Reconstruction of an effective magnon mean free path distribution from spin Seebeck measurements in thin films

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Supplementary information: estimation of the regularization parameter

The estimation of the regularization parameter (μ) was estimated using the L-curve criterion [1–3]. The L-curve criterion is a widely used tool for choosing the most adequate regularization parameter in an ill-posed (or inverse) problems of the form:

$$\min\left\{ \|A\cdot F-\alpha\|_2^2+\mu^2\|L\cdot F\|_2^2 \right\} \tag{S1}$$

where A is the ill-conditioned matrix which, in our problem, represents the part of the discretised integral defined in Eq. (6), F is the cumulative bulk LSEE, α_i is the normalized LSSE of the i^{th} measurement, μ is the regularization parameter and L is a second derivative operator, i.e., $L{\cdot}F = F_{i+1} - 2F_i + F_{i-1}$.

The first element of the Eq. S1 ensures the solution of the discretised integral defined in Eq. (6) and the second one enforces smoothness in the reconstructed distribution (F_{acc}) which is controlled by the regularization parameter μ.

The selection of most adequate μ is a hot topic of research in mathematics and several heuristic methods have been applied for this task, such as, e.g.: discrepancy principle, generalized cross-validation (GCV), *L-curve* criterion, the Gfrerer/Raus-method, the quasi-optimality method, to name a few [4]. Among these methods, the *L-curve* criterion is one of the most popular one due to its robustness, velocity and efficiency. Basically, this method balances the size of the discrepancy in the solution (residual norm, $\|A{\cdot}F_\mu - \alpha\|_2$) with the size of the solution (solution norm, $\|L{\cdot}F_\mu\|_2$) for different values of μ. As is displayed in Fig. S1b, the curve has an *L*-like shape composed by a flat and steep parts. The flat part represents solutions dominated by regularization errors and the steep perturbation errors, respectively. The corner represents a compromise between the data fitting and the smoothness of the solution [1–3]. Hansen showed that one fast method to obtain the corner point in the *L-curve* is through the analysis of its curvature [2], which is given by:

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$$k(\mu) = 2\frac{|\rho'(\mu)\eta''(\mu) - \rho''(\mu)\eta'(\mu)|}{\left(\rho'(\mu)^2 + \eta'(\mu)^2\right)^{3/2}} \tag{S2}$$

where $\eta\ (\mu) = \| L \cdot F_\mu \|_2$, $\rho\ (\mu) = \| A \cdot F_\mu - \alpha \|_2$ and η', ρ', η''and ρ'' are the first and second derivative of η and ρ with respect to regularization parameter μ, respectively. It is important to mention that in the original work of Hansen η and ρ are described in terms of the logarithm of the solution norm and residual norm, in concordance with *L-criterion*. However, in our case, we did not see very large variation of the solution norm, and the double logarithmic scaling does not yield improved results.

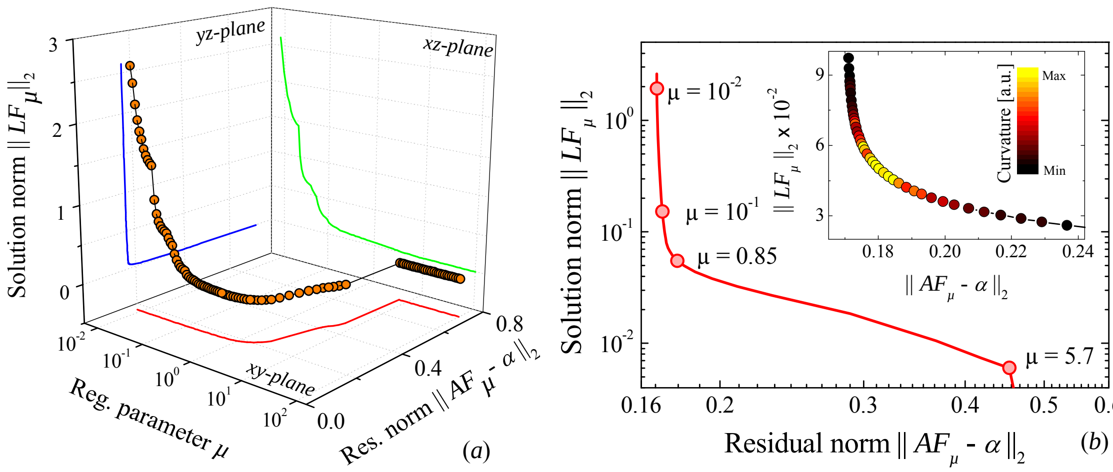

Figure S1 (a) Three-dimensional like representation of the solution norm ($\|LF_\mu\|_2$) as a function of the regularization parameter (μ) and the residual norm ($\|LF_\mu$ - $\alpha\|_2$). (b) Computed L-curve (or *zy* projection of (a)), different regularization parameters are marked with red solid dots. In the inset the curvature values are displayed in heat-like colour bar.

Figure S1a shows the three-dimensional-like (3D) representation of solution norm as a function of the regularization parameter and the residual norm. The 3D-like curve was computed using Fuchs-Sondheimer suppression function given in Eq. 3 and the LSSE experimental data measured at 250K. Some interesting features from this graph are represented in the different projections in the *xy*, *xz* and *yz* planes. The *xy* and *xz* projections give the variation of the residual norm and solution norm for different regularization parameters. We can see that very small μ-values introduce large errors in the solution norm, while large values of μ introduce large errors in the residual norm.

Finally, the *yz* projection is the computed *L-curve*. A detailed graph is showed in Figure S1b, where different values of the μ-parameter are displayed with red-solid dots and the curvature is presented in the inset. The heat-like colour bar represents the maximum (yellow) and minimum (black) values of the curvature. We can observe that the maximum of the curvature is located in the corner of the *L-curve* for $\mu \approx 0.85$.

Once the μ-parameter has been set, the reconstructed function is calculated as shown in Figure S2. The optimal μ-value is displayed in red-solid line, while small deviation around 10% of the maximum curvature is shown as light-red-like shadow.

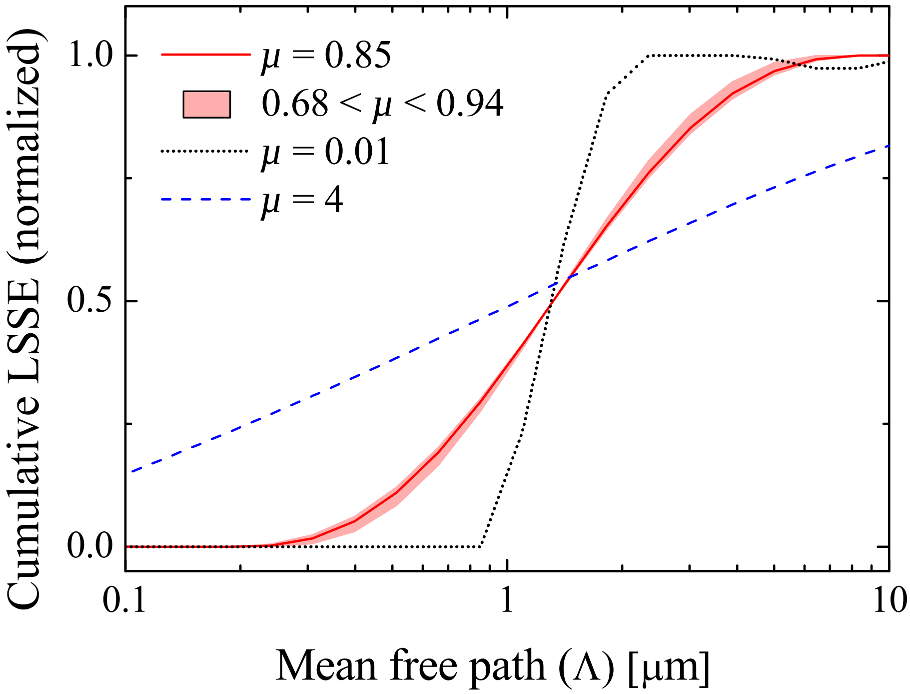

Figure S2 Reconstructed magnon LSEE-related MFP distribution for YIG using cross-plane Fuchs-Sondheimer and the experimental LSSE data measured at 250K. The light-red-like shadow represents 10% variation of the optimal regularization parameter (μ = 0.85, red solid line) in range of 0.68 < μ < 0.94. Other μ-values are also plotted with black-dotted (μ = 0.08) and blue-dashed (μ = 4) lines as an example.

As we can see in the reconstructed distribution the small deviation from the optimal value do not change significant the reconstructed function. However, larger deviation from the optimum value (blue-dashed and black-dotted lines) can introduce corresponding large deviations in the reconstructed distribution.

References

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