Some Aspects of Analogical Reasoning in Mathematical Creativity

Alison Pease, Markus Guhe, Alan Smaill

School of Informatics, University of Edinburgh,
Informatics Forum, 10 Crichton Street, Edinburgh, EH8 9AB, UK
{A.Pease,M.Guhe,A.Smaill}@ed.ac.uk
http://homepages.inf.ed.ac.uk/apease/research/index.html

Abstract. Analogical reasoning can shed light on both of the two key processes of creativity – generation and evaluation. Hence, it is a powerful tool for creativity. We illustrate this with three historical case studies of creative mathematical conjectures which were either found or evaluated via analogies. We conclude by describing our ongoing efforts to build computational realisations of these ideas.

1 Introduction

Analogical reasoning is an essential aspect of creativity [2] and computational realisations such as [15; 20] have placed it firmly in the computational creativity arena. However, investigation into analogical reasoning has been largely carried out in the context of problem solving in scientific or everyday domains. In particular, very few historical case studies of analogy (excepting [16; 18]) and no computational representations that we know of are in the mathematics domain. There may be features that distinguish mathematics from other domains, such as having a large number of objects as compared to relations, as opposed to domains typically studied by analogy researchers [18], and recent work in analogy [14] has suggested that current theories of analogical reasoning such as the structure mapping theory may require some modification if they are to generalise to mathematics. This largely theoretical paper explores roles that analogical reasoning has played in historical episodes of creativity in mathematics. Towards the end we also describe some computational aspects of our work.

2 A marriage of dimensions

Analogies between different geometrical dimensions, especially between two and three dimensions, date back to Babylonian times and have been particularly productive [16, p. 26]. The discovery of the Descartes–Euler conjecture, that for any polyhedron, the number of vertices (V) minus the number of edges

⋆ We are grateful to Alan Bundy for discussion on the continuity example in section 4. This work was supported by EPSRC grant EP/F035594/1.

(E) plus the number of faces (F) is equal to two is one such example: there are differing accounts of its discovery, but both involve analogy at some level. Euler's own account of his discovery suggests that analogy to two dimensional polygons helped to guide his search for a problem: in a letter to Christian Goldbach, Nov 1750 he wrote ... "there is no doubt that general theorems can be found for them [solids], *just as for plane rectilinear figures ...*" (our italics). The simple relationship that $V = E$ for two dimensional shapes prompted a search for an analogous relationship between edges, faces and vertices in three dimensional solids.

In Polya's reconstruction of this discovery [16, pp. 35-41] he suggested that the analogy was introduced to *evaluate*, as opposed to *generate* the conjecture. He developed a technique for using analogies to evaluate a conjecture: given analogical mappings and conjectures, Polya suggested that we adjust the representation in order to bring the relations closer [16, pp. 42-43]. In the Descartes–Euler example, the re-representation works by noting that vertices are 0D, edges 1D, faces 2D and polyhedron 3D, and then rewriting both conjectures in order of the increasing dimensions. In the polygonal case, $V = E$ then becomes $V-E+1 = 1$, and the polyhedral case $V - E + F = 2$ becomes $V - E + F - 1 = 1$. These two equations now look much more similar: in both of them the number of dimensions starts at zero on the left hand side of the equation, increases by one and has alternating signs. The right hand side is the same in both cases. Polya then suggests that since the two relations are very close and the first relation, for polygons, is true, then we have reason to think that the second relation may be true, and is therefore worthy of a serious proof effort.

3 An extremely daring conjecture

The Basel problem is the problem of finding the sum of the reciprocals of the squares of the natural numbers, *i.e.* finding the exact value of the infinite series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots$. In Euler's time this was a well known and difficult problem, thus in this example the initial problem already exists. Euler used analogical reasoning to find his conjectured solution $\frac{\pi^2}{6}$. To find this he rearranged known facts about finite series and polynomials in order to draw an analogy between finite and infinite series, and then applied a rule about finite series to infinite series, thus discovering what is referred to by Polya as an "extremely daring conjecture" [16, p. 18]. Euler then spent years evaluating both this conjecture and his analogous rule.

4 A split in the real continuum

Cauchy's conjecture and proof that "the limit of any convergent series of continuous functions is itself continuous" [3, p. 131] is another example in which a rule for one area is analogously assumed to hold for another area. In this case the source domain is *series* and the target *limits*, and the rule assumed to hold is that "what is true up to the limit is true at the limit". Lakatos [9, p. 128] states that throughout the eighteenth century this rule was assumed to hold and therefore any proof deemed unnecessary. However, this example is complicated: Cauchy's claim is generally regarded as obviously false, and the clarification of what was wrong is usually taken to be part of the more rigorous formalisation of the calculus developed by Weierstrass, involving the invention of the concept of *uniform convergence*.

This episode was treated by Lakatos in two different ways. Cauchy claimed that the function defined by pointwise limits of continuous functions must be continuous [3]. In fact, what we take to be counter-examples were already known when Cauchy made his claim, as Lakatos points out in his earlier analysis of the evolution of ideas involved [9, Appendix 1]. After discussion with Abraham Robinson, Lakatos then saw that there was an alternative analysis. Robinson was the founder of non-standard analysis, which found a way to rehabilitate talk of infinitesimals (for example, positive numbers greater than zero, but less than any "standard" real number (see [17], first edition 1966). Lakatos's alternative reading, presented in [10], is that Cauchy's proof was correct, but that his notion of (real) number was different from that adopted by mainstream analysis to this day. In analogy terminology, people who had different conceptions of the source domain were critiquing the target domain which Cauchy developed.

5 Computational considerations

We are exploring these ideas computationally in two ways. Firstly, we are using Lakoff and Núñez's notion of mathematical metaphor [11]. Lakoff and Núñez consider that the different notions of "continuum" outlined in §4 correspond to a discretised Number-Line blend (in the case of the Dedekind-Weierstrass reals); a discretised line as the result of "Spaces are Sets of Points" metaphor, where *all* the points on the line are represented (in the case where infinitesimals are present); or by a naturally continuous (physical) line [11, p. 288]. This approach provides promising avenues for the understanding of the relationships between the written representation of the mathematical theories, in this case mostly in natural language, the mathematical structures under consideration, and the geometrical or physical notions that informed the mathematical development. We are using the framework of Information Flow [1] to be more precise about what constitutes metaphors (and blends), by looking at the possible metaphorical relationships in terms of *infomorphisms* between domains. In [7; 8] we show how Information Flow theory [1] can be used to formalise the basic metaphors for arithmetic that ground the notions in embodied human experience (grounding metaphors). This gives us a form of implementation of aspects of the theory evolution involved here. We are extending this to Fauconnier and Turner's conceptual blending [4] and Goguen's Unified Concept theory [6].

Secondly, Schwering *et al.* have developed a mathematically sound framework for analogy making and a symbolic analogy model; heuristic-driven theory projection (HDTP) [19]. Analogies are established via a generalisation of source and target domain. Anti-unification is used to compare formulae of source and target for structural commonalities, then terms with a common generalisation are associated in the analogical mapping. HDTP matches functions and predicates with same and different labels as well matching formulae with different structure. In particular, one of its features is a mechanism for re-representing a domain in order to build an analogy. We are using this system to generate the domain of basic arithmetic from Lakoff and Núñez's four grounding metaphors.

6 Conclusions

Analogy was used in the first example to find, or generate a problem, for which values could subsequently be conjectured. Also, importantly, it was used to aid *evaluation* (thus forming an essential tool in McGraw's "central loop of creativity" [13]). This is particularly interesting given that humans are not very good at making judgements, particularly in historically creative domains, and is not a generally noted use of analogy. The importance of re-representation in order to make a more convincing analogy, making sure that any preconditions for the re-representation are satisfied is also clear in all of these examples. In our second case study the original problem, to find an exact value for the sum of the reciprocals of the squares, was invented independently of the analogy which was used to solve it. Euler then tested his application of the rule he used, and this rule itself, rather than the solution to the Basel problem, became the major contribution of Euler's work in this area. The freedom with which one can apply a rule from one domain to another depends on the extent to which the second domain has already been developed. In Euler's time, while the modern mathematical concept of infinity was not developed, infinite series were an established concept, and thus Euler's work with infinite series had to fit with the structure already developed. In other examples, the target domain is much less developed and the analogiser may be able to *define* the domain such that a desired rule holds. Examples include the operations of addition/subtraction on the reals as analogous to multiplication/division: both are commutative and associative, both have an identity (though a different one) and both admit an inverse operation. Alternatively, it may not be possible to define a target domain such that a particular rule from a source domain holds: in Hamilton's development of quaternions he wanted to develop a new type of number which was analogous to complex numbers but consisted of triples. He was unable to define multiplication on triples, but did discover a way of defining it for quadruples as $i^2 = j^2 = k^2 = ijk = -1$. However, his multiplication was non-commutative, although it was still associative and distributive. Another possibility is that an analogiser may actively *wish* to create a target domain in which a rule from a source domain is broken. One example is the development by Martínez [12] of a system in which the traditional rule that $(-1)(-1) = +1$ is changed to $(-1)(-1) = -1$, resulting in a new mathematical system.

Focusing on analogy as a way of developing new mathematical ideas raises questions about how novel these ideas can be. By definition, an analogy-generated idea must share some sort of similarity with another domain, familiar to the creator. Thus the criterion of novelty, accepted as necessary for creative output seems to be under threat. In this paper we leave aside such considerations: since we consider the examples we give to be both unambiguously creative and unambiguously based on analogy, it seems that any definition of novelty must not exclude analogy. We intend to address this question in a future paper.

Bibliography

[1] J. Barwise and J. Seligman. *Information Flow: The logic of distributed systems*. CUP, 1997.

[2] M. Boden. Computer models of creativity. *AI Mag.*, 30(3):23–34, Fall Issue, 2009.

[3] A. L. Cauchy. *Cours d'Analyse de l'Ecole Polyechnique*. de Bure, Paris, 1821.

[4] Gilles Fauconnier and Mark Turner. *The Way We Think: Conceptual blending and the mind's hidden complexities*. Basic Books, New York, 2002.

[5] D. Gentner. Structure-mapping: a theoretical framework for analogy. *Cognitive Science*, (7):155–170, 1983.

[6] Joseph Goguen. What is a concept? In F. Dau et al., editors, *Conceptual Structures: Common Semantics for Sharing Knowledge, Vol. 3596*, LNAI. Springer, 2005.

[7] M. Guhe, A. Smaill, and A. Pease. A formal cognitive model of mathematical metaphors. In *KI 2009: Advances in AI. 32nd Ann. German Conf. on AI, Germany, Sept., 2009*, pages 323–330. LNAI 5803, Berlin, 2009. Springer.

[8] M. Guhe, A. Smaill, and A. Pease. Using information flow for modelling mathematical metaphors. In *Proc. of the 9th Int. Conf. on Cognitive Modeling*, 2009.

[9] I. Lakatos. *Proofs and Refutations*. Cambridge University Press, UK, 1976.

[10] I. Lakatos. Cauchy and the continuum: the significance of non-standard analysis for the history and philosophy of mathematics. In J. Worral and C. Currie, editors, *Mathematics, science and epistemology*, chapter 3, pages 43–60. CUP, UK, 1978.

[11] George Lakoff and Rafael Núñez. *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. Basic Books, New York, 2000.

[12] A. A. Martínez. *Negative Math: How Mathematical Rules Can Be Positively Bent*. Princeton University Press, 2005.

[13] G. McGraw and D. Hofstadter. Perception and creation of diverse alphabetic styles. *AISBQ*, (85):42–49, 1993.

[14] A. Pease, M. Guhe, and A. Smaill. Analogy formulation and modification in geometry. In *Proc. of the 2nd Int. Conf. on Analogy*, pages 358–364, 2009.

[15] F. C. Pereira. *Creativity and AI: A Conceptual Blending Approach*. Applications of Cognitive Linguistics (ACL). Mouton de Gruyter, Berlin, December 2007.

[16] G. Polya. *Mathematics and plausible reasoning (Vol. 1): Induction and analogy in mathematics*. Princeton University Press, Princeton, USA, 1954.

[17] A. Robinson. *Non-standard Analysis*. Princeton University Press, Princeton, New Jersey, 1996. Revised Edition.

[18] D. Schlimm. Bridging theories with axioms: Boole, Stone, and Tarski. *Perspectives on Mathematical Practices*, II, July, 2008.

[19] A. Schwering et al. Syntactic principles of heuristic-driven theory projection. *Special Issue on Analogies - Integrating Cognitive Abilities in Journal of Cognitive Systems Research*, 10(3):251–269, 2009.

[20] T. Veale. Re-representation and creative analogy: A lexico-semantic perspective. *New Generation Computing*, 24(3):pp. 223–240, 2006.