figure 3 to go through each of the local maxima. According to (4.3), each local maximum occurs at a half-integer value of $F$.
- Figure 4 shows the accuracy of the approximate representation in (4.3). The solid curve (——) is the same curve shown in Figure 2, based on direct numerical evaluation of $N!Q_N(F)$, and scaled as in Figure 2. The dashed curve (- - - -) shows the sum of the two terms in square brackets in (4.3).
- As Figure 4 shows, the relative error in the representation in (4.3) is less than $10^{-4}$, at least in the range of $15 < F < 40$. A detailed comparison of the numerical values of the two curves at each local minimum, given in [8], shows that the relative discrepancy between the two curves is $O(10^{-5})$ at integer values of $F$. The subplot inset in the lower right of Figure 4 shows how the two curves compare in a small region near F = 30.
- As $F \to +\infty$ , the quantity in square brackets in (4.3) tends to 1. Therefore the numerical factor $\left(\sqrt{\frac{\pi}{2}}\right)$ in (4.3) has a relative error of no more than $O(10^{-5})$, at least in the range of $15 \le F \le 40$.

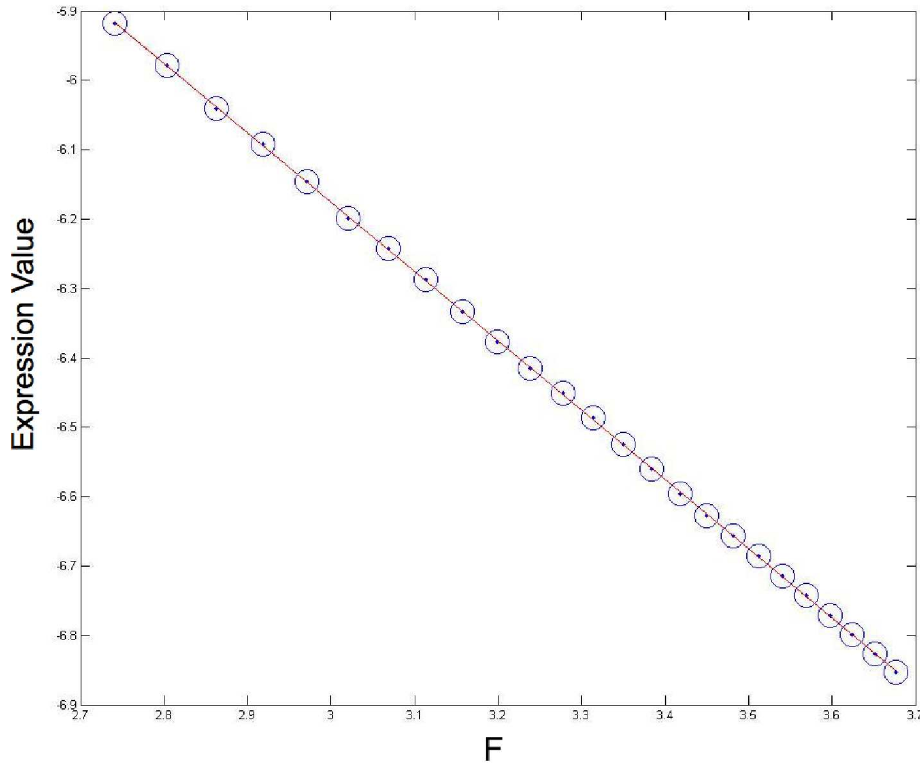


**Figure 3.** An observed linear relation between $ln(ln(\frac{1}{R}))$ and $ln(F)$. The formula for the best-fit straight line is given in (4.1).

## 5. Higher order terms in the convergent series

To this point, our asymptotic analysis has considered only the $Y_1(F)$ term in the convergent series representation in (2.7) of the relevant solution of (2.4). This leaves open the question of whether the small oscillations shown in Figure 2 might get cancelled or otherwise overwhelmed by the higher order terms in the series. Each higher order term in the convergent series is defined recursively by (2.7*c*). $Y_1(F)$ is given explicitly by (3.2), so substituting it into (2.7*c*) defines $Y_2(F)$, after which substituting $Y_2(F)$ into (2.7*c*) defines $Y_3(F)$, and so on. These calculations are tedious, but exact. Each iteration of this procedure introduces a new remainder term, and these grow in complication even though (2.7*f*) guarantees that $|Y_n(F)|$ decreases as $n$ increases. Here are the first four terms of the convergent series. For simplicity, we seek an accurate representation of the solution of (2.4) near $F = 4$, so that the optimal truncation of the usual asymptotic series occurs after only a few terms.

$$Y_0(F) = A, \tag{5.1$a$}$$

$$Y_1(F) = \frac{5}{36}\,(A)\left[-F^{-1} + F^{-2} - 2!F^{-3} + 3!F^{-4} - \{4!Q_4\,(F)\}\right], \tag{5.1$b$}$$

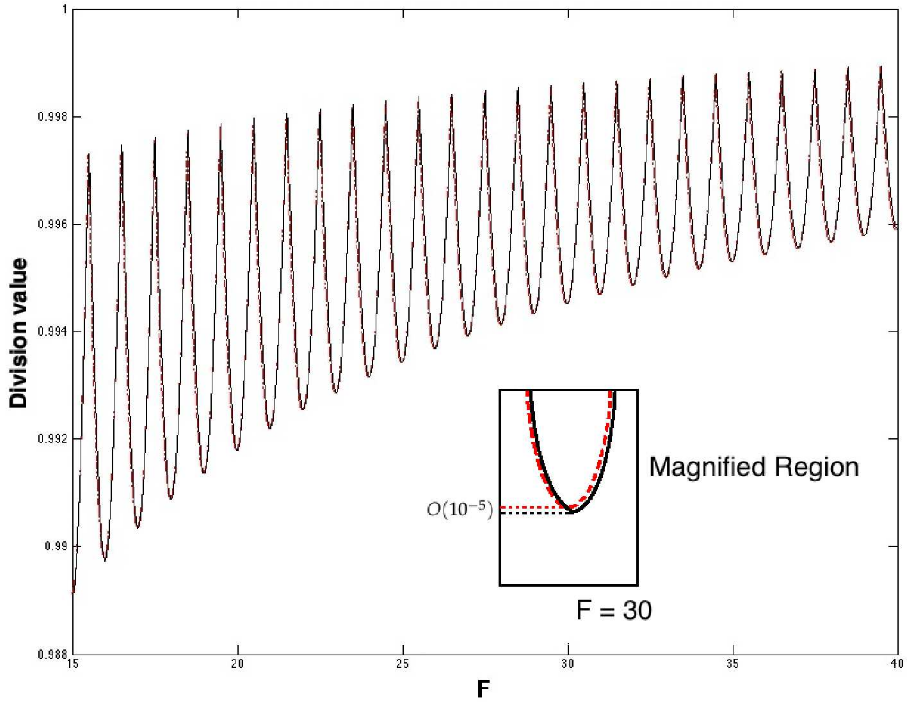


**Figure 4.** Comparison of $N!Q_N(F)$, as scaled in Figure 2, with the quantity given in square brackets in (4.3). The solid curve (——) is the same curve as in Figure 2; the dashed curve (- - - -) represents the curve-fitted predictions given in square brackets in (4.3). The subplot in Figure 4 magnifies the region near F = 30.

$$
\begin{aligned}
Y_2(F) = \left(\frac{5}{36}\right)^2 (A) \Bigg[ &+\frac{1}{2}F^{-2} - \frac{4}{3}F^{-3} + \frac{9}{2}F^{-4} - \left(\frac{4}{5}\right)\{4!Q_4(F)\} \\
&+4! \int_F^{\infty} \left(1 - e^{F-f}\right) f^{-2} Q_4(f) df \Bigg],
\end{aligned}
\tag{5.1c}
$$

$$
\begin{aligned}
Y_3(F) = \left(\frac{5}{36}\right)^3 (A) \Bigg[ &-\frac{1}{6}F^{-3} + \frac{5}{6}F^{-4} - \left(\frac{127}{720}\right)\{4!Q_4(F)\} \\
&+\frac{4}{5}\int_F^{\infty} \left(1 - e^{F-f}\right) f^{-2} \{4!Q_4(f)\} df \\
&- \int_F^{\infty} \left(1 - e^{F-f}\right) f^{-2} \left( \int_f^{\infty} \left(1 - e^{f-t}\right) t^{-2} \{4!Q_4(t)\} dt \right) df \Bigg].
\end{aligned}
\tag{5.1d}
$$

This partial list of terms in the convergent series for $Y(F)$ exhibits important structural features of the convergent series.

- Eq'n (2.7$f$) gives a rigorous bound for each $Y_n(F)$ in the convergent infinite series. Eq'n (5.1) shows that the first term in asymptotic expansion of each $Y_n(F)$ matches this bound exactly. Therefore, for each $Y_n(F)$, the first term in its asymptotic series not only represents the dominant term as $F \to \infty$ , but also its magnitude is an upper bound for that $Y_n(F)$, for all $F > 0$.

- Because the first term in the representation of each $Y_n(F)$ matches its upper bound, and because the maximum of $|Y_n(F)|$ decreases as $n$ increases, the term $(F^{-1})$ can appear only in the formula for $Y_1(F)$, $(F^{-2})$ can appear only in the formulae for $\{Y_1(F), Y_2(F)\}$, $(F^{-3})$ can appear only in the formulae for $\{Y_1(F), Y_2(F), Y_3(F)\}$, and so on. A consequence of this structure is that for fixed $F$, the finite series of integer powers of $F^{-1}$ gets systematically shorter as one proceeds to higher $Y_n(F)$.
- The terms within the square brackets in (5.1*b*) alternate in sign: even powers of $F^{-1}$ are positive, while odd powers of $F^{-1}$ are negative. This pattern of alternating signs in $Y_1(F)$ does not depend on the choice of $N = 4$ for the example in (5.1). Instead, it follows from the fact that the algebraic terms in $Y_1(F)$ are obtained by repeated integration by parts. For the same reason, the coefficient of the remainder term in (5.1*b*), $\{4!Q_4(F)\}$, is negative; if we had chosen $N = 5$, the coefficient of $\{5!Q_5(F)\}$ would have been positive.
- Meanwhile, the leading algebraic term in each $Y_n(F)$ alternates in sign as $(n)$ increases because of the negative sign outside the integral in (2.7*c*). This, along with the fact that the leading algebraic term in the representation of $Y_{n+1}(F)$ is one power of $F^{-1}$ higher than that in $Y_n(F)$, guarantees that the signs of the even and odd terms in $Y_{n+1}(F)$ match those in $Y_n(F)$. For example, the coefficient of $F^{-2}$ is positive in $Y_1(F)$ and $Y_2(F)$; the coefficient of $F^{-3}$ is negative in $Y_1(F)$, $Y_2(F)$ and $Y_3(F)$, and so on. Therefore, the algebraic terms in the higher $Y_n(F)$ reinforce the signs of the algebraic terms in $Y_1(F)$ – there is no cancellation from the higher $Y_n(F)$. Again, this pattern does not depend on the choice of $N = 4$ for the example in (5.1).
- This pattern of signs also applies to the remainder term in $Y_1(F)$. As noted above, the coefficient of $\{4!Q_4(F)\}$ is negative in (5.1*b*) because $N = 4$ is even. Then $\{4!Q_4(F)\}$ reappears in $Y_2(F)$ and $Y_3(F)$, always with a negative sign. Therefore the small oscillations that first appear in $Y_1(F)$ are reinforced by their reappearance in higher order $Y_n(F)$ – there is no cancellation of the oscillations due to higher order terms in the convergent series.
- See Appendix B for a brief discussion of the remainder term for every $Y_n(F)$ with $n > 1$.

## 6. Summary and Conclusions

This paper was inspired by earlier work by Berry & Howls [5], who proposed a new method to construct highly accurate approximate solutions of Schrödinger-type ordinary differential equations (ODEs), of the form shown in (1.6). Our objective has been to provide a variation on their method that avoids some ambiguities in their approach. Like them, we use the classical Airy equation (1.5) as our model problem, and like them, we begin by using a change of variables developed by Dingle [9] to transform the original problem into a new, equivalent ODE. Our approach differs from that of Dingle [9] or of Berry & Howls [5] in that we construct an exact solution of the transformed ODE, in the form of a bounded, convergent series for $F > 0$ (in (2.4), or for $z > 0$ in (1.5)). This convergent series is quite different from the hyperasymptotic series of Berry & Howls [5]. Then we represent each term in our (exact) convergent series with an (approximate) hyperasymptotic series, which again is different from that in [5].

Our most important result is the discovery that the hyperasymptotic series for $Y(F)$, which is equivalent to a hyperasymptotic series for the usual Airy function, contains oscillations with exponentially small amplitudes, even though $Ai(z)$ itself is not oscillatory for $z > 0$. These oscillations first appear in the (exact) remainder term for $Y_1(F)$, the first term in the convergent series representation of $Y(F)$. The oscillations also appear in subsequent terms in the convergent series, and their $(+/-)$ signs are such that these oscillations add constructively to the oscillations in $Y_1(F)$ – there is no cancellation. These oscillations are an essential ingredient in any hyperasymptotic representation of $Y(F)$.

The explanation of these small oscillations is fairly simple – they arise as a consequence of the procedure to construct a hyperasymptotic series. Recall that the first step in the procedure is to build a standard asymptotic (and divergent) series that approximates the function in question in some limit (e.g., $F \to \infty$ for $Y(F)$). Because this series diverges, one truncates the series after $N$ terms, and the optimal value of $N$ depends on $F$. This optimal $N$ is a new variable, so there are two variables $(F, N)$ in whatever series is appended to the original series. But $F$ changes continuously while $N$, which takes only integer values, changes discretely. So the error in approximating the function in question by a truncated asymptotic series exhibits small oscillations, with a period of approximately 1, as $F$ moves through the region where $N$ is the optimal truncation, and then crosses into the region where $(N+1)$ is optimal. At the boundary of these two regions, the error in approximating the function with the truncated series is a local maximum. The small oscillations observed in Figures 2 and 4 are just what are needed to correct for the oscillatory accuracy of the truncated asymptotic series. There is nothing special about the Airy function here – one should expect to see oscillations like these in hyperasymptotic representations of many smooth, non-oscillatory functions.

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Data accessibility. This research involves no data.

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## Appendix A: Upper and lower bounds on $N!Q_N(F)$

### Upper bound:

An upper bound on $N!Q_N(F)$ follows from three basic inequalities.

- Stirling's formula (eq'n 6.1.38 of [2]):

$$N! < \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N}}. \tag{A1}$$

- $Q_N(F)$ can be bounded in two different ways:

$$Q_N(F) = \int_F^\infty e^{F-f} f^{-(N+1)} df < \int_F^\infty f^{-(N+1)} df = \frac{1}{NF^N}; \tag{A2}$$

$$Q_N(F) = \int_F^\infty e^{F-f} f^{-(N+1)} df < \frac{1}{F^{N+1}} \int_F^\infty e^{F-f} df = \frac{1}{F^{N+1}}. \tag{A3}$$

First, suppose $N > F$. Then from (A1) and (A3),

$$N!\, Q_N(F) < \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N+\frac{1}{12N}} \frac{1}{F^{N+1}},$$

$$\Rightarrow \qquad N!\, Q_N(F) < \sqrt{2\pi} e^{\frac{1}{12N}} \frac{1}{\sqrt{F}} \left(\frac{N}{F}\right)^{N+\frac{1}{2}} e^{-N}. \tag{A4}$$

On the right side of (A4), set $N = F + a$, with $0 < a < F$. Then (A4) can be written as

$$N!\, Q_N(F) < \sqrt{2\pi} e^{\frac{1}{12N}} \left(\frac{e^{-F}}{\sqrt{F}}\right) e^{-a} \left(1 + \frac{a}{F}\right)^F \left(1 + \frac{a}{F}\right)^{a+\frac{1}{2}}. \tag{A5}$$

Define

$$P(F,a) := e^{-a}\left(1+\frac{a}{F}\right)^{F}\left(1+\frac{a}{F}\right)^{a+\frac{1}{2}} > 0. \tag{A6}$$

Note that as $F \to \infty$, the product of the first two terms tends to 1, and so does the last term. To determine whether $P(F,a)$ tends to 1 from above or below, calculate $\partial_F P(F,a)$:

$$P(F,a) := e^{\left[-a+F\cdot ln\left(1+\frac{a}{F}\right)+\left(a+\frac{1}{2}\right)ln\left(\left(1+\frac{a}{F}\right)\right)\right]},$$

$$\Rightarrow \quad \partial_F P(F,a) = P(F,a)\left[ln\left(1+\tfrac{a}{F}\right)+\tfrac{F}{1+\frac{a}{F}}\left(\tfrac{-a}{F^2}\right)+\tfrac{a+\frac{1}{2}}{1+\frac{a}{F}}\left(\tfrac{-a}{F^2}\right)\right],$$

For $|a/F| < 1$, $ln\left(1+\frac{a}{F}\right) < \frac{a}{F}$, so

$$\partial_F P(F,a) < P(F,a)\left[\tfrac{a}{F}+\tfrac{1}{1+\frac{a}{F}}\left(\tfrac{-a}{F}\right)+\tfrac{a+\frac{1}{2}}{1+\frac{a}{F}}\left(\tfrac{-a}{F^2}\right)\right].$$

$$\Rightarrow \quad \partial_F P(F,a) < P(F,a)\left[\tfrac{-1}{2F}\cdot\tfrac{\frac{s}{F}}{1+\frac{a}{F}}\right] < 0. \tag{A7}$$

for any $F > 0$, $0 < a < F$, so $P(F,a)$ is a decreasing function of $F$. Therefore, if $F \geq 3$ and $0 < a < F$, then

$$P(F,a) < P(3,a) = e^{-a}\left(1+\tfrac{a}{3}\right)^{3+a+\frac{1}{2}}.$$

Next show that $\partial_a P(3,a) > 0$, so $P(3,a)$ is an increasing function of $a$, for $a > 0$. So $P(3,a)$ grows without bound $a$ increases. However, our objective is minimize the error, $N!Q_N(F)$, so we can restrict $a$ to $0 \leq a \leq 1$. Therefore

$$P(F,a) < P(3,a) < P\left(3,1\right) = e^{-1}\left(\tfrac{4}{3}\right)^{\frac{9}{2}}. \tag{A8}$$

Combining everything, if follows that for $N > F \geq 3$, with $0 \leq a = N - F \leq 1$, then

$$N!\,Q_N(F) < \sqrt{2\pi}e^{\frac{1}{12N}}\left(\frac{e^{-F}}{\sqrt{F}}\right)e^{-1}\left(\frac{4}{3}\right)^{\frac{9}{2}} < 3.50\left(\frac{e^{-F}}{\sqrt{F}}\right). \tag{A9}$$

This is consistent with (3.3).

The calculation for $3 \leq N < F$ follows the same logic, but one starts with (A1) and (A2), instead of (A1) and (A3), and one sets $N = F - b$, with $0 \leq b < F$. Then one obtains

$$N!\,Q_N(F) < \sqrt{2\pi}e^{\frac{1}{12N}}\frac{1}{\sqrt{F}}\left(\frac{N}{F}\right)^{N-\frac{1}{2}}e^{-N}, \tag{A10}$$

instead of (A4). Then one obtains

$$N!\,Q_N(F) < \sqrt{2\pi}e^{\frac{1}{12N}}\left(\frac{e^{-F}}{\sqrt{F}}\right)e^{b}\left(1-\frac{b}{F}\right)^{F}\left(1-\frac{b}{F}\right)^{-\left(b+\frac{1}{2}\right)}, \tag{A11}$$

instead of (A5). Finally, one finds that for $3 \leq N < F$, and $0 < b < 1$,

$$N!\,Q_N(F) < \sqrt{2\pi}e^{1+\frac{1}{36}}\left(\frac{2}{3}\right)^{\frac{3}{2}}\left(\frac{e^{-F}}{\sqrt{F}}\right) < 3.82\left(\frac{e^{-F}}{\sqrt{F}}\right), \tag{A12}$$

instead of (A10). Both (A10) and (A12) are consistent with (3.3).

## Lower bound:

A lower bound on the error term is obtained from two basic inequalities.

- Stirling's formula (eq'n 6.1.38 of [2]):

$$N! > \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N}. \tag{A13}$$

- A lower bound on $Q_N(F)$

$$Q_N(F) = \int_F^\infty e^{F-f} f^{-(N+1)} df > \frac{1}{(F+1)^{N+1}} \int_F^{F+1} e^{F-f} df = \frac{(1-e^{-1})}{(F+1)^{N+1}}. \tag{A14}$$

Set $N = F + c$, $|c| \leq 1$, combine (A13) with (A14), then do some rearrangements to obtain

$$N! Q_N(F) > \sqrt{2\pi}\left(1 - e^{-1}\right)\left(\frac{e^{-(F+1)}}{\sqrt{F+1}}\right) e^{1-c}\left(\frac{F+1-(1-c)}{F+1}\right)^{F+1-(\frac{1}{2}-c)}. \tag{A15}$$

Define

$$P(F,c) := e^{1-c}\left(1 - \tfrac{1-c}{F+1}\right)^{F+1}\left(1 - \tfrac{1-c}{F+1}\right)^{-(\frac{1}{2}-c)} > 0. \tag{A16}$$

As above, the product of the first two terms in (A15) tends to 1 as $(F+1) \to \infty$, and so does the third term. To determine whether $P(F,c)$ tends to 1 from above or below, calculate $\partial_F P(F,c)$:

$$\begin{aligned}
\partial_F P(F,c) &= P(F,c)[\ln(1 - \frac{1-c}{F+1}) + \frac{F+1-(\frac{1}{2}-c)}{1-\frac{1-c}{F+1}}(\frac{1-c}{(F+1)^2})] \\
&= \frac{P(F,c)}{(1-\frac{1-c}{F+1})}[(1 - \frac{1-c}{F+1})\ln(1 - \frac{1-c}{F+1}) + (1 - \frac{\frac{1}{2}-c}{F+1})(\frac{1-c}{F+1})].
\end{aligned}$$

For $|c| < 1$ and $F \geq 3$, it follows that $0 < \frac{1-c}{F+1} < 1$, so the natural log has a convergent series. For convenience, define

$$\begin{aligned}
A &:= (1 - \frac{1-c}{F+1})\ln(1 - \frac{1-c}{F+1})) \\
&= (1 - \frac{1-c}{F+1})\sum_{n=1}^{\infty}\frac{1}{n}(\frac{1-c}{F+1})^n
\end{aligned}$$

$$\Rightarrow A = -(\frac{1-c}{F+1}) + \frac{1}{2}(\frac{1-c}{F+1})^2 + \frac{1}{6}(\frac{1-c}{F+1})^3 + \frac{1}{12}(\frac{1-c}{F+1})^4 + \cdots + \frac{1}{n(n-1)}(\frac{1-c}{F+1})^n + \ldots.$$

Also define

$$B := (1 - \frac{\frac{1}{2}-c}{F+1})(\frac{1-c}{F+1}) = (\frac{1-c}{F+1}) - (\frac{(\frac{1}{2}-c)(1-c)}{(F+1)^2}).$$

Therefore

$$\begin{aligned}
A + B &= \frac{c(1-c)}{(F+1)^2} + \sum_{n=3}^{\infty}\frac{1}{n(n-1)}(\frac{1-c}{F+1})^n, \\
&= (\frac{1-c}{F+1})\Big[\frac{c}{F+1} + \sum_{m=2}^{\infty}\frac{1}{(m+1)m}(\frac{1-c}{F+1})^m\Big].
\end{aligned}$$

- If $-1 < c < 0$, $F \geq 3$, then $A + B < 0$, so $\partial_F P(F,c) < 0$, and $P(F,c)$ decreases as $F$ increases. Thus, in this case the infimum of $P(F,c)$ is 1, approached as $F \to \infty$.
- If $0 \leq c \leq 1$, $F \geq 3$ then $A + B > 0$, so $\partial_F P(F,c) > 0$, and $P(F,c)$ increases as $F$ increases. Therefore the minimum of $P(F,c)$ occurs at $F = 3$. This minimum of $P(3,c)$ occurs near $c = 0.46$, and is approximately

$$min\{P(3,c)\} \sim 0.9663.$$

Combining this with (A15) and (A16) yields

$$N!Q_N(F) > \sqrt{2\pi}(1 - e^{-1})(0.966)\Big(\frac{e^{-(F+1)}}{\sqrt{F+1}}\Big) > 1.53\Big(\frac{e^{-(F+1)}}{\sqrt{F+1}}\Big). \tag{A17}$$

This is lower bound in (3.4).



## Appendix B: Upper bounds on the remainder term in $Y_n(F)$, $n \geq 2$

The analysis in Appendix A provides upper and lower bounds on the remainder term for $Y_1(F)$, *i.e.*, on the term that remains after approximating $Y_1(F)$ with a finite series of algebraic terms, truncated at the optimal stopping point for the specific $F$ of interest. But as we show next, that analysis is not sufficient to obtain an upper bound on the remainder term for $Y_2(F)$, or the remainder term for $Y_n(F)$ for any $n > 1$.

Consider eq'ns (5.1). The final term in (5.1*b*) is the remainder for $Y_1(F)$. It appears once in (5.1*b*), but it appears twice in (5.1*c*), once as a term in the finite series for $Y_2(F)$, and then again inside an integral in the final term in (5.1*c*), which is the remainder for $Y_2(F)$. Then this remainder for $Y_2(F)$ appears twice in (5.1*d*), once as a term in the finite series for $Y_3(F)$, then again inside an integral, which is the remainder for $Y_3(F)$.

This pattern persists for all integer $n > 1$. By construction, the final term in the formula for $Y_{n-1}(F)$ is the remainder term for $Y_{n-1}(F)$. Then that term appears twice in the formula for $Y_n(F)$, once as a term in the finite series for $Y_n(F)$, and then again inside an integral that is the remainder term for $Y_n(F)$. Each of these new remainder terms is an integral from the value fixed value of F of interest (*e.g.*, $F = 4$ in the example in (5.1)) to infinity. The analysis in Appendix A is valid for values of $F$ near a specified value, but for $n \geq 2$, the remainder term for $Y_n(F)$ requires knowledge of $N!$ and of $Q_N(F)$ for all $F$ larger than the specified, fixed value. The purpose of this Appendix is to provide this information.

Some notation:

- According to (3.2*b*), $N!Q_N(F)$ denotes the minimum value of this product at a fixed $F$, when the product is minimized over integer values of $N$.
- For fixed $F > 0$, with $N$ chosen to minimized $N!Q_N(F)$, the remainder term for $Y_2(F)$ requires knowledge of $N!$ and also of $Q_N(F)$ for all $F > N$, because of the integral in (5.1*c*). We denote the product of $N!$ and of $Q_N(F)$ for all $F > N$, by

$$N! \bullet Q_N(F),$$

in order to distinguish this product from the one minimized over integer $N > 0$.

Let $N$ be a positive integer. Then simple bounds on $N! \bullet Q_N(F)$ are obtained by multiplying (A1) and (A3):

$$0 < N! \bullet Q_N(F) < \sqrt{2\pi}\exp(\frac{1}{12N})(\frac{e^{-N}}{\sqrt{N}})(\frac{N}{F})^{N+1}. \tag{B1}$$

Comments about (B1):

- The decay rate of $N! \bullet Q_N(F)$ as $F \to \infty$ is always $KF^{-(N+1)}$, for some $K > 0$. This decay rate allows one to bound the integral in (5.1*c*), and also to obtain a bound on the decay rate of the new integral in (5.1*d*). This process continues for all higher order $Y_n(F)$, where the decay rate as $F \to \infty$ of the remainder term for $Y_n(F)$ determines the corresponding decay rate of the remainder term for $Y_{n+1}(F)$. It is easy to show that if the remainder term for $Y_n(F)$ decays like $F^{-p}$ as $F \to \infty$, then the decay rate for $Y_{n+1}(F)$ decays like $F^{-(p+1)}$ as $F \to \infty$.
- The remainder term in $Y_1(F)$ is $N!Q_N(F)$. The upper bound of $N!Q_N(F)$ in (3.3), its lower bound in (3.4) and its estimate in (3.5) all contain the factor $(\frac{e^{-F}}{\sqrt{F}})$, which guarantees that $N!Q_N(F)$ becomes exponentially small as $F \to \infty$. In (B1), the corresponding factor is $(\frac{e^{-N}}{\sqrt{N}})$, because $F$ and $N$ are not linked together in (B1) as they are in (3.3), (3.4), (3.5). Even so, the lower limit of the integral in $Q_N(F)$ is never smaller than $(N-1)$. For $|F-N| < 1$, $3 \leq F < \infty$, $N! \bullet Q_N(F)$ is bounded as in (3.3). For $3 \leq N < F$, an upper bound that is uniformly valid is

$$N! \bullet Q_N(F) < 13.05(\frac{e^{-N}}{\sqrt{N}}). \tag{B2}$$

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