Relative-Interior Solution for (Incomplete) Linear Assignment Problem with Applications to Quadratic Assignment Problem

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Abstract

We study the set of optimal solutions of the dual linear programming formulation of the linear assignment problem (LAP) to propose a method for computing a solution from the relative interior of this set. Assuming that an arbitrary dual-optimal solution and an optimal assignment are available (for which many efficient algorithms already exist), our method computes a relative-interior solution in linear time. Since LAP occurs as a subproblem in the linear programming relaxation of quadratic assignment problem (QAP), we employ our method as a new component in the family of dual-ascent algorithms that provide bounds on the optimal value of QAP. To make our results applicable to incomplete QAP, which is of interest in practical use-cases, we also provide a linear-time reduction from incomplete LAP to complete LAP along with a mapping that preserves optimality and membership in the relative interior. Our experiments on publicly available benchmarks indicate that our approach with relative-interior solution is frequently capable of providing superior bounds and otherwise is at least comparable.

1 Introduction

The NP-hard *quadratic assignment problem* (QAP) is a well-studied problem of combinatorial optimization with many real-world applications, such as facility location, scheduling, data analysis, ergonomic design, or problems originating in computer vision [7, 9, 17]. Informally, the QAP seeks to find a bijection between two finite sets of equal size that minimizes the objective which is a sum of unary and binary functions that depend on the values of the bijection. This problem becomes the polynomially solvable *linear assignment problem* (LAP) if the objective contains only unary functions (i.e., the binary functions are identically zero). There exist many algorithms for solving QAP exactly and heuristics for providing good solutions that are not guaranteed to be optimal. For a comprehensive introduction to LAP and QAP, we refer to [7, 9] and restrict our further attention only to the works that are most relevant to our approach.

Related Work We follow up on two methods for obtaining bounds on QAP, namely [42] and [19]. Both of these methods are based on *block-coordinate ascent*[1] (BCA) in the dual problem. BCA [40] is a well-known iterative method for (approximate) maximization of (generally constrained) multivariate functions. In each iteration, this method chooses a subset (i.e., a block) of variables and maximizes the objective over this subset of variables while keeping the other variables constant and staying within the feasible set. By repeating this iteration for different subsets, one attains a better and better solution to the original optimization problem. However, this method need not attain (or even converge to) the optimal solution of the problem.

In [42], BCA was used to approximately optimize a dual linear programming (LP) relaxation of QAP. The novel LP relaxation considered in [42] has the following features. If the dual linear program is restricted to one set of variables, it corresponds to the dual LP relaxation of the *weighted*

[1]Formally, QAP is defined in [42] as a maximization problem, so [42] in fact considers block-coordinate descent as the dual is a minimization problem. However, the approach of [42] can be easily adapted to the minimization version of QAP, which we consider in our descriptions.

constraint satisfaction problem (WCSP, equivalent to the MAP inference problem in graphical models [28, 39]). If the dual linear program is restricted to another set of variables, it becomes the LP formulation of LAP. To obtain bounds on the optimal value of QAP and good proposals for its solution, [42] introduced the *Hungarian Belief Propagation* (Hungarian-BP) algorithm that (approximately) optimizes the dual linear program by BCA: the subproblem corresponding to LAP is solved exactly by the Hungarian method [20] and the dual variables corresponding to the WCSP subproblem are improved by BCA algorithm MPLP [15]. After each iteration of Hungarian-BP, an assignment (i.e., a feasible solution for the QAP) is generated by solving the LAP subproblem. If the assignment is not proved to be optimal for the QAP (by comparing its cost to the bound given by the dual objective), [42] employs Hungarian-BP in a branch-and-bound scheme to either prove optimality or find a better assignment. To speed up this method for the cost of possible non-optimality of the returned assignment, one can limit the number of explored branches.

A related method was used in [19] where both subproblems were optimized only approximately by BCA and, similarly to [42], an assignment is generated after each iteration. More importantly, [19] found that the quality of the generated assignments is significantly improved if their sequence is gradually fused into a single one. This method constitutes the current state of the art for computer-vision instances.

For practical purposes, especially in computer vision [17], QAP can be generalized to the setting where the sets do not have the same size and some elements may remain unassigned (possibly for some cost). This gives rise to the *incomplete QAP* (IQAP) where one seeks an injective partial mapping (also called partial bijection) from one set to another that minimizes an objective consisting of both unary and binary functions, as in QAP. Again, if the objective of IQAP is restricted to contain only unary functions, one obtains the polynomially solvable *incomplete LAP* (ILAP). The IQAP has been formally defined and a reduction between it and QAP has been shown in [17, Section A1]. However, such a straightforward reduction (based on introducing multiple additional 'dummy' labels) need not be competitive with tackling IQAP directly if the underlying problem is large-scale and sparse. In contrast, the formulation of the optimization problem in [19] in fact corresponds to IQAP (although it is referred to as QAP in [19]). Naturally, the LP relaxation of IQAP contains a subproblem corresponding to ILAP instead of LAP.

Here, we combine the approaches [42, 19] with a recent result in the theory of BCA from [40]. It was shown that, when solving a block-subproblem, one should choose an optimizer from the relative interior of the set of optimizers, which is called the *relative-interior rule*. In [40], this rule is shown not to be worse (in a precise sense) than any other choice of the optimizer. However, even with the relative-interior rule, the fixed points of BCA need not be optimal.

Contribution In our work, we augment the approach of [42] by choosing an optimizer from the relative interior of the set of optimizers for the ILAP subproblem. To this end, we propose and implement an algorithm that is capable of providing such a solution. Our motivation for this is that, by the results of [40], the fixed points of BCA algorithm conforming to the relative-interior rule are not worse than if this rule is not adhered to. Our experimental results indicate that such an approach can indeed typically attain better dual objective (i.e., a lower bound on the optimal solution). Moreover, optimizing the ILAP subproblem exactly can improve the bound in cases where BCA cannot. Our method thus extends a long line of research focused on computing bounds on the optimal value of QAP [1, 14, 21, 7, 16].

Structure of the Paper We proceed as follows. We begin in Section 2 by defining the LAP and characterizing solutions of its (dual) LP formulation that are in the relative interior of the set of optimizers. These results allow us to devise an algorithm for computing such solutions. In particular, we constructively show that one can compute a relative-interior solution from an arbitrary optimal solution in linear time (assuming that an optimal assignment for the LAP is also available). Section 3 presents ILAP together with a linear-time reduction of ILAP to LAP. To be able to compute relative-interior solutions for the LP formulation of ILAP, we define a closed-form mapping that computes such a solution based on a relative-interior solution of the LP formulation of the constructed LAP. Next, Section 4 formally defines IQAP and its considered LP relaxation whose subproblems correspond to LP relaxation of WCSP and LP formulation of ILAP. Finally, Section 5 precisely describes the compared methods and overviews our experimental results.

Up to certain details, we follow the notation of [19] throughout our exposition.

2 LAP and Relative-Interior Solution

Let $\mathcal{V}$ be a finite set of *vertices* and $\mathcal{L}$ be a finite set of *labels* such that $|\mathcal{V}| = |\mathcal{L}|$ and $\mathcal{V} \cap \mathcal{L} = \emptyset$. Next, for each $v \in \mathcal{V}$, let $\mathcal{L}_v \subseteq \mathcal{L}$ be the set of *allowed labels* for vertex v and $\theta_v \colon \mathcal{L}_v \to \mathbb{R}$ be a cost function. The *linear assignment problem* (LAP) [7] is the optimization problem

$$\min \sum_{v\in\mathcal{V}} \theta_v(x_v) \tag{1a}$$

$$\forall v \in \mathcal{V} : x_v \in \mathcal{L}_v \tag{1b}$$

$$\forall \ell \in \mathcal{L} : \sum_{v\in\mathcal{V}} [\![x_v = \ell]\!] = 1 \tag{1c}$$

where $[\![\cdot]\!]$ denotes the Iverson bracket, i.e., $[\![\Psi]\!] = 1$ if Ψ is true and $[\![\Psi]\!] = 0$ otherwise. In words, the task is to find a bijection $x \colon \mathcal{V} \to \mathcal{L}$ such that each vertex is assigned an allowed label and the objective (1a) is minimized. Note, (1) need not always be feasible, but we will assume in the sequel that it is.

Remark 1. *In the usual setting [20], we have $\mathcal{L}_v = \mathcal{L}$ for each $v \in \mathcal{V}$ and the problem is thus always feasible. Although one can assume this without loss of generality (by setting a high cost $\theta_v(\ell)$ to disallowed labels [7]), we use the formalism above to be consistent with recent literature on QAP [19] that is introduced later.*

The LAP has a natural LP formulation that can be stated as the left-hand problem of the primal-dual pair

$$\min \sum_{\substack{v\in\mathcal{V}\\ \ell\in\mathcal{L}_v}} \theta_v(\ell)\mu_v(\ell) \qquad\qquad \max \sum_{v\in\mathcal{V}} \alpha_v + \sum_{\ell\in\mathcal{L}} \beta_\ell \tag{2a}$$

$$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v : \qquad \mu_v(\ell) \geq 0 \qquad\qquad \alpha_v + \beta_\ell \leq \theta_v(\ell) \tag{2b}$$

$$\forall v \in \mathcal{V} : \qquad \sum_{\ell\in\mathcal{L}_v} \mu_v(\ell) = 1 \qquad\qquad \alpha_v \in \mathbb{R} \tag{2c}$$

$$\forall \ell \in \mathcal{L} : \qquad \sum_{v\in\mathcal{V}_\ell} \mu_v(\ell) = 1 \qquad\qquad \beta_\ell \in \mathbb{R} \tag{2d}$$

where $\mathcal{V}_\ell = \{v \in \mathcal{V} \mid \ell \in \mathcal{L}_v\}$ is the set of vertices for which the label ℓ is allowed. On the right, we wrote the dual linear program. We note that the dual variables are written on the same lines as the primal constraints to which they correspond and vice versa – this applies to each primal-dual pair that we consider in this paper.

It is known [5, 25, 7] that the constraint matrix of the primal (2) (on the left) is totally unimodular, so the vertices of the polyhedron that constitutes the feasible set are integral and correspond to feasible solutions of (1). Since the minimum of a linear function on a polyhedron is always attained in at least one vertex of the polyhedron, the optimal objectives of (1) and (2) coincide.

2.1 Relative Interior of the Set of Optimal Solutions

We will assume that the sets $\mathcal{V}$, $\mathcal{L}$, $\{\mathcal{L}_v\}_{v\in\mathcal{V}}$, and costs θ are fixed in this and the following subsection to simplify formulations of statements. In this subsection, we provide characterizations of solutions of the primal and dual (2) that are in the relative interior of the set of primal and dual optimizers, respectively.

Formally, the *relative interior* of a convex set S, denoted $\operatorname{ri} S$, is the topological interior of S relative to the affine hull of S [18]. In the sequel, we will not use the definition of relative interior directly, but only rely on its properties. It is important to note that, for any convex set S, $\operatorname{ri} S \subseteq S$. Moreover, it is known that $\operatorname{ri} S = \emptyset$ if and only if $S = \emptyset$ [18].

Since we assumed that (1) is feasible, the primal (2) is also feasible and bounded. By strong duality, the dual (2) is also feasible and bounded. Both the primal and the dual thus have a non-empty set of optimizers which implies that the relative interiors of the sets of optimizers are non-empty too.

Recall that the solutions that lie in the relative interior of optimizers of any primal-dual pair can be characterized using the *strict complementary slackness* condition [41]. We formulate this condition for the case of the previously stated primal-dual pair (2) in the following theorem.

Theorem 1 ([41])**.** *Let μ and (α, β) be feasible for the primal and dual* (2)*, respectively. The following are equivalent:*

(a) μ and (α, β) is in the relative interior of optimal solutions of the primal and dual, respectively,

(b) $\forall v \in \mathcal{V},\ \ell \in \mathcal{L}_v : (\mu_v(\ell) > 0 \iff \alpha_v + \beta_\ell = \theta_v(\ell))$.

Next, we provide a more tangible description of optimal solutions from the relative interior using the notion of minimally-assignable pairs $\{v, \ell\}$. For $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v$, we say that the pair $\{v, \ell\}$ is *minimally assignable* if there exists an assignment x optimal for (1) with $x_v = \ell$.

Proposition 1. *Let μ be feasible for the primal* (2)*. The following are equivalent:*

(a) μ is in the relative interior of optimizers of the primal (2)*,*

(b) $\forall v \in \mathcal{V},\ \ell \in \mathcal{L}_v :$ ($\mu_v(\ell) > 0$ if and only if the pair $\{v, \ell\}$ is minimally assignable).

Proof. Let $X = \{x^1, \ldots, x^n\} \subseteq \mathcal{L}^{\mathcal{V}}$ be the set of assignments optimal for (1). Next, define μ^i for each $i \in \{1, \ldots, n\}$ by

$$\forall v \in \mathcal{V},\ \ell \in \mathcal{L}_v : \mu^i_v(\ell) = [\![x^i_v = \ell]\!] \tag{3}$$

and let $M = \{\mu^1, \ldots, \mu^n\}$. Due to integrality of the primal (2), the set of solutions optimal for the primal (2) is the convex hull of M, i.e.,

$$\operatorname{conv} M = \Big\{ \sum_{i=1}^n a_i \mu^i \;\Big|\; a \in \mathbb{R}^n,\ \sum_{i=1}^n a_i = 1,\ \forall i \in \{1, \ldots n\} : a_i \geq 0 \Big\}. \tag{4}$$

By [18, Remark 2.1.4], we have that the relative interior of the polyhedron (4) (i.e., the relative interior of optimal solutions of the primal (2)) is

$$\operatorname{ri} \operatorname{conv} M = \Big\{ \sum_{i=1}^n a_i \mu^i \;\Big|\; a \in \mathbb{R}^n,\ \sum_{i=1}^n a_i = 1,\ \forall i \in \{1, \ldots n\} : a_i > 0 \Big\} \tag{5}$$

where the only difference to (4) is that each coefficient a_i is required to be positive.

We begin with (a) $\implies$ (b), so let $\mu^* \in \operatorname{ri} \operatorname{conv} M$, $v \in \mathcal{V}$, and $\ell \in \mathcal{L}_v$ be arbitrary. We distinguish two cases:

- If the pair $\{v, \ell\}$ is not minimally assignable, then we have $x_v \neq \ell$ for all assignments x optimal for (1). Consequently, $\mu^i_v(\ell) = 0$ for all $i \in \{1, \ldots, n\}$ and $\mu^*_v(\ell) = \sum_{i=1}^n a_i \mu^i_v(\ell) = 0$.
- If the pair $\{v, \ell\}$ is minimally assignable, then there exists an assignment x^{i^*} optimal for (1) with $x^{i^*}_v = \ell$ and thus $\mu^{i^*}_v(\ell) = 1$. We have that $\mu^*_v(\ell) = \sum_{i=1}^n a_i \mu^i_v(\ell) > 0$ because all the terms $a_i \mu^i_v(\ell)$ are non-negative and $a_{i^*} \mu^{i^*}_v(\ell) > 0$ due to $a_{i^*} > 0$.

We continue to prove (b) $\implies$ (a). Let μ' and (α', β') be from the relative interior of optimizers of the primal and dual, respectively, and let μ^* be feasible for the primal (2). If μ^* satisfies condition (b), then, for all $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v$,

$$\mu^*_v(\ell) > 0 \iff \{v, \ell\} \text{ is minimally assignable} \iff \mu'_v(\ell) > 0 \iff \alpha'_v + \beta'_\ell = \theta_v(\ell) \tag{6}$$

where the first equivalence is statement (b), the second equivalence follows from (a) $\implies$ (b) and the fact that μ' is in the relative interior of optimizers of the primal. The third equivalence in (6) is strict complementary slackness for (α', β') and μ' and follows from Theorem 1. Consequently, μ^* is in the relative interior of optimizers of the primal because it satisfies strict complementary slackness with (α', β') by composing the equivalences in (6). $\square$

Combining Proposition 1 with Theorem 1 results in the following corollary:

Corollary 1. *Let (α, β) be feasible for the dual* (2)*. The following are equivalent:*

(a) (α, β) is in the relative interior of optimizers of the dual (2)*,*

(b) $\forall v \in \mathcal{V},\ \ell \in \mathcal{L}_v :$ ($\alpha_v + \beta_\ell = \theta_v(\ell)$ if and only if the pair $\{v, \ell\}$ is minimally assignable).

For the purposes of the sequel, we introduce the notion of perfectly-matchable edges. Formally, let $\mathcal{E} \subseteq \{\{v, \ell\} \mid v \in \mathcal{V},\, \ell \in \mathcal{L}_v\}$ so that $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ is a bipartite graph. By a *perfect matching* in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$, we mean a bijection $x\colon \mathcal{V} \to \mathcal{L}$ such that $\{v, x_v\} \in \mathcal{E}$ for all $v \in \mathcal{V}$. An edge $\{v, \ell\} \in \mathcal{E}$ is *perfectly matchable* in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ if $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ has a perfect matching x with $x_v = \ell$.

With different choices of the edge set $\mathcal{E}$, there are several connections between perfect matchings in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ and assignments feasible or optimal for (1) which we outline in the remaining part of this subsection.

For example, with $\mathcal{E} = \{\{v, \ell\} \mid v \in \mathcal{V},\, \ell \in \mathcal{L}_v\}$, the set of perfect matchings in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ coincides with the feasible set of (1). With this choice of $\mathcal{E}$, an edge $\{v, \ell\} \in \mathcal{E}$ is perfectly matchable in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ if and only if there exists an assignment x feasible for (1) with $x_v = \ell$.

Recall that, for (α, β) feasible for the dual (2), the bipartite graph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ where

$$\mathcal{E}(\alpha, \beta) = \{\{v, \ell\} \mid v \in \mathcal{V},\, \ell \in \mathcal{L}_v,\, \alpha_v + \beta_\ell = \theta_v(\ell)\} \tag{7}$$

is called the *equality subgraph* [23, 2]. The following lemma connects perfectly-matchable and minimally-assignable edges in the context of LAP and the equality subgraph.

Lemma 1. *Let (α, β) be optimal for the dual (2). For all $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v$, $\{v, \ell\}$ is minimally assignable if and only if $\{v, \ell\}$ is perfectly matchable in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$.*

Proof. Let $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v$ be arbitrary. If $\{v, \ell\}$ is minimally assignable, there exists an assignment x optimal for (1) such that $x_v = \ell$. Analogously to (3), μ defined by $\mu_v(\ell) = [\![x_v = l]\!]$ is optimal for the primal (2) and thus satisfies complementary slackness with the dual-optimal solution (α, β). Consequently, $\alpha_{v'} + \beta_{x_{v'}} = \theta_{v'}(x_{v'})$ holds for all $v' \in \mathcal{V}$, hence $\{v', x_{v'}\} \in \mathcal{E}(\alpha, \beta)$ for all $v' \in \mathcal{V}$, x is a perfect matching in the equality subgraph, and $\{v, \ell\}$ is perfectly matchable.

For the converse relation, if $\{v, \ell\}$ is perfectly matchable, there exists a perfect matching x in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ with $x_v = \ell$ by definition. This assignment x is optimal for (1) since μ defined as above satisfies complementary slackness with (α, β). □

We are now able to formulate Theorem 2, which is the important result of this subsection.

Theorem 2. *Let (α, β) be feasible for the dual (2).*

(a) (α, β) is optimal for the dual if and only if the equality subgraph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ has a perfect matching,

(b) (α, β) is in the relative interior of optimizers of the dual if and only if the equality subgraph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ has a perfect matching and each edge of the equality subgraph is perfectly matchable.

Proof. Statement (a) follows from complementary slackness: for a perfect matching x, we define a feasible solution μ of primal (2) as in (3). This pair of solutions satisfies complementary slackness, so μ is optimal for the primal and (α, β) is optimal for the dual. Conversely, if (α, β) is optimal for the dual and x is an optimal assignment for (1), then μ defined by (3) based on x is optimal for the primal and satisfies complementary slackness conditions, i.e., $\alpha_v + \beta_{x_v} = \theta_v(x_v)$ holds for all $v \in \mathcal{V}$ due to $\mu_v(x_v) = 1 > 0$, so x is a perfect matching in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$.

Statement (b) is obtained by combining Lemma 1 with Corollary 1. □

2.2 Obtaining a Relative-Interior Solution from an Optimal Solution

Let us now focus on obtaining a solution belonging to the relative interior of optimal solutions of the dual (2). For this, we assume that an arbitrary dual-optimal solution (α, β) and an assignment $x\colon \mathcal{V} \to \mathcal{L}$ optimal for (1) are at our disposal. Both can be obtained, e.g., using the Hungarian method for solving LAP [20]. Our method for computing a relative-interior solution is based on changing the given dual solution (α, β) so that it remains optimal but non-perfectly-matchable edges are removed from the equality subgraph by making the corresponding dual constraint (2b) hold with strict inequality. In other words, only perfectly-matchable edges remain in the equality subgraph, which corresponds to being in the relative interior of dual optimizers by Theorem 2b. Next, we focus on how to perform this task in detail.

We base our method for obtaining a relative-interior solution on the paper [33] where (aside from other results) a method for finding all perfectly-matchable edges in a bipartite graph was proposed. Although the technique of [33] is applicable to any bipartite graph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$, we will apply it to the equality subgraph, i.e., we have $\mathcal{E} = \mathcal{E}(\alpha, \beta)$.

Following [33], the first step of the procedure is to construct the directed graph $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}))$ where[2]

$$\mathcal{F}_x(\mathcal{E}) = \{(u, v) \mid u, v \in \mathcal{V},\ u \neq v,\ \{u, x_v\} \in \mathcal{E}\}. \tag{8}$$

The motivation for constructing the graph $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}))$ is the following. An edge $\{v, \ell\}$ is perfectly matchable in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ if and only if $x_v = \ell$ or there exists an alternating cycle in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ w.r.t. x that contains $\{v, \ell\}$ [33].[3] Each alternating cycle w.r.t. x in the undirected graph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ corresponds to a directed cycle in $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}))$ and vice versa. We thus have the following result from [33].

Theorem 3 ([33])**.** *Let $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ be a bipartite graph with a perfect matching x. Let $\{v, \ell\} \in \mathcal{E}$ and $u \in \mathcal{V}$ be such that $x_u = \ell$. Edge $\{v, \ell\}$ is perfectly matchable in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ if and only if $x_v = \ell$ or $(v, u) \in \mathcal{F}_x(\mathcal{E})$ is a part of a directed cycle in $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}))$.*

To simplify notation in this subsection, we abbreviate $\mathcal{F} = \mathcal{F}_x(\mathcal{E})$ if there is no ambiguity. First, we focus on deciding which edges of $(\mathcal{V}, \mathcal{F})$ belong to some directed cycle. As proposed in [33], this can be achieved by computing the strongly connected components of $(\mathcal{V}, \mathcal{F})$, which can be done in linear time [32, 31]. We remind[4] the reader that a directed graph is *strongly connected* if there exists a directed path between each ordered pair of its vertices. The *strongly connected components* of a directed graph are its maximal subgraphs (w.r.t. their set of vertices) that are strongly connected. Since the strongly connected components are vertex-induced subgraphs, we will identify the strongly connected components with the sets of vertices that induce them. Let $\{V_1, \dots, V_n\}$ be the strongly connected components of $(\mathcal{V}, \mathcal{F})$ so that $\{V_1, \dots, V_n\}$ is a partition of $\mathcal{V}$. The *condensation* of $(\mathcal{V}, \mathcal{F})$ is a directed acyclic graph with the set of vertices $\{V_1, \dots, V_n\}$ (denoted by subsets of vertices of the original graph). The condensation contains an edge (V_i, V_j) if and only if $i \neq j$ and $\exists u \in V_i,\ v \in V_j : (u, v) \in \mathcal{F}$. Since the condensation is acyclic, it has a topological ordering.

Without loss of generality, let $\{V_1, \dots, V_n\}$ be a topological ordering of the condensation. To obtain a relative-interior solution (based on the previously mentioned optimal solution (α, β)), we process the components in a reversed topological order and, sequentially for each V_i with ingoing edges in the condensation, update

$$\forall v \in V_i : \alpha_v := \alpha_v + \delta/2 \tag{9a}$$
$$\forall l \in L_i : \beta_\ell := \beta_\ell - \delta/2 \tag{9b}$$

where $L_i = \{x_v \mid v \in V_i\}$ and

$$\delta = \begin{cases} \min\{\theta_v(\ell) - \alpha_v - \beta_\ell \mid v \in V_i,\ \ell \in \mathcal{L}_v \setminus L_i\} & \text{if this set is non-empty} \\ 1 & \text{otherwise} \end{cases}. \tag{10}$$

We provide a formal overview of this procedure in Algorithm 1. Correctness and time complexity of this algorithm is given by Theorem 4, which requires an auxiliary lemma.

Lemma 2. *Let (α, β) be optimal for the dual (2). Let $\{V_1, \dots, V_n\}$ be the strongly connected components of $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}(\alpha, \beta))$. Let $i \in \{1, \dots, n\}$. Perform the update on lines 5–6 of Algorithm 1 for i and denote the resulting values of the dual variables by (α', β'). If there are no outgoing edges from the component V_i in the condensation of $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}(\alpha, \beta)))$, then*

(a) (α', β') is dual optimal,

(b) $\mathcal{E}(\alpha', \beta') = \mathcal{E}(\alpha, \beta) \setminus \{\{v, \ell\} \in \mathcal{E}(\alpha, \beta) \mid \ell \in L_i,\ v \notin V_i\}$,

(c) $\mathcal{F}_x(\mathcal{E}(\alpha', \beta')) = \mathcal{F}_x(\mathcal{E}(\alpha, \beta)) \setminus \{(v, u) \in \mathcal{F}_x(\mathcal{E}(\alpha, \beta)) \mid v \in \mathcal{V} \setminus V_i,\ u \in V_i\}$.

[2] As usual, we denote edges of undirected graphs as (unordered) 2-element sets (e.g., $\{u, v\}$) and edges of directed graphs as (ordered) 2-tuples (e.g., (u, v)).

[3] Recall [33] that an alternating cycle in $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ w.r.t. x is a sequence of (non-repeating) edges $\{v_1, \ell_1\}, \{v_2, \ell_2\}, \dots, \{v_{2n}, \ell_{2n}\}$ from $\mathcal{E}$ satisfying the following three conditions: (i) for all odd i: $\ell_i = \ell_{i+1}$ and $x_{v_i} \neq \ell_i$, (ii) for all even i: $v_i = v_{i+1}$ and $x_{v_i} = \ell_i$, (iii) $v_{2n} = v_1$.

[4] For a more detailed overview of graph-theoretical notions, we refer to the book [35] or papers [33, 32].

input: instance of LAP, optimal solution (α, β) for dual (2), assignment x optimal for (1).
output: solution from the relative interior of optimizers of the dual (2).

1 Construct graph $(\mathcal{V}, \mathcal{F})$ by computing $\mathcal{F} := \mathcal{F}_x(\mathcal{E}(\alpha, \beta))$ (see (8)).
2 Compute the strongly connected components and the condensation of $(\mathcal{V}, \mathcal{F})$.
3 Find a topological ordering $\{V_1, \dots, V_n\}$ of the condensation of $(\mathcal{V}, \mathcal{F})$.
4 **for** $i \in \{n, \dots, 1\}$ (in decreasing order) **do**
5   **if** component i has ingoing edges in the condensation **then**
6   Perform updates (9) where δ is (10) and $L_i = \{x_v \mid v \in V_i\}$.
7 **return** (α, β)

Algorithm 1: Computing a solution from the relative-interior of optimizers of dual (2).

Proof. If condition on line 5 is not satisfied, then the strongly connected component V_i is an isolated vertex in the condensation because it has no outgoing edges (by our assumption in the lemma) and no ingoing edges (by condition on line 5). Statements (a)-(c) are thus trivially satisfied by $(\alpha', \beta') = (\alpha, \beta)$.

For the remaining part, let condition on line 5 be satisfied, i.e., $(u, v) \in \mathcal{F}_x(\mathcal{E}(\alpha, \beta))$ for some $u \in \mathcal{V} \setminus V_i$ and $v \in V_i$. Consequently, we have $\{u, x_v\} \in \mathcal{E}(\alpha, \beta)$, i.e., there is at least one edge between $\mathcal{V} \setminus V_i$ and L_i in $\mathcal{E}(\alpha, \beta)$.

Next, see that there are no edges between V_i and $\mathcal{L} \setminus L_i$ in $\mathcal{E}(\alpha, \beta)$ – for contradiction, let $v \in V_i$, $\ell \in \mathcal{L} \setminus L_i$, and $\{v, \ell\} \in \mathcal{E}(\alpha, \beta)$. Denoting by $u \in \mathcal{V}$ the vertex with $x_u = \ell$ yields that $(v, u) \in \mathcal{F}_x(\mathcal{E}(\alpha, \beta))$ by (8). See that $u \in \mathcal{V} \setminus V_i$ because $\ell \in \mathcal{L} \setminus L_i$, which implies that there is an outgoing edge from the component V_i. This is contradictory with the fact that V_i has no outgoing edges by our assumption in this lemma. This also implies that $\delta > 0$ because $\alpha_v + \beta_\ell < \theta_v(\ell)$ holds for each $v \in V_i$ and $\ell \in \mathcal{L}_v \setminus L_i$.

We proceed to show by case analysis that the new values (α', β') satisfy statements (b) and (c). Let $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v$.

- If $v \in V_i$ and $\ell \in \mathcal{L}_v \setminus L_i$, then $\{v, \ell\} \notin \mathcal{E}(\alpha, \beta)$, as discussed previously. By definition of δ, we have $\delta \le \theta_v(\ell) - \alpha_v - \beta_\ell$ and thus $\alpha'_v + \beta'_\ell = \alpha_v + \beta_\ell + \delta/2 < \theta_v(\ell)$, i.e., $\{v, \ell\} \notin \mathcal{E}(\alpha', \beta')$.
- If $v \in V_i$ and $\ell \in L_i$ (or $v \in \mathcal{V} \setminus V_i$ and $\ell \in \mathcal{L} \setminus L_i$), then $\{v, \ell\} \in \mathcal{E}(\alpha, \beta)$ if and only if $\{v, \ell\} \in \mathcal{E}(\alpha', \beta')$ due to $\alpha_v + \beta_\ell = \alpha'_v + \beta'_\ell$ by (9).
- If $v \in \mathcal{V} \setminus V_i$ and $\ell \in L_i$, then the vertex $u \in V_i$ with $x_u = \ell$ is in a different component than v. By $\delta > 0$, we have that $\alpha'_v + \beta'_\ell < \alpha_v + \beta_\ell \le \theta_v(\ell)$, so $\{v, \ell\} \notin \mathcal{E}(\alpha', \beta')$ and $(v, u) \notin \mathcal{F}_x(\mathcal{E}(\alpha', \beta'))$.

Optimality of (α', β') follows from the fact that the objective does not change by the update (9), i.e., $\sum_{v \in \mathcal{V}} \alpha_v + \sum_{\ell \in \mathcal{L}} \beta_\ell = \sum_{v \in \mathcal{V}} \alpha'_v + \sum_{\ell \in \mathcal{L}} \beta'_\ell$. Feasibility follows from the previous case analysis. □

Theorem 4. *Algorithm 1 returns a dual-optimal solution from the relative interior of optimizers. Moreover, the time complexity of Algorithm 1 is $O(\sum_{v \in \mathcal{V}} |\mathcal{L}_v|)$, i.e., linear in the size of the input.*

Proof. To show correctness, we proceed by induction based on Lemma 2. For this, let us denote by $(\alpha^{n+1}, \beta^{n+1})$ the initial values of the dual variables and by (α^i, β^i) their values after the update on line 6 was performed with i. In case that the update is skipped for i (due to unsatisfied condition on line 5), we define $(\alpha^i, \beta^i) = (\alpha^{i+1}, \beta^{i+1})$. We will prove that the following holds for any $i \in \{1, \dots, n\}$:

(a) (α^i, β^i) is dual optimal,

(b) $\mathcal{E}^i = \mathcal{E}^{n+1} \setminus \bigcup_{j=i}^{n} \{\{v, \ell\} \in \mathcal{E}^{n+1} \mid \ell \in L_j, v \notin V_j\}$,

(c) $\mathcal{F}_x(\mathcal{E}^i) = \mathcal{F}_x(\mathcal{E}^{n+1}) \setminus \bigcup_{j=i}^{n} \{(v, u) \in \mathcal{F}_x(\mathcal{E}^{n+1}) \mid v \in \mathcal{V} \setminus V_j,\ u \in V_j\}$

where we abbreviated $\mathcal{E}(\alpha^i, \beta^i)$ to $\mathcal{E}^i$ for each $i \in \{1, \ldots, n+1\}$.

First, we show that the assumptions of Lemma 2 are satisfied whenever we process any component V_i. For the base case, note that there can be no outgoing edges from V_n as it is last in the topological ordering, so the assumptions of Lemma 2 are satisfied. For the inductive step, based on statement (c) in Lemma 2, the edges leading to processed components are removed, so whenever a component V_i should be processed, it has no outgoing edges since all its successor components must have been processed beforehand.

The statements (a)-(c) in this proof thus follow from inductively combining the statements (a)-(c) in Lemma 2. When all components are processed, we have that (α^1, β^1) is optimal by (a) and

$$\mathcal{E}^1 = \{\{v, \ell\} \in \mathcal{E}^{n+1} \mid \exists j \in \{1, \ldots, n\} : (v, \ell) \in V_j \times L_j\} \tag{11a}$$

$$\mathcal{F}_x(\mathcal{E}^1) = \{(u, v) \in \mathcal{F}_x(\mathcal{E}^{n+1}) \mid \exists j \in \{1, \ldots, n\} : u, v \in V_j\} \tag{11b}$$

by (b) and (c), respectively. Note that $\mathcal{F}_x(\mathcal{E}^1) \subseteq \mathcal{F}_x(\mathcal{E}^{n+1})$, so $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}^1))$ is a subgraph of the directed graph $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}^{n+1}))$. Moreover, $\mathcal{F}_x(\mathcal{E}^1)$ contains precisely those edges that are within the strongly connected components of $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}^{n+1}))$. By Theorem 3, each edge of $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}^1))$ belongs to some directed cycle and thus each edge in $\mathcal{E}^1 = \mathcal{E}(\alpha^1, \beta^1)$ is perfectly matchable. By Theorem 2b, (α^1, β^1) is in the relative interior of optimizers of the dual.

Concerning the time complexity, this is clear for the construction of $(\mathcal{V}, \mathcal{F}_x(\mathcal{E}^{n+1}))$ and also for lines 2–3 that can be performed in linear time, e.g., by Tarjan's algorithm [32]. Next, for each i, δ can be computed in at most $O(\sum_{v \in V_i} |\mathcal{L}_v|)$ operations and update of (α, β) can be performed in $O(|V_i|)$ operations. All in all, the loop on lines 4–6 takes at most $O(\sum_{v \in \mathcal{V}} |\mathcal{L}_v|)$ operations. Note that $\sum_{v \in \mathcal{V}} |\mathcal{L}_v| \geq |\mathcal{V}|$. □

Example 1. *Let us consider the LAP with* $\mathcal{V} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$, $\mathcal{L} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}\}$*, costs* θ *defined in Figure 1a, and a dual-optimal solution* (α, β) *defined by* $\alpha = (2, 2, 3, 3, 3)$, $\beta = (1, 1, 1, 4, 4)$ *(also indicated in Figure 1a). Here, we assume for simplicity that* $\mathcal{L}_v = \mathcal{L}$ *for each* $v \in \mathcal{V}$*. Following Algorithm 1, we will show how to change this dual-optimal solution* (α, β) *(which is in the relative boundary) to obtain a solution that belongs to the relative interior of dual optimizers.*

The equality subgraph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ *with a perfect matching* x *is shown in Figure 1b. Based on the equality subgraph and the chosen perfect matching, one can construct the directed graph* $(\mathcal{V}, \mathcal{F})$ *that is shown in Figure 1c. The directed graph* $(\mathcal{V}, \mathcal{F})$ *contains 2 directed cycles, each corresponding to an augmenting cycle in* $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$*:*

- *cycle* (b, d), (d, b) *corresponding to augmenting cycle* $\{\mathsf{b}, \mathsf{B}\}$, $\{\mathsf{B}, \mathsf{d}\}$, $\{\mathsf{d}, \mathsf{A}\}$, $\{\mathsf{A}, \mathsf{b}\}$,
- *cycle* (c, e), (e, c) *corresponding to augmenting cycle* $\{\mathsf{c}, \mathsf{C}\}$, $\{\mathsf{C}, \mathsf{e}\}$, $\{\mathsf{e}, \mathsf{D}\}$, $\{\mathsf{D}, \mathsf{c}\}$.

Consequently, all of these aforementioned edges (along with edges $\{v, x_v\}$ *for all* $v \in \mathcal{V}$*) are perfectly matchable in the equality subgraph. All the other edges in the equality subgraph (e.g.,* $\{\mathsf{a}, \mathsf{A}\}$ *or* $\{\mathsf{d}, \mathsf{C}\}$*) are not perfectly matchable. The perfectly-matchable edges in the equality subgraph are marked by circles in Figure 1a.*

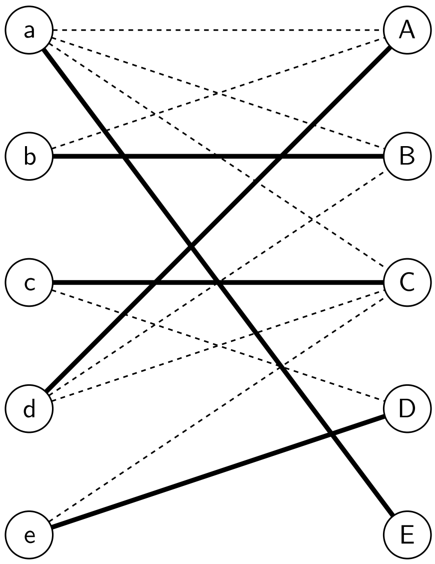
The condensation of the directed graph $(\mathcal{V}, \mathcal{F})$ *is shown in Figure 1d and the partition of the vertices* $\{V_1, V_2, V_3\}$ *also corresponds to a topological ordering of the condensation. We will now process the strongly connected components in reversed topological order:*

1. *For* $i = 3$*, we have* $V_3 = \{\mathsf{c}, \mathsf{e}\}$, $L_3 = \{\mathsf{C}, \mathsf{D}\}$*, and* $\delta = 4$*. The minimal value of* δ *is attained, e.g., for* $v = \mathsf{e}$ *and* $\ell = \mathsf{A}$*. By (9), we increase* α_{c} *and* α_{e} *by 2 and decrease* β_{C} *and* β_{D} *by 2. This results in* $\alpha = (2, 2, 5, 3, 5)$ *and* $\beta = (1, 1, -1, 2, 4)$*.*
2. *For* $i = 2$*, we have* $V_2 = \{\mathsf{b}, \mathsf{d}\}$, $L_3 = \{\mathsf{B}, \mathsf{A}\}$*, and* $\delta = 2$*. The minimal value of* δ *is attained, e.g., for* $v = \mathsf{b}$ *and* $\ell = \mathsf{E}$*. Note that we use the previously updated values of dual variables to compute* δ*, not the initial ones. By (9), we increase* α_{b} *and* α_{d} *by 1 and decrease* β_{B} *and* β_{A} *by 1. This yields* $\alpha = (2, 3, 5, 4, 5)$ *and* $\beta = (0, 0, -1, 2, 4)$*.*
3. *For* $i = 1$*, the strongly connected component* $V_1 = \{\mathsf{a}\}$ *has no ingoing edges and is thus not processed.*

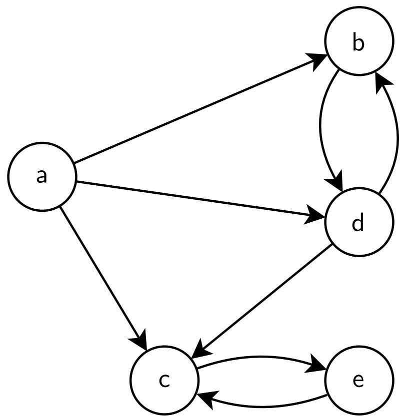
By Theorem 2b, the current dual solution $\alpha = (2, 3, 5, 4, 5)$, $\beta = (0, 0, -1, 2, 4)$ *is in the relative interior of dual optimizers because the equality subgraph* $(\mathcal{V} \cup \mathcal{L}, \mathcal{E}(\alpha, \beta))$ *(computed for these values of dual variables) contains only perfectly-matchable edges.*

	A $\beta_{\mathsf{A}} = 1$	B $\beta_{\mathsf{B}} = 1$	C $\beta_{\mathsf{C}} = 1$	D $\beta_{\mathsf{D}} = 4$	E $\beta_{\mathsf{E}} = 4$
a $\alpha_{\mathsf{a}} = 2$	3	3	3	7	⑥
b $\alpha_{\mathsf{b}} = 2$	③	③	9	9	8
c $\alpha_{\mathsf{c}} = 3$	9	10	④	⑦	11
d $\alpha_{\mathsf{d}} = 3$	④	④	4	8	11
e $\alpha_{\mathsf{e}} = 3$	8	9	④	⑦	13

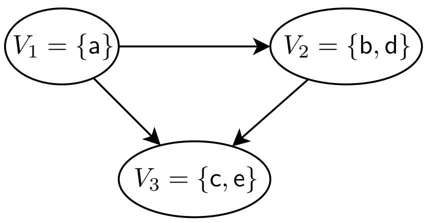
(a) Costs θ for individual edges of the complete bipartite graph $(\mathcal{V} \cup \mathcal{L}, \mathcal{E})$ and initial values of dual variables (α, β). Edges that are present in the equality subgraph are highlighted in gray. The costs of the edges that are perfectly matchable in the equality subgraph (i.e., minimally assignable for the LAP defined by these costs) are circled.



(b) Equality subgraph for the initial values of the dual variables with a highlighted perfect matching.



(c) Oriented graph $(\mathcal{V}, \mathcal{F})$ defined by the equality subgraph and perfect matching x in Figure 1b.



(d) Condensation of the oriented graph $(\mathcal{V}, \mathcal{F})$ from Figure 1c.

Figure 1: Illustrations to Example 1.

3 Incomplete LAP: Reduction to LAP and Relative-Interior Solution

In this section, we introduce the *incomplete LAP* (ILAP) and a reduction from ILAP to LAP that makes it possible to solve ILAP using widely available LAP algorithms. Moreover, this reduction allows to easily construct a solution of the LP formulation of ILAP based on a solution of the LP formulation of LAP. Importantly, we discuss that one can easily map a relative-interior solution of the LP formulation of LAP to a relative-interior solution of the LP formulation of ILAP.

The ILAP is the extension of LAP where each vertex from $\mathcal{V}$ need not be assigned a label, i.e., some vertices may be left unassigned for a vertex-specific cost. As in [19], we capture this by extending the set of labels by the *dummy label* $\#$ that can be assigned to arbitrarily many vertices.

Formally, let $\mathcal{V}$ be a finite set of vertices and $\mathcal{L}$ be a finite set of labels with $\# \in \mathcal{L}$ and $\mathcal{V} \cap \mathcal{L} = \emptyset$. For each $v \in \mathcal{V}$, $\mathcal{L}_v \subseteq \mathcal{L}$ with $\# \in \mathcal{L}_v$ is the set of allowed labels for vertex v. Finally, as in LAP, $\theta_v \colon \mathcal{L}_v \to \mathbb{R}$ is a cost function for each $v \in \mathcal{V}$. In this setting, ILAP is the optimization problem

$$\min \sum_{v \in \mathcal{V}} \theta_v(x_v) \tag{12a}$$

$$\forall v \in \mathcal{V} : x_v \in \mathcal{L}_v \tag{12b}$$

$$\forall \ell \in \mathcal{L} \setminus \{\#\} : \sum_{v \in \mathcal{V}} [\![x_v = \ell]\!] \leq 1. \tag{12c}$$

Remark 2. *As opposed to LAP from Section 2, we do not require $|\mathcal{V}| = |\mathcal{L}|$ for ILAP. Next, the equality sign in* (1c) *changed to the inequality sign in* (12c)*, i.e., not every label needs to be assigned to some vertex. Also, the constraint* (12c) *does not limit the number of vertices assigned to the dummy label. Thus, in contrast to LAP, no question of feasibility arises because the assignment defined by $x_v = \#$ for all $v \in \mathcal{V}$ is always feasible.*

Remark 3. *There exist many variants and extensions of LAP [7, 38, 4, 3, 10, 24, 27]. The closest problem to ILAP is the* rectangular LAP *[7, 6, 4] (also called asymmetric [3] or unbalanced LAP [27]) where different sizes of the partitions are also assumed (with $|\mathcal{V}| < |\mathcal{L}|$) but the dummy label is not considered because each vertex is required to be assigned to some non-dummy label. The* coverage-sensitive many-to-many min-cost bipartite matching *(CSM) problem from [22] is more general than ILAP as it allows specifying a (possibly infinite) cost for each label $\mathcal{L}$ that depends on how many vertices are assigned this label.*

Analogously to LAP, the ILAP can be also formulated as a linear program that is the left-hand problem of the primal-dual pair

$$
\begin{array}{rlll}
 & \min \sum\limits_{\substack{v \in \mathcal{V} \\ \ell \in \mathcal{L}_v}} \theta_v(\ell)\mu_v(\ell) & \max \sum\limits_{v \in \mathcal{V}} \alpha_v + \sum\limits_{\ell \in \mathcal{L} \setminus \{\#\}} \beta_\ell & \text{(13a)} \\
\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v : & \mu_v(\ell) \geq 0 & \alpha_v + [\![\ell \neq \#]\!] \beta_\ell \leq \theta_v(\ell) & \text{(13b)} \\
\forall v \in \mathcal{V} : & \sum\limits_{\ell \in \mathcal{L}_v} \mu_v(\ell) = 1 & \alpha_v \in \mathbb{R} & \text{(13c)} \\
\forall \ell \in \mathcal{L} \setminus \{\#\} : & \sum\limits_{v \in \mathcal{V}_\ell} \mu_v(\ell) \leq 1 & \beta_\ell \leq 0. & \text{(13d)}
\end{array}
$$

Again, we wrote the dual linear program on the right. Note that the dual variables β_ℓ are defined only for $\ell \in \mathcal{L} \setminus \{\#\}$, so the dual constraint (13b) does not contain β_ℓ if $\ell = \#$. $\mathcal{V}_\ell$ is defined as in (2).

It is not hard to show that the primal (13) (on the left-hand side) is integral and that any its integral feasible solution corresponds to a feasible solution of (12) with the same objective. This fact follows immediately, e.g., from [30, Theorem 5.21] or [7, Theorem 4.2].

3.1 Reduction of ILAP to LAP

Next, we describe a reduction of ILAP to LAP that allows us to solve ILAP instances and also obtain a relative-interior solution of the dual (13) based on our previous results from Section 2.

A similar reduction was informally mentioned in [36] and also discussed in [27, Section 1.3] (which however transforms rectangular LAP to LAP instead of ILAP to LAP). The reduction provided in [17, Section A1] is also to some extent similar to the one considered here but it neither preserves the sparsity pattern nor splits the costs, so the resulting cost matrix is not symmetric.

Reduction For the purpose of showing our reduction formally, let $\mathcal{P} = (\mathcal{V}, \mathcal{L}, \#, \{\mathcal{L}_v\}_{v\in\mathcal{V}}, \theta)$ define an instance of ILAP. We define the instance of LAP $\mathcal{P}' = (\mathcal{V}', \mathcal{L}', \{\mathcal{L}'_v\}_{v\in\mathcal{V}'}, \theta')$ by

$$\mathcal{L}' = \mathcal{V}' = \mathcal{V} \cup \mathcal{L} \setminus \{\#\} \tag{14a}$$

$$\forall v \in \mathcal{V}': \qquad \mathcal{L}'_v = \begin{cases} (\mathcal{L}_v \setminus \{\#\}) \cup \{v\} & \text{if } v \in \mathcal{V} \\ \mathcal{V}_v \cup \{v\} & \text{if } v \in \mathcal{L} \end{cases} \tag{14b}$$

$$\forall v \in \mathcal{V}',\, \ell \in \mathcal{L}'_v: \qquad \theta'_v(\ell) = \begin{cases} \theta_v(\ell)/2 & \text{if } v \in \mathcal{V},\, \ell \in \mathcal{L} \\ \theta_\ell(v)/2 & \text{if } v \in \mathcal{L},\, \ell \in \mathcal{V} \\ \theta_v(\#) & \text{if } v \in \mathcal{V},\, \ell \in \mathcal{V} \\ 0 & \text{if } v \in \mathcal{L},\, \ell \in \mathcal{L} \end{cases}. \tag{14c}$$

If a sparse encoding is used for the sets $\mathcal{L}_v$, the size of the instance $\mathcal{P}'$ scales linearly with the size of $\mathcal{P}$. In this subsection, we analyze this reduction and assume that the instances $\mathcal{P}$ and $\mathcal{P}'$ are fixed for brevity.

Remark 4. *Strictly speaking, we required in Section 2 the partitions $\mathcal{V}'$ and $\mathcal{L}'$ to be disjoint, which is not satisfied by* (14a)*. However, there is no ambiguity since the sets of allowed labels $\{\mathcal{L}'_v\}_{v\in\mathcal{V}'}$ are defined for elements of $\mathcal{V}'$ and the costs θ' are denoted asymetrically, i.e., for $\theta'_v(\ell)$, we have $v \in \mathcal{V}'$ and $\ell \in \mathcal{L}'$ (and analogously for the primal variables μ in* (2)*). By distinguishing $\mathcal{V}'$ from $\mathcal{L}'$ in equations, we resolve any ambiguity.*

To provide the reader with an intuitive understanding of the reduction, let us now informally comment on it. For an assignment $x\colon \mathcal{V} \to \mathcal{L}$ feasible for ILAP $\mathcal{P}$, one can define an assignment $x'\colon \mathcal{V}' \to \mathcal{L}'$ feasible for LAP $\mathcal{P}'$ by

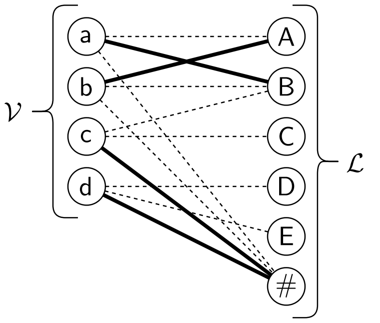
$$x'_{v'} = \begin{cases} x_{v'} & \text{if } v' \in \mathcal{V} \text{ and } x_{v'} \neq \# \\ v' & \text{if } v' \in \mathcal{V} \text{ and } x_{v'} = \# \\ u \text{ where } u \in \mathcal{V} \text{ is such that } x_u = v' & \text{if } v' \in \mathcal{L} \text{ and } \exists u \in \mathcal{V} : x_u = v' \\ v' & \text{if } v' \in \mathcal{L} \text{ and } \forall u \in \mathcal{V} : x_u \neq v' \end{cases} \tag{15}$$

for all $v' \in \mathcal{V}'$. In words, if $v \in \mathcal{V}$ is assigned to $x_v = \ell \neq \#$, then $x'_v = \ell$ and $x'_\ell = v$. On the other hand, if $v \in \mathcal{V}$ is assigned to the dummy label $x_v = \#$, then it is assigned to itself by x', i.e., $x'_v = v$. If some label $\ell \in \mathcal{L} \setminus \{\#\}$ is not assigned to any vertex, i.e., $x_v \neq \ell$ for all $v \in \mathcal{V}$, then ℓ is also assigned to itself by x', i.e., $x'_\ell = \ell$.

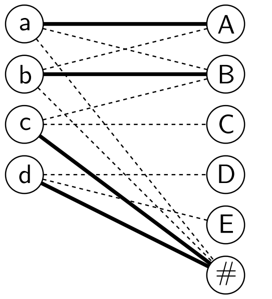
By (14c), the cost for assigning a vertex $v \in \mathcal{V}$ to itself in $\mathcal{P}'$ is equal to assigning the vertex to the dummy label # in $\mathcal{P}$, i.e., $\theta_v(\#)$. The cost for assigning a label $\ell \in \mathcal{L} \setminus \{\#\}$ to itself in $\mathcal{P}'$ is zero. Consequently, since the other costs are halved, the objective value of x' for LAP $\mathcal{P}'$ is equal to the objective value of x for ILAP $\mathcal{P}$. We prove later in Proposition 3 that the optimal values of $\mathcal{P}$ and $\mathcal{P}'$ are equal.

Example 2. *To exemplify this reduction, let $\mathcal{V} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ and $\mathcal{L} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \#\}$. The allowed labels are given by $\mathcal{L}_\mathsf{a} = \mathcal{L}_\mathsf{b} = \{\mathsf{A}, \mathsf{B}, \#\}$, $\mathcal{L}_\mathsf{c} = \{\mathsf{B}, \mathsf{C}, \#\}$, and $\mathcal{L}_\mathsf{d} = \{\mathsf{D}, \mathsf{E}, \#\}$. This instance is depicted in Figure 2a where the allowed labels are connected by edges to the corresponding vertices.*

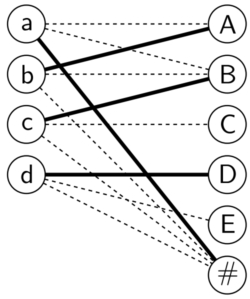
The bold edges in Figure 2a define an assignment $x\colon \mathcal{V} \to \mathcal{L}$ feasible for this instance, i.e., $x_\mathsf{a} = \mathsf{B}$, $x_\mathsf{b} = \mathsf{A}$, and $x_\mathsf{c} = x_\mathsf{d} = \#$. Based on this instance $\mathcal{P}$ of ILAP, one can define the LAP $\mathcal{P}'$ by (14)*, which is shown in Figure 2d. Elements of $\mathcal{V}'$ and $\mathcal{L}'$ are on the left and right part of the bipartite graph, respectively. As in Figure 2a, the sets $\{\mathcal{L}'_{v'}\}_{v'\in\mathcal{V}'}$ are depicted by drawing an edge $\{v', \ell'\}$ for each $v' \in \mathcal{V}'$ and $\ell' \in \mathcal{L}'_{v'}$. We have, e.g., $\mathcal{L}'_\mathsf{a} = \{\mathsf{A}, \mathsf{B}, \mathsf{a}\}$ and $\mathcal{L}'_\mathsf{B} = \{\mathsf{B}, \mathsf{a}, \mathsf{b}, \mathsf{c}\}$. Figure 2d also shows the assignment $x'\colon \mathcal{V}' \to \mathcal{L}'$ defined by* (15) *based on the aforementioned assignment $x\colon \mathcal{V} \to \mathcal{L}$. This assignment corresponds to a perfect matching in the bipartite graph shown in Figure 2d.*



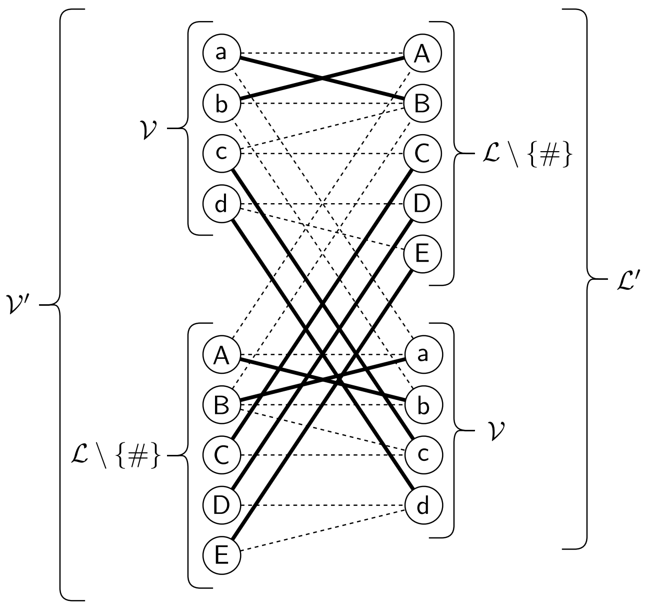
(a) Bipartite graph with partitions $\mathcal{V}$ and $\mathcal{L}$ that defines an instance of ILAP (up to the costs θ) and a feasible assignment x for this instance (determined by bold edges).



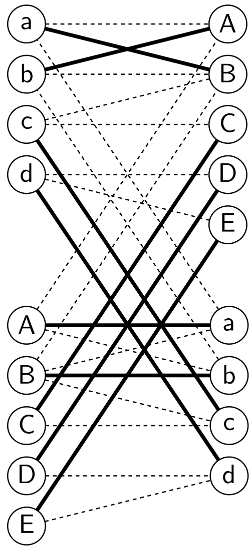
(b) Assignment x^2 feasible for the same ILAP instance as in Figure 2a.



(c) Assignment x^3 feasible for the same ILAP instance as in Figure 2a.



(d) Instance of LAP obtained by the reduction (14) from the ILAP instance determined by Figure 2a. The highlighted perfect matching x' is obtained by (15) from the assignment in Figure 2a.



(e) A different assignment x'' feasible for the LAP instance from Figure 2d.

Figure 2: Illustrations to Example 2.

See that no vertex $v \in \mathcal{V}$ is assigned to label C, D, *or* E *in Figure 2a. Consequently, these labels are assigned to themselves in Figure 2d. Similarly, vertices* c *and* d *are assigned to the dummy label # in Figure 2a and thus, they are assigned to themselves in Figure 2d. Informally speaking, for the other vertices and labels, the lower part of the diagram is obtained by mirroring the upper part horizontally.*

Figure 2e shows another assignment $x'' \colon \mathcal{V}' \to \mathcal{L}'$ feasible for this LAP instance. This assignment differs from x' only in the lower part of the LAP instance and the resulting perfect matching is thus not 'symmetric'[5]*. From the lower part, one can extract the assignment $x^2 \colon \mathcal{V} \to \mathcal{L}$ that is shown in Figure 2b and is also feasible for the aforesaid ILAP instance.*

Although one can map each assignment x feasible for the ILAP $\mathcal{P}$ to an assignment x' feasible for the LAP $\mathcal{P}'$ via (15), not every assignment x' feasible for the LAP $\mathcal{P}'$ can be mapped to a unique assignment for the ILAP $\mathcal{P}$. To be precise, one can generally extract two different ILAP assignments from each assignment feasible for the LAP. Recalling Example 2, this is the situation

[5] More precisely, the bijection x' defined by Figure 2d is an involution, i.e., it is self-inverse. The bijection x'' defined by Figure 2e is not an involution.

of the LAP assignment shown in Figure 2e that can be decomposed into the two ILAP assignments in Figures 2a and 2b. Proposition 2 analyzes the objective values of such decomposed assignments.

Remark 5. *Seen from the other side, not every two assignments feasible for ILAP $\mathcal{P}$ can be combined in this way to result in a feasible assignment for LAP $\mathcal{P}'$. A sufficient condition for the possibility to combine assignments $x, y\colon \mathcal{V} \to \mathcal{L}$ feasible for the ILAP $\mathcal{P}$ is that $\{x_v \mid v \in \mathcal{V}\} = \{y_v \mid v \in \mathcal{V}\}$ and $\{v \in \mathcal{V} \mid x_v = \#\} = \{v \in \mathcal{V} \mid y_v = \#\}$, i.e., if the sets of assigned labels are the same and the sets of vertices assigned to the dummy label are the same.*

To illustrate this, let us continue in Example 2. The assignments $x^2, x^3\colon \mathcal{V} \to \mathcal{L}$ defined by Figures 2b and 2c, respectively, do not satisfy these conditions because $\{x^2_v \mid v \in \mathcal{V}\} = \{\mathsf{A}, \mathsf{B}, \#\} \neq \{x^3_v \mid v \in \mathcal{V}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{D}, \#\}$ and even $\{v \in \mathcal{V} \mid x^2_v = \#\} = \{\mathsf{c}, \mathsf{d}\} \neq \{v \in \mathcal{V} \mid x^3_v = \#\} = \{\mathsf{a}\}$. In this case, if the lower and upper part of the LAP $\mathcal{P}'$ is assigned based on x^2 and x^3, respectively, such a matching cannot be completed to a feasible assignment for LAP $\mathcal{P}'$ (i.e., to a perfect matching in the bipartite graph shown in Figure 2d).

Proposition 2. *Let x' be an assignment feasible for $\mathcal{P}'$. Define assignments $x^1, x^2\colon \mathcal{V} \to \mathcal{L}$ by*

$$\forall v \in \mathcal{V} : x^1_v = \begin{cases} x'_v & \text{if } x'_v \in \mathcal{L} \setminus \{\#\} \\ \# & \text{otherwise (i.e., } x'_v = v) \end{cases} \tag{16a}$$

$$\forall v \in \mathcal{V} : x^2_v = \begin{cases} x'^{-1}_v & \text{if } x'^{-1}_v \in \mathcal{L} \setminus \{\#\} \\ \# & \text{otherwise (i.e., } x'^{-1}_v = v) \end{cases} \tag{16b}$$

where $x'^{-1}\colon \mathcal{L}' \to \mathcal{V}'$ is the inverse of x' (recall that x' is a bijection). Assignments x^1 and x^2 are feasible for the ILAP $\mathcal{P}$. If Θ' is the cost of x' for LAP $\mathcal{P}'$ and Θ_1 and Θ_2 are the costs of x^1 and x^2 for ILAP $\mathcal{P}$, respectively, then $2\Theta' = \Theta_1 + \Theta_2$.

Proof. To show feasibility of x^1, note that x'_v, $v \in \mathcal{V}$ are unique and that if $x'_v \in \mathcal{L}$, then $x'_v \in \mathcal{L}_v$. In detail, this is due to $x'_v \in \mathcal{L}'_v = (\mathcal{L}_v \setminus \{\#\}) \cup \{v\}$, so, if $x'_v \notin \mathcal{L}$, then $x'_v = v$. The proof for x^2 is analogous since $\mathcal{V}'_v = (\mathcal{L}_v \setminus \{\#\}) \cup \{v\}$ and $x'^{-1}_v \in \mathcal{V}'_v$ for all $v \in \mathcal{V}$.

For the next statement, see that

$$2\Theta' = 2\sum_{v \in \mathcal{V}'} \theta'_v(x'_v) = 2\sum_{\substack{v \in \mathcal{V} \\ x'_v \neq v}} \overbrace{\theta'_v(x'_v)}^{\theta_v(x'_v)/2} + 2\sum_{\substack{v \in \mathcal{V} \\ x'_v = v}} \overbrace{\theta'_v(x'_v)}^{\theta_v(\#)} + 2\sum_{\substack{\ell \in \mathcal{L}\setminus\{\#\} \\ x'_\ell \neq \ell}} \overbrace{\theta'_\ell(x'_\ell)}^{\theta_{x'_\ell}(\ell)/2} + 2\sum_{\substack{\ell \in \mathcal{L}\setminus\{\#\} \\ x'_\ell = \ell}} \overbrace{\theta'_\ell(x'_\ell)}^{0} \tag{17a}$$

$$= \underbrace{\sum_{\substack{v \in \mathcal{V} \\ x'_v \neq v}} \theta_v(x'_v) + \sum_{\substack{v \in \mathcal{V} \\ x'_v = v}} \theta_v(\#)}_{\Theta_1} + \underbrace{\sum_{\substack{v \in \mathcal{V} \\ x'^{-1}_v = v}} \theta_v(\#) + \sum_{\substack{v \in \mathcal{V} \\ x'^{-1}_v \in \mathcal{L}\setminus\{\#\}}} \theta_v(x'^{-1}_v)}_{\Theta_2} \tag{17b}$$

where equality (17a) holds because the sets of indices in the individual sums form a partition of $\mathcal{V}'$. The values in the upper brackets in (17a) follow from the definition of θ' in (14c). The equality (17b) holds because $x'_v = v \iff x'^{-1}_v = v$ and

$$\{(x'_\ell, \ell) \mid \ell \in \mathcal{L} \setminus \{\#\},\, x'_\ell \neq \ell\} = \{(v, x'^{-1}_v) \mid v \in \mathcal{V},\, x'^{-1}_v \in \mathcal{L} \setminus \{\#\}\} \tag{18}$$

which is due to $\mathcal{L}'_\ell = \mathcal{V}_\ell \cup \{\ell\}$ for all $\ell \in \mathcal{L} \setminus \{\#\}$, so the term $x'_\ell \neq \ell$ in the first bracket in (18) is equivalent to $x'_\ell \in \mathcal{V}$. By (16), the parts enclosed in the brackets in (17b) correspond to Θ_1 and Θ_2. □

Properties of Optimal Solutions

Proposition 3. *The optimal values of ILAP $\mathcal{P}$ and LAP $\mathcal{P}'$ are equal.*

Proof. We prove this claim by showing that, for any assignment feasible for $\mathcal{P}$, we can construct an assignment feasible for $\mathcal{P}'$ that has the same or better objective and vice versa.

As discussed above, for any assignment x feasible for the ILAP $\mathcal{P}$, we can construct an assignment x' via (14) that is feasible for the LAP $\mathcal{P}'$ and has the same objective.

For the other direction, let x' be a feasible assignment for the LAP $\mathcal{P}'$ with cost Θ'. By Proposition 2, one can construct the assignments x^1 and x^2 whose objective values are Θ_1 and Θ_2, respectively. Without loss of generality, let $\Theta_1 \leq \Theta_2$. Then, $\Theta' = (\Theta_1 + \Theta_2)/2 \geq \Theta_1$ and assignment x^1 is feasible for $\mathcal{P}$ and has the same or better objective than x' for $\mathcal{P}'$. □

Remark 6. *Let x^1 and x^2 be obtained from x' as in Proposition 2. It follows from Proposition 2 (combined with Proposition 3) that the assignment x' is optimal for LAP $\mathcal{P}'$ if and only if both x^1 and x^2 are optimal for ILAP $\mathcal{P}$. Consequently, if ILAP $\mathcal{P}$ has n optimal solutions, then LAP $\mathcal{P}'$ has at most n^2 optimal solutions. Note that LAP $\mathcal{P}'$ need not have exactly n^2 optimal solutions as not every pair of optimal solutions of ILAP $\mathcal{P}$ can be combined together (recall Remark 5).*

To compare with other reductions connected to LAP, e.g., the well-known reduction from rectangular LAP to LAP in [7, Section 5.4.4] (also mentioned in [4]) increases the overall number of optimal solutions $(|\mathcal{L}|-|\mathcal{V}|)!$ times where $|\mathcal{V}|$ and $|\mathcal{L}|$ are the partition sizes of the rectangular LAP (with $|\mathcal{V}| < |\mathcal{L}|$) [4]. The reduction of IQAP to QAP in [17, Section A1] (and the analogous reduction of ILAP to LAP) increases the number of optimal solutions at least $\max\{|\mathcal{L}|, |\mathcal{V}|\}!/||\mathcal{L}|-|\mathcal{V}||!$ times.

Now, we will discuss how to transform a solution feasible for the LP formulation of LAP $\mathcal{P}'$ to a solution feasible for the LP formulation of ILAP $\mathcal{P}$. The mappings for both primal and dual solutions are introduced in Theorem 5 where we show that they preserve not only optimality but also membership in the relative interior of optimizers. For the proof of this theorem, we require an auxiliary lemma.

Lemma 3. *Let μ' be from the relative interior of optimizers of the primal* (2) *for LAP $\mathcal{P}'$ (i.e., from the relative interior of optimizers of the primal* (22)*, stated later). Then, for all $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v \setminus \{\#\}$, $\mu'_v(\ell) > 0 \iff \mu'_\ell(v) > 0$.*

Proof. Let μ^* be defined by

$$v \in \mathcal{V} : \mu^*_v(v) = \mu'_v(v) \tag{19a}$$
$$\forall \ell \in \mathcal{L} \setminus \{\#\} : \mu^*_\ell(\ell) = \mu'_\ell(\ell) \tag{19b}$$
$$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v \setminus \{\#\} : \mu^*_v(\ell) = \mu'_\ell(v) \tag{19c}$$
$$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v \setminus \{\#\} : \mu^*_\ell(v) = \mu'_v(\ell). \tag{19d}$$

By symmetry in the definition of μ^*, it is easy to see that μ^* is also feasible for the primal (22). Moreover, the objective values coincide for μ^* and μ', so μ^* is also optimal.

We prove the claim by contradiction.[6] Let (α', β') be from the relative interior of optimizers of the dual (22). Without loss of generality, suppose that there is $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v \setminus \{\#\}$ such that $\mu'_v(\ell) > 0$ and $\mu'_\ell(v) = 0$. By strict complementary slackness, $\alpha'_\ell + \beta'_v < \theta_v(\ell)/2$ (see (22e)). By definition of μ^*, we have $\mu^*_\ell(v) = \mu'_v(\ell) > 0$ and μ^* does not satisfy complementary slackness with (α', β'). This is contradictory with optimality of μ^*. ☐

Theorem 5. *Let μ' and (α', β') be feasible for the primal and dual* (2)*, respectively, for the LAP instance $\mathcal{P}'$. Define μ by*

$$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v : \mu_v(\ell) = \begin{cases} (\mu'_v(\ell) + \mu'_\ell(v))/2 & \text{if } \ell \neq \# \\ \mu'_v(v) & \text{if } \ell = \# \end{cases} \tag{20}$$

and (α, β) by

$$\forall v \in \mathcal{V} :\ \alpha_v = \alpha'_v + \beta'_v \tag{21a}$$
$$\forall \ell \in \mathcal{L} \setminus \{\#\} :\ \beta_\ell = \alpha'_\ell + \beta'_\ell. \tag{21b}$$

Then it holds that

(a) μ is feasible for the primal (13)*,*

(b) (α, β) is feasible for the dual (13)*,*

(c) if μ' is optimal for primal (2)*, then μ is optimal for primal* (13)*,*

(d) if (α', β') is optimal for dual (2)*, then (α, β) is optimal for dual* (13)*,*

(e) if μ' is in the relative interior of optimizers of primal (2)*, then μ is in the relative interior of optimizers of primal* (13)*,*

[6] Lemma 3 can be also proved using Proposition 1: by symmetry of the problem $\mathcal{P}'$, there is an optimal assignment with $x_v = \ell$ if and only if there is an optimal assignment with $x_\ell = v$.

(f) if (α', β') is in the relative interior of optimizers of dual (2)*, then (α, β) is in the relative interior of optimizers of dual* (13)*.*

Proof. For clarity of this proof, we write the LP formulation (2) of the LAP $\mathcal{P}'$. Together with the corresponding dual, this reads

$$
\begin{aligned}
& && \min\ p(\mu') && \max\ d(\alpha', \beta') && \text{(22a)}\\
\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v \setminus \{\#\}: & && \mu'_v(\ell) \geq 0 && \alpha'_v + \beta'_\ell \leq \theta_v(\ell)/2 && \text{(22b)}\\
\forall v \in \mathcal{V}: & && \mu'_v(v) \geq 0 && \alpha'_v + \beta'_v \leq \theta_v(\#) && \text{(22c)}\\
\forall \ell \in \mathcal{L} \setminus \{\#\}: & && \mu'_\ell(\ell) \geq 0 && \alpha'_\ell + \beta'_\ell \leq 0 && \text{(22d)}\\
\forall \ell \in \mathcal{L} \setminus \{\#\},\, v \in \mathcal{V}_\ell: & && \mu'_\ell(v) \geq 0 && \alpha'_\ell + \beta'_v \leq \theta_v(\ell)/2 && \text{(22e)}\\
\forall v \in \mathcal{V}: & && \mu'_v(v) + \sum_{\ell \in \mathcal{L}_v \setminus \{\#\}} \mu'_v(\ell) = 1 && \alpha'_v \in \mathbb{R} && \text{(22f)}\\
\forall l \in \mathcal{L} \setminus \{\#\}: & && \mu'_\ell(\ell) + \sum_{v \in \mathcal{V}_\ell} \mu'_\ell(v) = 1 && \alpha'_\ell \in \mathbb{R} && \text{(22g)}\\
\forall v \in \mathcal{V}: & && \mu'_v(v) + \sum_{\ell \in \mathcal{L}_v \setminus \{\#\}} \mu'_\ell(v) = 1 && \beta'_v \in \mathbb{R} && \text{(22h)}\\
\forall \ell \in \mathcal{L} \setminus \{\#\}: & && \mu'_\ell(\ell) + \sum_{v \in \mathcal{V}_\ell} \mu'_v(\ell) = 1 && \beta'_\ell \in \mathbb{R} && \text{(22i)}
\end{aligned}
$$

where the primal and dual objectives are defined by

$$p(\mu') = \sum_{\substack{v \in \mathcal{V} \\ \ell \in \mathcal{L} \setminus \{\#\}}} \theta_v(\ell)\big(\mu'_v(\ell) + \mu'_\ell(v)\big)/2 + \sum_{v \in \mathcal{V}} \theta_v(\#)\mu'_v(v) \tag{23a}$$

$$d(\alpha', \beta') = \sum_{v \in \mathcal{V}} (\alpha'_v + \beta'_v) + \sum_{\ell \in \mathcal{L} \setminus \{\#\}} (\alpha'_\ell + \beta'_\ell). \tag{23b}$$

Note that lines (22b)-(22e), (22f)-(22g), and (22h)-(22i) correspond to lines (2b), (2c), and (2d), respectively.

We begin by proving (a). For any $v \in \mathcal{V}$, it holds that

$$\sum_{\ell \in \mathcal{L}_v} \mu_v(\ell) = \Big(\mu'_v(v) + \sum_{\ell \in \mathcal{L}_v \setminus \{\#\}} \mu'_v(\ell)\Big)/2 + \Big(\mu'_v(v) + \sum_{\ell \in \mathcal{L}_v \setminus \{\#\}} \mu'_\ell(v)\Big)/2 = 1 \tag{24}$$

where we used (20) and primal constraints (22f) and (22h). Analogously, for any $\ell \in \mathcal{L} \setminus \{\#\}$,

$$\sum_{v \in \mathcal{V}_\ell} \mu_v(\ell) = \sum_{v \in \mathcal{V}_\ell} \mu'_v(\ell)/2 + \sum_{v \in \mathcal{V}_\ell} \mu'_\ell(v)/2 \leq 1 \tag{25}$$

which again follows from (20) and primal constraints (22g) and (22i). Non-negativity of μ follows trivially from non-negativity of μ'.

Next, we proceed with (b). To show that (α, β) satisfies the dual constraint (13b), we consider two cases, depending on whether $\ell = \#$. First, if $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v \setminus \{\#\}$, then $\alpha_v + \beta_\ell = \alpha'_v + \beta'_v + \alpha'_\ell + \beta'_\ell \leq \theta_v(\ell)$ where the equality is given by (21) and the inequality follows from the two corresponding inequalities in (22b) and (22e). Second, if $v \in \mathcal{V}$ and $\ell = \#$, then $\alpha_v = \alpha'_v + \beta'_v \leq \theta_v(\#)$ where the equality again follows from (21) and the inequality from (22c). Finally, for the other dual constraint (13d), we have $\beta_\ell = \alpha'_\ell + \beta'_\ell \leq 0$ by (21) and (22d).

To show (c) and (d), recall that the optimal values of ILAP $\mathcal{P}$ and LAP $\mathcal{P}'$ and their LP formulations coincide by Proposition 3 and integrality of the primal linear programs. By (a) and (b), (20) and (21) map a feasible solution of the primal and dual (22) to a feasible solution of the primal and dual (13), respectively. The claim follows because these mappings preserve the objective, which can be verified by plugging the definitions (20) and (21) to (23) and comparing to (13a).

To prove (e) and (f), let μ' and (α', β') be from the relative interior of optimizers of the primal and dual (22), respectively. We will prove that μ defined by (20) and (α, β) defined by (21) are in the relative interior of optimizers of the primal and dual (13), respectively, by showing that they satisfy strict complementary slackness.

We start by proving it for the constraints (13d). See that, for all $\ell \in \mathcal{L} \setminus \{\#\}$,

$$\beta_\ell = 0 \iff \alpha'_\ell + \beta'_\ell = 0 \iff \mu'_\ell(\ell) > 0 \iff \sum_{v \in \mathcal{V}_\ell} \mu'_v(\ell) + \sum_{v \in \mathcal{V}_\ell} \mu'_\ell(v) < 2 \iff \sum_{v \in \mathcal{V}_\ell} \mu_v(\ell) < 1 \tag{26}$$

where the first equivalence is given by definition of β in (21), the second equivalence follows from strict complementarity of μ' and (α', β') (see (22d)), and the third equivalence follows from constraints (22g) and (22i). Definition of μ in (20) yields the last equivalence in (26).

Next, we proceed with the constraints (13b). For all $v \in \mathcal{V}$ and $\ell \in \mathcal{L}_v \setminus \{\#\}$, the following equivalences hold:

$$\alpha'_\ell + \beta'_v = \theta_v(\ell)/2 \iff \mu'_\ell(v) > 0 \iff \mu'_v(\ell) > 0 \iff \alpha'_v + \beta'_\ell = \theta_v(\ell)/2. \tag{27}$$

The first and last equivalences in (27) are given by strict complementarity of μ' and (α', β') (see (22e) and (22b)) and the middle equivalence follows from Lemma 3. These equivalences imply $\mu_v(\ell) > 0 \iff \alpha_v + \beta_\ell = \theta_v(\ell)$ by definition of μ and (α, β) in (20) and (21), respectively. For $v \in \mathcal{V}$ and $\ell = \#$, we have that

$$\mu_v(\#) > 0 \iff \mu'_v(v) > 0 \iff \alpha'_v + \beta'_v = \theta_v(\#) \iff \alpha_v = \theta_v(\#) \tag{28}$$

where the first and last equivalences hold by definition of μ and (α, β) and the middle equivalence holds by strict complementarity of μ' and (α', β') (see (22c)). □

4 Incomplete QAP and Subproblems of Its LP Relaxation

We define the *incomplete quadratic assignment problem* (IQAP) [19, 17] as follows. Let $(\mathcal{V}, \mathcal{E})$ be an undirected loopless graph where $\mathcal{V}$ is a finite set of vertices and $\mathcal{E} \subseteq \{\{u, v\} \mid u, v \in \mathcal{V},\, u \neq v\}$ is a set of edges. For clarity of notation, we abbreviate $\{u, v\}$ to uv in this section. Next, let $\mathcal{L}$ be a finite set of labels such that $\# \in \mathcal{L}$ and, for each $v \in \mathcal{V}$, let $\mathcal{L}_v \subseteq \mathcal{L}$ with $\# \in \mathcal{L}_v$ be the set of allowed labels for vertex v. Finally, let $\theta_v \colon \mathcal{L}_v \to \mathbb{R}$ and $\theta_{uv} \colon \mathcal{L}_u \times \mathcal{L}_v \to \mathbb{R}$ be the cost functions for each $v \in \mathcal{V}$ and $uv \in \mathcal{E}$, respectively (adopting that $\theta_{uv}(\ell, k) = \theta_{vu}(k, \ell)$).

In this setting, the IQAP reads

$$\min \sum_{v \in \mathcal{V}} \theta_v(x_v) + \sum_{uv \in \mathcal{E}} \theta_{uv}(x_u, x_v) \tag{29a}$$

$$\forall v \in \mathcal{V} : x_v \in \mathcal{L}_v \tag{29b}$$

$$\forall \ell \in \mathcal{L} \setminus \{\#\} : \sum_{v \in \mathcal{V}} [\![x_v = \ell]\!] \leq 1 \tag{29c}$$

and differs from the ILAP (12) only by the quadratic term in the objective (29a).

Remark 7. *To compare, the (complete) QAP does not generally require a dummy label (i.e., we remove $\#$ from $\mathcal{L}$ and from each $\mathcal{L}_v$, $v \in \mathcal{V}$) and seeks an assignment $x \colon \mathcal{V} \to \mathcal{L}$ minimizing* (29a) *such that $x_u \neq x_v$ for each distinct $u, v \in \mathcal{V}$. In this case, we typically also have $|\mathcal{V}| = |\mathcal{L}|$ so that x is a bijection between $\mathcal{V}$ and $\mathcal{L}$ [42, 7].*

An LP relaxation of IQAP is the left-hand problem of the primal-dual pair

	Primal	Dual	
	$\min \sum_{v \in \mathcal{V}, \ell \in \mathcal{L}_v} \theta_v(\ell)\mu_v(\ell) + \sum_{uv \in \mathcal{E}, (k,\ell) \in \mathcal{L}_u \times \mathcal{L}_v} \theta_{uv}(k, \ell)\mu_{uv}(k, \ell)$	$\max \sum_{v \in \mathcal{V}} \alpha_v + \sum_{\ell \in \mathcal{L} \setminus \{\#\}} \beta_\ell$	(30a)
$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v :$	$\mu_v(\ell) \geq 0$	$\alpha_v + [\![\ell \neq \#]\!]\beta_\ell \leq \theta^\phi_v(\ell)$	(30b)
$\forall uv \in \mathcal{E},\, (k, \ell) \in \mathcal{L}_u \times \mathcal{L}_v :$	$\mu_{uv}(k, \ell) \geq 0$	$0 \leq \theta^\phi_{uv}(k, \ell)$	(30c)
$\forall v \in \mathcal{V} :$	$\sum_{\ell \in \mathcal{L}_v} \mu_v(\ell) = 1$	$\alpha_v \in \mathbb{R}$	(30d)
$\forall v \in \mathcal{V},\, u \in \mathcal{N}_v,\, \ell \in \mathcal{L}_v :$	$\sum_{k \in \mathcal{L}_u} \mu_{uv}(k, \ell) = \mu_v(\ell)$	$\phi_{v \to u}(\ell) \in \mathbb{R}$	(30e)
$\forall \ell \in \mathcal{L} \setminus \{\#\} :$	$\sum_{v \in \mathcal{V}_\ell} \mu_v(\ell) \leq 1$	$\beta_\ell \leq 0$	(30f)

where $\mathcal{N}_v = \{u \in \mathcal{V} \mid uv \in \mathcal{E}\}$ is the set of neighbors of vertex v in the graph $(\mathcal{V}, \mathcal{E})$ and

$$\theta_v^\phi(\ell) = \theta_v(\ell) + \sum_{u \in \mathcal{N}_v} \phi_{v \to u}(\ell) \tag{31a}$$

$$\theta_{uv}^\phi(k, \ell) = \theta_{uv}(k, \ell) - \phi_{v \to u}(\ell) - \phi_{u \to v}(k) \tag{31b}$$

are reparametrized costs [28, 19]. In contrast to the problems considered in the previous sections, the primal (30) is not integral and it is therefore not an LP formulation of IQAP. This is of course not surprising as the IQAP problem is NP-hard.

The LP relaxation (30) is implicit in [19, Section 5.2] and can be seen as an adaptation of the LP relaxation of QAP from [42] to the case of IQAP (also see [36, Section 3.3.1]). As already noted earlier, the beneficial feature of this relaxation is that block-variable subproblems of the dual correspond to dual LP formulation of ILAP and dual LP relaxation of WCSP. We thoroughly show this in the following two subsections.

4.1 ILAP Subproblem

Let us fix the ϕ variables in the dual (30). The dual restricted to the variables (α, β) together with the corresponding primal[7] reads

$$\min \sum_{\substack{v \in \mathcal{V} \\ \ell \in \mathcal{L}_v}} \theta_v^\phi(\ell) \mu_v(\ell) \qquad \max \sum_{v \in \mathcal{V}} \alpha_v + \sum_{\ell \in \mathcal{L} \setminus \{\#\}} \beta_\ell \tag{32a}$$

$$\forall v \in \mathcal{V},\, \ell \in \mathcal{L}_v: \qquad \mu_v(\ell) \geq 0 \qquad \alpha_v + [\![\ell \neq \#]\!] \beta_\ell \leq \theta_v^\phi(\ell) \tag{32b}$$

$$\forall v \in \mathcal{V}: \qquad \sum_{\ell \in \mathcal{L}_v} \mu_v(\ell) = 1 \qquad \alpha_v \in \mathbb{R} \tag{32c}$$

$$\forall \ell \in \mathcal{L} \setminus \{\#\}: \qquad \sum_{v \in \mathcal{V}_\ell} \mu_v(\ell) \leq 1 \qquad \beta_\ell \leq 0. \tag{32d}$$

It is easy to see that this is the LP formulation (13) of ILAP except that the costs are θ^ϕ instead of θ.

For the purpose of applying BCA to the dual (32), it is convenient to notice that, at optimum of the dual, we always have

$$\forall v \in \mathcal{V}: \alpha_v = \min_{\ell \in \mathcal{L}_v} \big(\theta_v^\phi(\ell) - [\![\ell \neq \#]\!] \beta_\ell\big). \tag{33}$$

Plugging (33) into the dual (32) results in the optimization problem

$$\max \sum_{v \in \mathcal{V}} \min_{\ell \in \mathcal{L}_v} \big(\theta_v^\phi(\ell) - [\![\ell \neq \#]\!] \beta_\ell\big) + \sum_{\ell \in \mathcal{L} \setminus \{\#\}} \beta_\ell \tag{34a}$$

$$\forall \ell \in \mathcal{L} \setminus \{\#\}: \beta_\ell \leq 0, \tag{34b}$$

which can be interpreted as the maximization of a concave piecewise-affine function over non-positive variables.

Coordinate Ascent To optimize (34) coordinate-wise, we derive coordinate-ascent updates for the individual β variables. For this, let $\ell \in \mathcal{L} \setminus \{\#\}$. The objective (34a) restricted to the variable β_ℓ reads (up to a constant)

$$\sum_{v \in \mathcal{V}_\ell} \min \big\{\theta_v^\phi(\ell) - \beta_\ell, \Phi_v(\ell)\big\} + \beta_\ell \tag{35}$$

where

$$\Phi_v(\ell) = \min_{\ell' \in \mathcal{L}_v \setminus \{\ell\}} \big(\theta_v^\phi(\ell') - [\![\ell' \neq \#]\!] \beta_{\ell'}\big) \tag{36}$$

[7] Since the primal-dual pair (32) is defined by restricting the set of dual variables in (30), the primal (32) contains only a subset of the primal constraints from (30). This also applies to the LP relaxation of WCSP (38) shown in Section 4.2.

are constants w.r.t. β_ℓ. Let b_1 and b_2 be the smallest and second smallest value among $\theta^\phi_v(\ell)-\Phi_v(\ell)$ for $v \in \mathcal{V}_\ell$, respectively.[8] Based on [40, Lemma 28 in supplement], the set of maximizers of (35) subject to $\beta_\ell \le 0$ is the interval $[\min\{b_1, 0\}, \min\{b_2, 0\}]$. To satisfy the relative-interior rule, we can choose, e.g., the midpoint of this interval, i.e., set

$$\beta_\ell := (\min\{b_1, 0\} + \min\{b_2, 0\})/2. \tag{37}$$

4.2 WCSP Subproblem

Analogously, let us fix variables β in the dual LP relaxation (30). The dual restricted to variables (α, ϕ) together with the corresponding primal reads

$$
\begin{aligned}
&\min \sum_{\substack{v\in\mathcal{V}\\ \ell\in\mathcal{L}_v}} \theta'_v(\ell)\mu_v(\ell) + \sum_{\substack{uv\in\mathcal{E}\\ (k,\ell)\in\mathcal{L}_u\times\mathcal{L}_v}} \theta_{uv}(k,\ell)\mu_{uv}(k,\ell) && \max \sum_{v\in\mathcal{V}} \alpha_v && \text{(38a)}\\
&\forall v\in\mathcal{V},\, \ell\in\mathcal{L}_v: \quad \mu_v(\ell)\ge 0 && \alpha_v \le \theta'^\phi_v(\ell) && \text{(38b)}\\
&\forall uv\in\mathcal{E},\, (k,\ell)\in\mathcal{L}_u\times\mathcal{L}_v: \quad \mu_{uv}(k,\ell)\ge 0 && 0\le \theta^\phi_{uv}(k,\ell) && \text{(38c)}\\
&\forall v\in\mathcal{V}: \quad \sum_{\ell\in\mathcal{L}_v}\mu_v(\ell) = 1 && \alpha_v\in\mathbb{R} && \text{(38d)}\\
&\forall v\in\mathcal{V},\, u\in\mathcal{N}_v,\, \ell\in\mathcal{L}_v: \quad \sum_{k\in\mathcal{L}_u}\mu_{uv}(k,\ell) = \mu_v(\ell) && \phi_{v\to u}(\ell)\in\mathbb{R} && \text{(38e)}
\end{aligned}
$$

where $\theta'_v(\ell) = \theta_v(\ell) - [\![\ell \ne \#]\!]\beta_\ell$ are constants for each $v \in \mathcal{V}$, $\ell \in \mathcal{L}_v$. This subproblem is equivalent to the LP relaxation of (pairwise) WCSP which is also known as the MAP inference problem in graphical models [42, 19, 28, 39]. The LP relaxation (38) was proposed independently multiple times (e.g., in [29] or [11]) and is often referred to as the basic LP relaxation of WCSP [34] or local polytope relaxation [28]. Note that this linear program was shown to be as hard to solve as any linear program [26], thus solving (30) is also at least as hard.

Remark 8. *There exist several equivalent formulations of the LP relaxation* (38) *[39]. As with* (32)*, for any dual-optimal solution* (α, ϕ) *of* (38)*,* (33) *holds, which results in a different form of the dual, namely*

$$
\begin{aligned}
&\max \sum_{v\in\mathcal{V}} \min_{\ell\in\mathcal{L}_v} \theta'^\phi_v(\ell) && \text{(39a)}\\
&\forall uv\in\mathcal{E},\, (k,\ell)\in\mathcal{L}_u\times\mathcal{L}_v : \theta^\phi_{uv}(k,\ell)\ge 0 && \text{(39b)}\\
&\forall v\in\mathcal{V},\, u\in\mathcal{N}_v,\, \ell\in\mathcal{L}_v : \phi_{v\to u}(\ell)\in\mathbb{R}. && \text{(39c)}
\end{aligned}
$$

Even though LP relaxation in this form was considered, e.g., in [11], computer-vision literature [19, 42, 39, 28, 37] typically considers the unconstrained form

$$
\begin{aligned}
&\max \sum_{v\in\mathcal{V}} \min_{\ell\in\mathcal{L}_v} \theta'^\phi_v(\ell) + \sum_{uv\in\mathcal{E}} \min_{(k,\ell)\in\mathcal{L}_u\times\mathcal{L}_v} \theta^\phi_{uv}(k,\ell) && \text{(40a)}\\
&\forall v\in\mathcal{V},\, u\in\mathcal{N}_v,\, \ell\in\mathcal{L}_v : \phi_{v\to u}(\ell)\in\mathbb{R}. && \text{(40b)}
\end{aligned}
$$

It is known [39, Remark 3] that the optimal values of (39) *and* (40) *coincide. In detail, one can introduce the constraint* $\sum_{(k,\ell)\in\mathcal{L}_u\times\mathcal{L}_v}\mu_{uv}(k,\ell) = 1$ *for each* $uv \in \mathcal{E}$ *into the primal* (38) *and eliminate the corresponding dual variables analogously to* (33)*. Note, such a change does not influence the optimal value because the additional constraints are already implied by primal constraints* (38d) *and* (38e)*.*

5 Compared Methods and Experimental Evaluation

Given the background from the previous sections, let us now summarize the algorithms that will be compared. A general algorithmic scheme is shown in Algorithm 2. There, the dual variables β and ϕ

[8] For clarity, if the minimal value is attained for multiple $v \in \mathcal{V}_\ell$, then $b_1 = b_2$. If $|\mathcal{V}_\ell| = 1$ or $\mathcal{V}_\ell = \emptyset$, then $b_2 = 0$ or $b_1 = b_2 = 0$, respectively. Note that, if $|\mathcal{V}_\ell| \ge 2$, then the objective (35) is increasing for $\beta_\ell \le b_1$, constant for $b_1 \le \beta_\ell \le b_2$, and decreasing for $\beta_\ell \ge b_2$.

input: instance of IQAP
output: lower bound on the optimal value
1 Initialize $\beta := 0$ and $\phi := 0$.
2 **repeat**
3 | Improve ϕ variables for fixed β variables.
4 | Improve β variables for fixed ϕ variables.
5 **until** time limit is not reached;
6 **return** current dual objective (30a) where α is (33)

Algorithm 2: General algorithmic scheme for obtaining a lower bound on IQAP.

are first initialized to be feasible (by assigning them zero values). Then, we iteratively improve the current dual solution by separately updating the ϕ and β variables. We do not consider the α variables since their value is assumed to be implicitly determined by (33) based on the current values of β and ϕ.

Since the common optimal value of the primal-dual pair (30) constitutes a lower bound on the optimal value of the original IQAP problem (29), any dual-feasible solution also provides a lower bound, which is determined by its objective. Algorithm 2 can be thus seen as gradually improving a lower bound on the optimal value of the IQAP problem.

Compared Algorithms To obtain a concrete algorithm, we need to define how the individual steps of Algorithm 2 are performed precisely:

- Concerning line 3, the ϕ variables are improved by one loop of MPLP algorithm in [42] and by one loop of MPLP++ in [19]. Since MPLP++ [37] is an improved[9] version of MPLP [15], we consider only MPLP++ in our experiments, i.e., line 3 is performed by running a single loop of MPLP++ algorithm.
- The update of the other dual variables on line 4 can be again done approximately by a single loop of BCA updates (as in [19]) or exactly, e.g., by the Hungarian method (as in [42]). In addition to these options, we consider relative-interior updates, which can be obtained by combining any exact method with our Algorithm 1.

As listed above, the only difference in the compared methods lies in how the update of the dual variables β is performed. To be precise, the three options for updating β variables yield the following algorithms:

- `BCA` performs a single loop of coordinate-wise updates of β_ℓ while adhering to the relative-interior rule [40]. In detail, we sequentially perform the updates (37) for all $\ell \in \mathcal{L} \setminus \{\#\}$. Note that, after a loop of updates[10], the resulting β need not be optimal for the ILAP subproblem (34).
- `Hung` updates the dual variables β by exactly solving the ILAP subproblem. For this, we use the reduction to LAP from Section 3.1, solve the resulting LAP by the Hungarian method [20], and obtain an optimal solution of the dual LP formulation of ILAP via (21).
- `Hung+RI` is the same as `Hung` but when the dual solution of the LAP is obtained, we shift it to the relative interior of optimizers using Algorithm 1. By Theorem 5, this choice of optimizer adheres to the relative-interior rule.

Remark 9. `BCA` *is analogous to the BCA algorithm considered in [19] and* `Hung` *is similar to Hungarian-BP [42] except that we use MPLP++ instead of MPLP. Another difference to the particular implementations of these methods lies in the fact that the precise form of the duals in [19] and [42] is different (and also different from the dual that we consider here, which is more compact). In more detail, our dual can be obtained from the ones in [19] or [42] by adding more*

[9]Although the fixed points of MPLP and MPLP++ coincide, it was shown that, if initialized at the same point, the dual objective after a single MPLP++ iteration is not worse than the dual objective after a single MPLP iteration [37, Section 5.1].

[10]It is likely that even if the updates were performed repeatedly, the objective need not converge to the optimum. This follows from the fact that the dual of LAP is not solvable by BCA [13, Section 5.1] (although the optimum was frequently reached in experiments), so the dual of ILAP is also likely not solvable by BCA.

constraints and eliminating variables. It is known [13] that the quality of fixed points of BCA highly depends on the precise problem formulation. In the aforementioned case, this change of formulation seems to have a negative impact on the quality of fixed points.

Problem Instances Used for Evaluation For evaluation, we used the recent computer-vision benchmark [17] (451 instances), as well as the operations-research benchmark QAPLIB [8] (132 instances). This resulted in 583 instances in total.[11] The sizes and densities of the instances from different groups are shown in Table 1 where the first 11 groups contain computer-vision instances and the remaining 15 groups are instances from QAPLIB. Due to the different nature of the instances, we separate them visually in our tables. The time limits for individual groups are set so that the methods have enough time to converge to a fixed point that would not be improved if the time limit was prolonged. We implemented the corresponding versions of Algorithm 2 in Matlab and performed the evaluation on a laptop with i7-7500U processor at 2.7 GHz and 8 GB RAM.

Comparison Results for Original Instances To evaluate the results, we aggregate the attained bounds group-wise in Table 2 that reports in how many instances each method attained the best bound and also shows the average bound. To avoid issues with numerical precision when computing the number of best bounds, we proceed as follows. Denoting by $B_{\texttt{BCA}}$, $B_{\texttt{Hung}}$, and $B_{\texttt{Hung+RI}}$ the attained bound by each method on a particular instance, a bound $B \in \{B_{\texttt{BCA}}, B_{\texttt{Hung}}, B_{\texttt{Hung+RI}}\}$ is considered best if $B \geq (1 + 10^{-10}) \cdot \max\{B_{\texttt{BCA}}, B_{\texttt{Hung}}, B_{\texttt{Hung+RI}}\}$. Note, all the bounds are negative.

Concerning the computer-vision instances, except for 3 groups, `Hung+RI` provided the best bound most frequently and solely attained the best average bounds. In groups *house-sparse* and *opengm*, all methods attained the same bound in each instance and are thus equally-performing (to be more precise, no method can improve the initial bound in *opengm*). A similar situation occurred for group *motor* where all methods obtained similar bounds and none is significantly ahead of the others. The methods `BCA` and `Hung` also perform relatively well (but not as `Hung+RI`) for groups *car* and *hotel*. In *house-dense* and *pairs*, a significant portion of the best bounds is taken by `Hung`.

For the QAPLIB instances, the situation is different because no method is able to improve the initial bound in any instance except for group *bur*. For instances in this group, `BCA` is still unable to improve the initial bound but `Hung` and `Hung+RI` can. Although `Hung` attained the best bound more frequently (always) and also has better average bound than `Hung+RI`, the differences in the values of the bound attained by these methods are relatively small when compared to `BCA`. In particular, the value $(B_{\texttt{Hung}} - B_{\texttt{Hung+RI}})/(B_{\texttt{Hung}} - B_{\texttt{BCA}})$ is always lower than 10^{-5}.

Comparison Results for Augmented Instances To possibly avoid trivial fixed points, we replace the initial value $\theta_{uv}(\ell, \ell) = 0$ by 10^7 for all $uv \in \mathcal{E}$ and $\ell \in (\mathcal{L}_v \cap \mathcal{L}_u) \setminus \{\#\}$, which was also done in [19]. This change does not influence the optimal value of the IQAP instance because, for any x feasible to (29) and any $uv \in \mathcal{E}$, $x_u \in \mathcal{L}_u \setminus \{\#\}$ implies $x_u \neq x_v$. The obtained results for such augmented instances are reported in Table 3. Except for groups *caltech-large* and *caltech-small*, the attained bounds are comparable or better for all methods after this change.

To provide a more detailed comment, the relative performance of the methods in the computer-vision instances does not change much, except for *flow* (in favour of `Hung`), *house-dense*, *hotel* (all methods become comparable), *opengm* (`Hung+RI` takes majority of best bounds), and *worms* (`BCA` takes approximately half of best bounds from `Hung+RI`). See that augmentation of the instances allowed each method to improve the initial bound in group *opengm* (in contrast to the results in Table 2), so each method can improve the initial bound in any of the 451 computer-vision instances.

Regarding QAPLIB, `BCA`, `Hung`, and `Hung+RI` can improve the initial bound in 39, 69, and 69 instances, respectively, out of the 132 QAPLIB instances. The bounds provided by `BCA` are never better than (and sometimes equal to) the bound computed by `Hung` or `Hung+RI`. In particular, for 9 groups, all methods provide identical bounds. Concerning the remaining 6 groups (i.e., *bur*, *chr*, *had*, *lipa*, *rou*, and *tai*), `Hung+RI` always achieves the best bound most frequently although `Hung` also takes some portion of best bounds in some groups (and sometimes attains a better bound than `Hung+RI`). Nevertheless, `Hung+RI` solely attains the best average bounds in 5 of these groups.

[11] We downloaded the computer-vision instances in `.dd` format from `https://vislearn.github.io/gmbench/datasets/`. The datasets considered in [19] constitute a strict subset of those in [17]. Namely, the groups *caltech-small*, *caltech-large*, and *house-sparse* are missing in [19]. We converted the QAPLIB benchmark to `.dd` format and subtracted large constants from the unary costs to guarantee a complete assignment when solving IQAP. Consequently, the bounds obtained for these transformed instances are not directly comparable to the original ones.

group	instances	time limit (in minutes)	$\|\mathcal{V}\|$	$\|\mathcal{L} \setminus \{\#\}\|$	$\|\mathcal{L}_v \setminus \{\#\}\|$	$\|\mathcal{E}\|$	density $\frac{2\|\mathcal{E}\|}{\|\mathcal{V}\|(\|\mathcal{V}\|-1)}$ (in percent)
caltech-large	9	5	36-219	51-341	1-62	612-7448	31.2-97.1
caltech-small	21	5	9-117	14-201	1-67	36-2723	30.6-100
car	30	5	19-49	19-49	19-49	46-131	11.1-26.9
flow	6	5	48-126	51-130	1-19	1100-5227	44.6-97.5
hotel	105	5	30	30	30	435	100
house-dense	105	5	30	30	30	435	100
house-sparse	105	5	30	30	30	79	18.2
motor	20	5	15-52	15-52	15-52	33-139	10.5-32.4
opengm	4	5	19-20	19-20	19-20	112-190	65.5-100
pairs	16	30	511-565	523-565	20-24	25173-34334	18.2-22.8
worms	30	90	558	1202-1427	20-127	2343-2363	1.51-1.52
bur	8	1	26	26	26	325	100
chr	14	1	12-25	12-25	12-25	11-24	8-16.7
els	1	1	19	19	19	171	100
esc	18	1	16-64	16-64	16-64	0-141	0-95.8
had	5	1	12-20	12-20	12-20	66-190	100
kra	3	1	30-32	30-32	30-32	165-435	33.3-100
lipa	16	5-30	20-90	20-90	20-90	179-4005	94.2-100
nug	15	1	12-30	12-30	12-30	66-435	66.4-100
rou	3	1	12-20	12-20	12-20	66-189	99.5-100
scr	3	1	12-20	12-20	12-20	28-62	32.6-42.4
sko	13	5	42-100	42-100	42-100	861-4950	100
ste	3	5	36	36	36	630	100
tai	26	5-60	10-100	10-100	10-100	45-4950	3.87-100
tho	2	1	30-40	30-40	30-40	435-780	100
wil	2	1	50-100	50-100	50-100	1225-4950	100

Table 1: Properties of instances.

group	instances	# best bound			average bound		
		BCA	Hung	Hung+RI	BCA	Hung	Hung+RI
caltech-large	9	1	0	**8**	-52723.332119	-46428.494418	**-46097.488226**
caltech-small	21	3	3	**15**	-13652.503528	-12090.593939	**-12033.44565**
car	30	19	17	**26**	-70.709679522	-70.630589459	**-70.552252184**
flow	6	1	1	**4**	-2987.5410857	-2924.2261306	**-2921.9594394**
hotel	105	60	77	**92**	-4333.2653764	-4328.4675597	**-4328.1062817**
house-dense	105	0	48	**57**	-3932.7580921	-3907.814637	**-3907.752873**
house-sparse	105	**105**	**105**	**105**	**-66.78286813**	**-66.78286813**	**-66.78286813**
motor	20	15	**17**	16	-63.432831899	**-63.347527022**	-63.35557555
opengm	4	**4**	**4**	**4**	**-192.5**	**-192.5**	**-192.5**
pairs	16	0	5	**11**	-72176.661389	-69497.975593	**-69496.092129**
worms	30	3	0	**27**	-49010.329086	-49002.558873	**-48991.016585**
bur	8	0	**8**	3	-270541453	**-270322255.87**	-270322256.31
chr	14	**14**	**14**	**14**	**-1719481.4286**	**-1719481.4286**	**-1719481.4286**
els	1	**1**	**1**	**1**	**-5884241157**	**-5884241157**	**-5884241157**
esc	18	**18**	**18**	**18**	**-11704.888889**	**-11704.888889**	**-11704.888889**
had	5	**5**	**5**	**5**	**-24416.8**	**-24416.8**	**-24416.8**
kra	3	**3**	**3**	**3**	**-1562613.3333**	**-1562613.3333**	**-1562613.3333**
lipa	16	**16**	**16**	**16**	**-9700506.25**	**-9700506.25**	**-9700506.25**
nug	15	**15**	**15**	**15**	**-36380.666667**	**-36380.666667**	**-36380.666667**
rou	3	**3**	**3**	**3**	**-2512323**	**-2512323**	**-2512323**
scr	3	**3**	**3**	**3**	**-4809050**	**-4809050**	**-4809050**
sko	13	**13**	**13**	**13**	**-1189818.4615**	**-1189818.4615**	**-1189818.4615**
ste	3	**3**	**3**	**3**	**-1177825536**	**-1177825536**	**-1177825536**
tai	26	**26**	**26**	**26**	**-22420280666**	**-22420280666**	**-22420280666**
tho	2	**2**	**2**	**2**	**-3413850**	**-3413850**	**-3413850**
wil	2	**2**	**2**	**2**	**-956250**	**-956250**	**-956250**

Table 2: Aggregated results for comparison of attained bound for the original instances. As mentioned in Footnote 11, the dual bounds on QAPLIB are shifted by a large negative value

group	instances	# best bound			average bound		
		`BCA`	`Hung`	`Hung+RI`	`BCA`	`Hung`	`Hung+RI`
caltech-large	9	1	1	**7**	-56373.951021	-47027.315143	**-46834.78052**
caltech-small	21	1	5	**15**	-15767.73333	-12348.888557	**-12298.78972**
car	30	17	16	**27**	-70.74753519	-70.626416902	**-70.561444436**
flow	6	2	**4**	**4**	-2905.6869569	-2890.2266147	**-2888.5287922**
hotel	105	91	98	**102**	-4300.6219448	-4298.2018252	**-4298.0750981**
house-dense	105	103	**105**	101	-3778.2422098	**-3778.1860114**	-3778.2952917
house-sparse	105	**105**	**105**	**105**	**-66.78286813**	**-66.78286813**	**-66.78286813**
motor	20	15	**18**	17	-63.453858889	**-63.343014952**	-63.353400646
opengm	4	0	1	**3**	-177.35139958	-177.16963725	**-177.04124018**
pairs	16	0	5	**11**	-70255.186636	-68396.600521	**-68391.594598**
worms	30	**17**	0	14	-48500.586699	-48504.052203	**-48498.000833**
bur	8	0	1	**7**	-270541453	-270288019.93	**-270288005.08**
chr	14	1	7	**10**	-1717755	-1713383.6	**-1713368.2986**
els	1	**1**	**1**	**1**	**-5884241157**	**-5884241157**	**-5884241157**
esc	18	**18**	**18**	**18**	**-11704.888889**	**-11704.888889**	**-11704.888889**
had	5	0	2	**3**	-23501.6	-22860.511374	**-22860.475974**
kra	3	**3**	**3**	**3**	**-1550480**	**-1550480**	**-1550480**
lipa	16	0	4	**12**	-9589444.3125	**-9297605.1371**	-9297781.9666
nug	15	**15**	**15**	**15**	**-36135.866667**	**-36135.866667**	**-36135.866667**
rou	3	0	0	**3**	-2512323	-2457885.0268	**-2457799.5439**
scr	3	**3**	**3**	**3**	**-4762288.6667**	**-4762288.6667**	**-4762288.6667**
sko	13	**13**	**13**	**13**	**-1189818.4615**	**-1189818.4615**	**-1189818.4615**
ste	3	**3**	**3**	**3**	**-1177825536**	**-1177825536**	**-1177825536**
tai	26	10	16	**21**	-22420261916	-22420186983	**-22420186927**
tho	2	**2**	**2**	**2**	**-3413850**	**-3413850**	**-3413850**
wil	2	**2**	**2**	**2**	**-956250**	**-956250**	**-956250**

Table 3: Aggregated results for comparison of attained bound for augmented instances (where we set $\theta_{uv}(\ell, \ell) := 10^7$, as explained at the end of Section 5). As mentioned in Footnote 11, the dual bounds on QAPLIB are shifted by a large negative value

6 Conclusion

Our theoretical results provide a characterization of the relative interior of the set of optimal solutions of the LP formulation of LAP. Using this characterization, we were able to provide a linear-time algorithm for computing such a solution (Algorithm 1). Moreover, we analyzed a reduction from ILAP to LAP along with a closed-form mapping that preserves optimality and membership in the relative interior (Section 3.1 and Theorem 5).

We employed the aforementioned results in a BCA method that computes a bound on the optimal value of the IQAP. Based on our experiments, our method with relative-interior solution is capable of providing the best bounds most frequently and otherwise the bounds are competitive to the other considered methods.

Concerning future work, our approach might be more useful in pruning the search space during branch-and-bound search where exact solution is sought thanks to the higher quality of the computed bounds. This may apply not only to the (I)QAP, but also to other problems where LAP naturally occurs as a subproblem, such as the travelling salesperson problem. Finally, although we used Hungarian algorithm for computing an optimal solution to the LAP, there is large potential to make this computation faster using Sinkhorn [12] or auction [3] algorithms.

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