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Amalgams of Inverse Semigroups and C*-Algebras

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Abstract. An amalgam of inverse semigroups $[S, T, U]$ is full if U contains all of the idempotents of S and T. We show that for a full amalgam $[S, T, U]$, $C^*(S *_U T) \cong C^*(S) *_{C^*(U)} C^*(T)$. Using this result, we describe certain amalgamated free products of C^*-algebras, including finite-dimensional C^*-algebras, the Toeplitz algebra, and the Toeplitz C^*-algebras of graphs.

1. Introduction

Inverse semigroups are playing an increasingly prominent role in the theory of C^*-algebras. This paper connects certain amalgams of inverse semigroups and of C^*-algebras. Using this connection, we describe amalgams of various C^*-algebras.

The first work on amalgamated free products of C^*-algebras that we know of is due to Blackadar [4]. Shortly thereafter, Larry Brown noted in [5] that for countable discrete groups G and H with a common subgroup K, $C^*(G *_K H) \cong C^*(G) *_{C^*(K)} C^*(H)$. The obvious generalization for inverse semigroups is not true, even for finite inverse semigroups, without some restriction; see, for instance, Example 2.2 below. In Section 2, we prove an analogous result for full amalgams of discrete inverse semigroups; namely,

$$C^*(S *_U T) \cong C^*(S) *_{C^*(U)} C^*(T).$$

We apply this result to describe the structure of certain amalgams of C^*-algebras. First, we describe amalgams of finite-dimensional C^*-algebras over the natural diagonal matrices in Section 3. These amalgams turn out to be direct sums of matrix algebras over the C^*-algebras of free groups. The ranks of the free groups and the sizes of the matrix algebras are easily computed using graphs arising from Bass-Serre theory [16]. These methods extend to direct sums of matrix algebras over group C^*-algebras.

Section 4 gives some structural results for amalgams of a strongly E^*-unitary inverse semigroup with itself. These results allow us to apply work of Khoshkam and Skandalis [20] and of Milan [27] to decompose certain amalgams of C^*-algebras as either crossed products or partial crossed products of abelian C^*-algebras and groups. Specifically, Section 5 shows that a full amalgam of the Toeplitz algebra with itself is strongly Morita equivalent to a crossed product of an abelian C^*-algebra and a group, while the amalgam of a Toeplitz graph C^*-algebra with itself over the natural diagonal is isomorphic to a partial crossed product of an abelian C^*-algebra and a group.

We remark that the structure of amalgamated free products of semigroups or of inverse semigroups is far from understood in general. For example, it is known that the word problem for an amalgamated free product $S_1 *_U S_2$ of semigroups (in the category of semigroups) may be undecidable even if S_1, S_2 and U are finite semigroups [34]. On the other hand, the word problem for an amalgamated free product $S_1 *_U S_2$ of finite inverse semigroups in the category of inverse semigroups is decidable [7]. It follows from results of Bennett [2] that the word problem for $S_1 *_U S_2$ is decidable if U is a full inverse subsemigroup of the inverse semigroups S_1 and S_2.

The structure of amalgamated free products of C^*-algebras has been studied extensively by Pedersen in [31], which also includes an excellent introduction and bibliography.

Next, we review the background we need. For more information, see [19], [21], or [32] for introductions to inverse semigroups; see [11] or [13], for example, for more on C^*-algebras.

Amalgamated free products may be defined in any category by the standard universal property. Given objects U, S_1, and S_2, with monomorphisms $i_j : U \to S_j$, $j = 1, 2$ in some category, the *free product of* S_1 *and* S_2*, amalgamated over* U*,* is an object T and morphisms $\psi_i : S_i \to T$ with $\psi_1 i_1 = \psi_2 i_2$ so that for any object R and morphisms $\varphi_j : S_j \to R$ with $\varphi_1 i_1 = \varphi_2 i_2$, there is a unique morphism $\lambda : T \to R$ so that the following diagram commutes:

$$
\begin{tikzcd}
 & S_1 \arrow[dr, "\psi_1"] \arrow[drr, bend left, "\varphi_1"] & & \\
U \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & T \arrow[r, dotted, "\lambda"] & R \\
 & S_2 \arrow[ur, "\psi_2"'] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}
\tag{1.1}
$$

If it exists, the object T is unique up to isomorphism and is denoted $S_1 *_U S_2$. The tuple $[S_1, S_2, U, i_1, i_2]$ is called an *amalgam* in the category: in all cases of interest in this paper, the monomorphisms i_1, i_2 will be embeddings. We will often use $[S_1, S_2, U]$ and think of U as contained in S_1 and S_2.

Inverse semigroups. An inverse semigroup is a semigroup S such that for each $s \in S$ there exists a unique element $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. Every inverse semigroup S is evidently (von Neumann) regular; i.e., for each $s \in S$ there exists $t \in S$ such that $s = sts$. Inverse semigroups can be characterized as those regular semigroups whose idempotents commute [21, Theorem 1.1.3]. Inverse semigroups may also be viewed as an equationally defined class of semigroups with an involution $s \mapsto s^{-1}$ so that $ss^{-1}s = s$ and $ss^{-1}tt^{-1} = tt^{-1}ss^{-1}$ for all s and t [32, Theorem VIII.1.1].

We denote the set of idempotents of an inverse semigroup S by $E(S)$, or E, if S is clear: $E(S)$ is a commutative idempotent semigroup, (i.e., a semilattice) relative to the product in S. There is a natural partial order on an inverse semigroup S defined by $a \leq b$ (for $a, b \in S$) if and only if there exists $e \in E(S)$ such that $a = eb$. The smallest congruence σ on S for which S/σ is a group is generated by collapsing this partial order. Note that if S has a zero, then $S/\sigma \cong \{0\}$.

An inverse subsemigroup T of an inverse semigroup S is called a *full* subsemigroup of S if it contains all of the idempotents of S (i.e., $E(T) = E(S)$). An amalgam $[S_1, S_2, U]$ of inverse semigroups is called a *full* amalgam if U is a full inverse subsemigroup of S_1 and S_2.

It is a non-trivial fact that the category of inverse semigroups has the *strong amalgamation property*: if $[S_1, S_2, U, i_1, i_2]$ is an amalgam of inverse semigroups, then in the notation of the definition above, the morphisms ψ_i are monomorphisms (embeddings) and $\psi_1(S_1) \cap \psi_2(S_2)$ equals the image of U [17]. This property fails in general in the category of semigroups [8, p. 139].

In [16], the authors use Bass-Serre theory to describe the structure of the maximal subgroups of $S_1 *_U S_2$ in the case where $[S_1, S_2, U, i_1, i_2]$ is a full amalgam. We will use these results in Sections 3 and 5.

An inverse semigroup S may or may not have an identity element 1 or a zero element 0. If S has an identity, we refer to it as an inverse monoid. If S does not have a zero, we may adjoin one, obtaining the inverse semigroup with zero $S^0 = S \cup \{0\}$ with the obvious multiplication making 0 the zero element.

A representation of an inverse semigroup S is a homomorphism of semigroups $\rho : S \to B(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$, such that ρ sends the inverse operation of the semigroup to the adjoint operation of $B(\mathcal{H})$. Each $T \in \rho(S)$ satisfies $TT^*T = T$ and $T^*TT^* = T^*$, and so T is a partial isometry in $B(\mathcal{H})$. In fact, every inverse semigroup can be faithfully represented as a semigroup of partial isometries on some Hilbert space [12].

C^*-algebras. One can define $C^*(S)$ so that it has the universal property that each representation of S lifts to a unique representation of $C^*(S)$. Precisely, there is a monomorphism $i : S \to C^*(S)$ so that, for each representation $\rho : S \to B(\mathcal{H})$, there is a unique representation $\tilde{\rho} : C^*(S) \to B(\mathcal{H})$ with $\tilde{\rho} \circ i = \rho$. It follows from the uniqueness that if two representations of $C^*(S)$ agree on S, then they are equal. For details, see [12, Section 1]. Of course, for a finite inverse semigroup S, $C^*(S)$ is the complex inverse semigroup algebra, $\mathbb{C}S$.

For an inverse semigroup S with a zero, 0, it is natural to restrict to representations that send 0 to the zero operator. If we modify the universal property of $C^*(S)$ to consider only such representations, then we obtain the contracted C^*-algebra, $C_0^*(S)$. This can be identified with the quotient of $C^*(S)$ by the ideal generated by 0, which is a copy of the complex numbers. That is, $C^*(S) \cong C_0^*(S) \oplus \mathbb{C}$. We can define $\mathbb{C}_0S$ similarly.

Let $\mathcal{P}$ be a semilattice of projections in a C^*-algebra $\mathcal{A}$; that is, $\mathcal{P}$ is closed under products. Note that $\mathcal{P}$ is always commutative, as two projections in $\mathcal{A}$ whose product is a projection must commute. Define $\mathrm{PI}(\mathcal{P})$ to be the set of all partial isometries X in $\mathcal{A}$, i.e., elements satisfying $X = XX^*X$ and $X^* = X^*XX^*$, such that

(1) $X^*X, XX^* \in \mathcal{P}$; and
(2) $X^*\mathcal{P}X \subseteq \mathcal{P}$ and $X\mathcal{P}X^* \subseteq \mathcal{P}$.

Observe that if $\mathcal{A}$ is unital and $1_A \in \mathcal{P}$, then Condition (2) gives $XX^* = X1_AX^* \in \mathcal{P}$ and $X^*X \in \mathcal{P}$ similarly, so we can omit Condition (1) from the definition in this case.

Proposition 1.1. *If $\mathcal{P}$ is a semilattice of projections in a C^*-algebra $\mathcal{A}$, then* $\mathrm{PI}(\mathcal{P})$ *is an inverse semigroup with idempotents $\mathcal{P}$. Also, if $S \subset \mathcal{A}$ is an inverse semigroup with $E(S) \subseteq \mathcal{P}$, then $S \subseteq$* $\mathrm{PI}(\mathcal{P})$.

Proof. If $X, Y \in \mathrm{PI}(\mathcal{P})$, then XY is a partial isometry, as X^*X and YY^* are in $\mathcal{P}$ and so commute. Clearly, $(XY)^*\mathcal{P}XY = Y^*(X^*\mathcal{P}X)Y \subseteq \mathcal{P}$ and $(XY)^*XY = Y^*(X^*X)Y$ is in $\mathcal{P}$, as $X^*X \in \mathcal{P}$ and $Y^*\mathcal{P}Y \subseteq \mathcal{P}$. We can verify that $XY\mathcal{P}(XY)^* \subseteq \mathcal{P}$ and $XY(XY)^* \in \mathcal{P}$ similarly, so $XY \in \mathrm{PI}(\mathcal{P})$. If $X \in \mathrm{PI}(\mathcal{P})$, so is X^*, and X^* is an inverse for X. Finally, if X is an idempotent in $\mathrm{PI}(\mathcal{P})$, then it is easy to check that X is a projection and hence $X = XX^* \in \mathcal{P}$. As a regular semigroup whose idempotents commute, $\mathrm{PI}(\mathcal{P})$ is an inverse semigroup.

For S as above, each $X \in S$ satisfies $XX^*X = X$, $X^*XX^* = X^*$, conjugates $E(S)$ into itself, and has both X^*X and XX^* in $E(S)$. Thus $S \subseteq \mathrm{PI}(\mathcal{P})$. □

In particular, it follows that if $\Psi : S \to \mathcal{A}$ is a representation of an inverse semigroup in a C^*-algebra, then $\Psi(S) \subseteq \mathrm{PI}(\Psi(E))$.

2. AMALGAMS

Before turning to our main theorem, we first point out a related result.

Theorem 2.1. *Suppose $S = [S_1, S_2, U]$ is an amalgam of inverse semigroups with U a full inverse subsemigroup of both S_1 and S_2. Then, in the category of complex algebras,*

$$\mathbb{C}S \cong \mathbb{C}S_1 *_{\mathbb{C}U} \mathbb{C}S_2.$$

Example 2.2. The conclusion of Theorem 2.1 is not true without some condition on the amalgam. Let S and T be different copies of the two-element semilattice; i.e., $S = \{e, 0\}$ with $e^2 = e$ and all other products equal to 0, and $T = \{f, 0\}$ is similar. Letting $U = \{0\}$, we see that the inverse semigroup amalgam, $S *_U T$, has four elements e, f, ef, and 0 and $\mathbb{C}(S *_U T) = \mathbb{C}^4$. We also have $\mathbb{C}S = \mathbb{C}^2$, $\mathbb{C}T = \mathbb{C}^2$ and $\mathbb{C}U = \mathbb{C}$. However, the existence of inverse semigroup homomorphisms from $\mathbb{C}S$ and $\mathbb{C}T$ into a complex algebra does not force the images of e and f to commute. Thus, in general, there is no homomorphism from $\mathbb{C}(S *_U T) \cong \mathbb{C}^4$ to $\mathbb{C}S *_{\mathbb{C}U} \mathbb{C}T$.

Another complicating fact is that the functor $S \mapsto \mathbb{C}S$ from inverse semigroups to complex algebras behaves badly with respect to colimits. The difficulty is that the multiplicative semigroup of $\mathbb{C}S$ need not be an inverse semigroup. The construction of a complex algebra can be performed on an arbitrary semigroup, though, and we can use this to prove the result for full amalgams.

Proof of Theorem 2.1. Consider the amalgamated free product, in the category of semigroups, of inverse semigroups S_1 and S_2 over the inverse semigroup U, which we denote by $S_1 \star_U S_2$. Morphisms in the category of inverse semigroups are just semigroup morphisms [21, p. 30]. The functor that sends a semigroup M to $\mathbb{C}M$ has a right adjoint given by the forgetful functor (forget everything in $\mathbb{C}M$ except the multiplication). It follows that this functor preserves colimits [24, Dual of Theorem V.5.1]. Thus the diagram (1.1) does lift to the category of complex algebras, but from the category of semigroups, not that of inverse semigroups. That is,

$$\mathbb{C}(S_1 \star_U S_2) = \mathbb{C}S_1 *_{\mathbb{C}U} \mathbb{C}S_2.$$

Finally, [18, Theorem 2] asserts that for U a full inverse subsemigroup of both S_1 and S_2, then $S_1 \star_U S_2$ is an inverse semigroup. Thus, $S_1 \star_U S_2 = S_1 *_U S_2$, so

$$\mathbb{C}S_1 *_{\mathbb{C}U} \mathbb{C}S_2 = \mathbb{C}(S_1 *_U S_2),$$

as required. ◻

Given a full amalgam of inverse semigroups $[S_1, S_2, U, i_1, i_2]$, the inclusions $i_j : U \to S_j$ induce inclusions $I_j : C^*(U) \to C^*(S_j)$. Thus, we have an associated amalgam of C^*-algebras $[C^*(S_1), C^*(S_2), C^*(U)]$. We will always assume that the inclusions of this amalgam are induced by the inverse semigroup inclusions.

Theorem 2.3. *Suppose that $[S_1, S_2, U]$ is a full amalgam of inverse semigroups. Then*

$$C^*(S_1 *_U S_2) \cong C^*(S_1) *_{C^*(U)} C^*(S_2),$$

and if U has a zero, then

$$C_0^*(S_1 *_U S_2) \cong C_0^*(S_1) *_{C_0^*(U)} C_0^*(S_2).$$

Proof. We show that $C^*(S_1 *_U S_2)$ has the universal property of the C^*-algebraic amalgam $C^*(S_1) *_{C^*(U)} C^*(S_2)$ and so is isomorphic to it. Precisely, if $i_j : U \to S_j$, $\psi_j : S_j \to S_1 *_U S_2$ are the canonical injections, then we use the lifts $I_j : C^*(U) \to C^*(S_j)$ and $\Psi_j : C^*(S_j) \to C^*(S_1 *_U S_2)$.

Let $\mathcal{A}$ be a C^*-algebra and suppose that there are $*$-homomorphisms $\Phi_j : C^*(S_j) \to \mathcal{A}$, that agree on $C^*(U)$ (that is, $\Phi_1 \circ I_1 = \Phi_2 \circ I_2$). We will find an inverse semigroup and homomorphisms from each S_j into that inverse semigroup that induce Φ_j.

Let $\mathcal{P}$ be the image of $E(U)$ under $\Phi_j \circ I_j$. As $E(S_j) = I_j(E(U))$, it follows from Proposition 1.1 that $\Phi_j(S_j) \subseteq \mathrm{PI}(\mathcal{P})$ for $j = 1$ and $j = 2$. Let $\varphi_j : S_j \to \mathrm{PI}(\mathcal{P})$ be the restriction of Φ_j to S_j. Thus we have the diagram (1.1) in the category of inverse semigroups, with $R = \mathrm{PI}(\mathcal{P})$ and $T = S_1 *_U S_2$.

By the universal property, we have a unique map $\lambda : S_1 *_U S_2 \to \mathrm{PI}(\mathcal{P})$ that makes the diagram commute. Lifting λ gives a unique map η from $C^*(S_1 *_U S_2)$ into $C^*(\mathrm{PI}(\mathcal{P}))$. The inclusion map from $i : \mathrm{PI}(\mathcal{P}) \to \mathcal{A}$ is a representation and so lifts to a unique map $\zeta : C^*(\mathrm{PI}(\mathcal{P})) \to \mathcal{A}$. Letting $\Lambda = \zeta \circ \eta$, we have a map from $C^*(S_1 *_U S_2)$ into $\mathcal{A}$.

For $j = 1, 2$, we have $\Lambda \circ \Psi_j|_{S_j} = \zeta|_{S_j} \circ (\lambda \circ \psi_j) = i \circ \varphi_j = \Phi_j|_{S_j}$. Since a representation of $C^*(S_j)$ is determined by its action on S_j, $\Lambda \circ \Psi_j = \Phi_j$. That is, the following diagram commutes:

$$
\begin{tikzcd}
 & C^*(S_1) \arrow[dr, "\Psi_1"] \arrow[drr, bend left, "\Phi_1"] & & \\
C^*(U) \arrow[ur, "I_1"] \arrow[dr, "I_2"'] & & C^*(S_1 *_U S_2) \arrow[r, dotted, "\Lambda"] & \mathcal{A} \\
 & C^*(S_2) \arrow[ur, "\Psi_2"'] \arrow[urr, bend right, "\Phi_2"'] & &
\end{tikzcd}
\tag{2.1}
$$

To see that Λ is unique, suppose that replacing Λ with $\mu : C^*(S_1 *_U S_2) \to \mathcal{A}$ in this diagram also makes it commute. Then $\mu \circ \Psi_j$ and $\Lambda \circ \Psi_j$ agree on S_j, for $j = 1, 2$, so μ and Λ agree on a generating set of $S_1 *_U S_2$ and so agree on $S_1 *_U S_2$. But this implies $\mu = \Lambda$, as required. Thus, $C^*(S_1 *_U S_2)$ has the universal property for amalgamated free products of C^*-algebras and so is isomorphic to $C^*(S_1) *_{C^*(U)} C^*(S_2)$.

To obtain the result for the contracted algebras, one can either repeat the above proof for representations that take 0 to 0, or apply the first result and quotient out on both sides by the ideals associated to the common zero. We outline the latter approach. Consider the following commuting square:

$$
\begin{tikzcd}
 & C_0^*(S_1) \arrow[dr, "\Psi_1'"] & \\
C_0^*(U) \arrow[ur, "I_1'"] \arrow[dr, "I_2'"'] & & C_0^*(S_1 *_U S_2) \\
 & C_0^*(S_2) \arrow[ur, "\Psi_2'"'] &
\end{tikzcd}
\tag{2.2}
$$

Here the primed maps are the appropriate lifts of the i_j and ψ_i, as above. Adding a copy of $\mathbb{C}$ to each contracted C^*-algebra and extending the primed maps by mapping $\mathbb{C}$ to $\mathbb{C}$ gives the commuting square in diagram (2.1). Given $\Phi_j' : C_0^*(S_j) \to \mathcal{A}$, we can define $\Phi_j : C^*(S_j) \to \mathcal{A} \oplus \mathbb{C}$ by mapping the copy of $\mathbb{C}$ associated to the zero of S to the copy of $\mathbb{C}$ in the codomain algebra. The result above gives a unique map $\Lambda : C^*(S_1 *_U S_2) \to \mathcal{A} \oplus \mathbb{C}$, and, identifying $C^*(S_1 *_U S_2)$ with $C_0^*(S_1 *_U S_2) \oplus \mathbb{C}$, one can then show that if $\Lambda' = \Lambda|_{C_0^*(S_1 *_U S_2)}$, then the range of Λ' is contained in $\mathcal{A}$, and Λ' is the unique map making the appropriate diagram based on (2.2) commute. $\square$

3. Amalgams of Finite-Dimensional C^*-Algebras

As an application, we use Theorem 2.3 to describe amalgams of finite-dimensional C^*-algebras, i.e., direct sums of matrix algebras over $\mathbb{C}$, over the diagonal matrices. These methods easily extend to amalgams of direct sums of matrix algebras over (discrete) group C^*-algebras.

Given a group G and a natural number n, we define the Brandt inverse semigroup $B_n(G)$ as the set $\{(i, g, j) : 1 \le i, j \le n,\ g \in G\}$ together with 0, where we define the product of 0 with any element to be 0 and the product of (i, g, j) and (k, h, l) to be (i, gh, l) if $j = k$ and 0 otherwise. If G is the trivial group, we use B_n for $B_n(G)$; this is called a combinatorial Brandt inverse semigroup. Notice that B_n can be identified with the matrix units of $M_n = M_n(\mathbb{C})$, together with the zero matrix. Further, $\mathbb{C}B_n = M_n \oplus \mathbb{C}$, $\mathbb{C}_0 B_n = M_n$, and $C^*(B_n(G)) = M_n(C^*(G)) \oplus \mathbb{C}$.

Given two semigroups S and T, each with a zero 0, the 0-direct union of S and T is $S *_{\{0\}} T$. If S is the 0-direct union of finitely many combinatorial Brandt inverse semigroups $B_{n_1}, \dots, B_{n_k}$, then $C_0^*(S) = M_{n_1} \oplus \cdots \oplus M_{n_k}$. Since all finite-dimensional C^*-algebras are finite direct sums of matrix algebras, we can identify all finite dimensional C^*-algebras as C^*-algebras of inverse semigroups.

Suppose that $P = \bigoplus_{i=1}^r M_{m_i}$ and $Q = \bigoplus_{i=1}^s M_{n_i}$, where $\sum_i m_i = \sum_i n_i$. Using N for this common sum, we identify $\mathbb{C}^N$ with a natural abelian subalgebra of P and Q, namely the diagonal matrices. We can describe $P *_{\mathbb{C}^N} Q$ by recognizing P and Q as C^*-algebras of inverse semigroups as described above. If S is the 0-direct union of $B_{m_1}, \dots, B_{m_r}$, $C_0^*(S)$ is P. Similarly, $C_0^*(T)$ is Q for T the 0-direct union of $B_{n_1}, \ldots, B_{n_s}$. Moreover, $\mathbb{C}^N = C_0^*(U)$ for $U = E(S) = E(T)$. Thus, by Theorem 2.3,

$$P *_{\mathbb{C}^N} Q = C_0^*(S *_U T).$$

We apply the results of [16] to describe the maximal subgroups of this amalgam of inverse semigroups. We need one of the standard Green's relations for inverse semigroups: the $\mathcal{J}$-relation on a semigroup is defined by $u \mathcal{J} v$ if and only if u and v generate the same principal two-sided ideal of the semigroup [21, Section 3.2]. The non-zero $\mathcal{J}$-classes of the semigroups S and T correspond to the summands of P and Q. As S and T have trivial maximal subgroups, the construction of [16, p. 46] gives a graph of groups with trivial vertex and edge groups (that is, a graph). This graph has $r + s$ vertices, one for each summand of P and Q, and N edges, one for each matrix unit in $\mathbb{C}^N$. Moreover, the edge associated to a non-zero idempotent $e \in U$ connects the vertices associated to the summands of P and Q containing e.

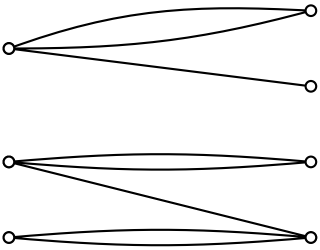
Let $W_1, \ldots, W_p$ be the components of this graph. For each W_i, let k_i be the number of edges in W_i, and q_i be the number of edges left over after removing a spanning tree from W_i. Each W_i corresponds to a non-zero $\mathcal{J}$-class in $S *_U T$, and the maximal subgroup of that $\mathcal{J}$-class is the free group of rank q_i, F_{q_i}. Thus $S *_U T$ is the 0-direct union of $B_{k_1}(F_{q_1}), \ldots, B_{k_p}(F_{q_p})$. For more details, see Example 3 of [16] and the subsequent discussion in [16]. Summarizing, we have the following result.

Theorem 3.1. *If $P = \bigoplus_{i=1}^r M_{m_i}$ and $Q = \bigoplus_{i=1}^s M_{n_i}$, where $\sum_i m_i = \sum_i n_i = N$, then*

$$P *_{\mathbb{C}^N} Q \cong \bigoplus_{i=1}^{p} M_{k_i}(C^*(F_{q_i})),$$

where $p, k_1, \ldots, k_p, q_1, \ldots, q_p$ are obtained from the graph above.

For example, if $P = M_3 \oplus M_3 \oplus M_2$ and $Q = M_2 \oplus M_1 \oplus M_2 \oplus M_3$, then the inverse semigroups are $B_3 *_{\{0\}} B_3 *_{\{0\}} B_2$ and $B_2 *_{\{0\}} B_1 *_{\{0\}} B_2 *_{\{0\}} B_3$. The resulting graph is



and so we have two components, with $k_1 = 3$, $q_1 = 1$, $k_2 = 5$, and $q_2 = 2$. Thus, $P *_{\mathbb{C}^8} Q \cong M_3(C^*(\mathbb{Z})) \oplus M_5(C^*(F_2))$.

Of course, Theorem 3.1 immediately gives the K-theory of such amalgams, first obtained by McClanahan in [26]. From the stability of K-groups and the short exact sequence on page 83 of [10], it follows that $K_0(M_k(C^*(F_q))) = \mathbb{Z}$ and $K_1(M_k(C^*(F_q))) = \mathbb{Z}^q$. Hence we obtain

$$K_0(P *_{\mathbb{C}^N} Q) = \mathbb{Z}^p, \quad K_1(P *_{\mathbb{C}^N} Q) = \mathbb{Z}^q,$$

where $q = q_1 + \cdots + q_p$. Haataja has shown (see [15, Section 4.3]) that this agrees with McClanahan's procedure for the computation of the K-groups [26, Proposition 7.1].

Of course, these methods also apply to amalgams of matrix algebras over group C^*-algebras, as these are the C^*-algebras of inverse semigroups of the form $B_n(G)$, for a fixed group G. We leave the details to the interested reader.

4. Some Special Amalgams

In this section, we look at the structure of special amalgams and describe the universal group of an inverse semigroup with zero, which is a suitable generalization of the maximal group homomorphic image. This enables us, in the next section, to describe certain special amalgams of C^*-algebras.

A *special amalgam* of inverse monoids is an amalgam $[S, S, U]$ of two copies of S over a common inverse submonoid U. More precisely, it is an amalgam $[S_1, S_2, U, i_1, i_2]$ together with an isomorphism $\theta : S_1 \to S_2$ such that $i_2 = \theta \circ i_1$. If G is a group, then the amalgamated free product $G *_U G$ is referred to as a "double" of the group G. The terminology "special amalgam" comes from universal algebra, where this concept has been well studied.

We need more of the Green's relations: $a \,\mathcal{H}\, b$ if and only if $aa^{-1} = bb^{-1}$ (i.e., $a \,\mathcal{R}\, b$) and $a^{-1}a = b^{-1}b$ (i.e., $a \,\mathcal{L}\, b$). For a full treatment of these relations, see, for example, [21, Section 3.2].

Lemma 4.1. *Let U be a full inverse submonoid of an inverse monoid S and consider the special amalgam $[S, S, U, i_1, i_2]$ with associated isomorphism $\theta : S \to S$. Then $a \,\mathcal{H}\, \theta(a)$ and $ab \,\mathcal{H}\, \theta(a)\theta(b) \,\mathcal{H}\, a\theta(b) \,\mathcal{H}\, \theta(a)b$ in $S *_U S$ for all $a, b \in S$.*

Proof. Since e is identified with $\theta(e)$ in $S *_U S$ for all idempotents $e \in E(S)$, it follows that $aa^{-1} = \theta(a)\theta(a)^{-1}$ and $a^{-1}a = \theta(a)^{-1}\theta(a)$ in $S *_U S$ and hence $a \,\mathcal{H}\, \theta(a)$ in $S *_U S$. It follows that $ab \,\mathcal{H}\, \theta(a)\theta(b)$ for all $a, b \in S$. Also, $ab \,\mathcal{R}\, abb^{-1}a^{-1}$, which is identified with $a\theta(b)\theta(b^{-1})a^{-1}$ in $S *_U S$, so $a\theta(b) \,\mathcal{R}\, ab$. Similarly, $a\theta(b) \,\mathcal{L}\, ab$ in $S *_U S$, so $ab \,\mathcal{H}\, a\theta(b)$. The proof that $ab \,\mathcal{H}\, \theta(a)b$ is similar. ◻

A subset U of a semigroup S is called a *unitary subset* of S if $s \in U$ whenever either $us \in U$ or $su \in U$ for some $u \in U$. An inverse semigroup S is called *E-unitary* if $E(S)$ is a unitary subsemigroup of S; equivalently, if $a \geq e$ for some $a \in S$, $e \in E(S)$, then $a \in E(S)$. The inverse semigroup S is said to be *F-inverse* if each σ-class has a maximum element in the natural partial order: every F-inverse semigroup is E-unitary. If S has a zero, these concepts can be modified to yield the concept of a 0-E-unitary inverse semigroup (also referred to as an E^*-inverse semigroup)—namely, $E(S) - \{0\}$ is a unitary subset—and the concept of a 0-F-inverse semigroup (also referred to as an F^*-inverse semigroup)—namely, each non-zero element of S is below a unique maximal element in the natural partial order.

For an inverse semigroup S with zero, consider pairs (G, φ), where G is a group and $\varphi : S \to G^0$ is a 0-morphism—that is, $\varphi^{-1}(0) = \{0\}$ and $\varphi(ab) = \varphi(a)\varphi(b)$ whenever $ab \neq 0$. (We use G^0 for $G \cup \{0\}$ with the obvious multiplication, and for a group morphism $\alpha : G \to H$, we use α^0 for the 0-morphism from G^0 to H^0 that sends 0 to 0 and agrees with α on G.) There is a largest group, the *universal group* $G(S)$ of S, with this property; that is, $(G(S), \varphi)$ has the property that if $\tau : S \to H^0$ is a 0-morphism, then there is a group morphism $\beta : G(S) \to H$ so that $\beta^0 \circ \varphi = \tau$. If S^0 is S with a zero adjoined, then $G(S^0)$ coincides with S/σ, the maximal group homomorphic image of S.

Let $\varphi : S \to G^0$ be a 0-morphism from S to G^0 for some group G, as in the definition above. Following [25], consider $\hat{S}$, the inverse semigroup given by $\{(s, g) : g = \varphi(s) \text{ if } s \neq 0\} \cup \{(0, g) : g \in G\}$ with the obvious multiplication. The maximal group image of $\hat{S}$ is G, with the map given by projection onto the second element of each ordered pair. Moreover, S and $\hat{S}$ have the same semilattice of idempotents, and S is the Rees quotient $S \cong \hat{S}/I$, where I is the ideal $I = \{(0, g) : g \in G\}$.

Proposition 4.2. *For $[S, T, U]$ an amalgam of inverse monoids with a common zero in U, $G(S *_U T) = G(S) *_{G(U)} G(T)$.*

The analogous result for the maximal group images, i.e., that

$$(S *_U T)/\sigma = (S/\sigma) *_{U/\sigma} (T/\sigma), \tag{4.1}$$

is well known, and the proof strategy below is a natural adaptation of the proof of (4.1). In fact, since $G(S^0) = S/\sigma$, equation (4.1) follows from Proposition 4.2.

Proof. Let A be the amalgam of $[S, T, U]$ in the category of inverse monoids and let K be the amalgam of $[G(S), G(T), G(U)]$ in the category of groups. We need to construct the 0-morphism that is the dotted arrow in the following diagram:

$$\begin{tikzcd}
 & G(S)^0 \arrow[rr] & & K^0 \\
S \arrow[ur] \arrow[rr] & & A \arrow[ur, dotted] & \\
 & G(U)^0 \arrow[uu] \arrow[rr] & & G(T)^0 \arrow[uu] \\
U \arrow[uu] \arrow[ur] \arrow[rr] & & T \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Observe that the front square involves inverse semigroup morphisms, the back square has group morphisms (with zeros added) and the diagonal arrows are 0-morphisms.

We have two 0-morphisms: $\alpha : S \to K^0$, the composition of the maps $S \to G(S)^0$ and $G(S)^0 \to K^0$, and $\beta : T \to K^0$, defined similarly. We define a map $\gamma : A \to K^0$ by sending 0 to 0 and sending a non-zero word $s_1 t_1 s_2 \ldots s_n t_n$, with $s_i \in S$, $t_i \in T$, to

$$\alpha(s_1)\beta(t_1)\alpha(s_2)\ldots\alpha(s_n)\beta(t_n).$$

Notice that since the zero is common to both S and T, if $s_1 t_1 \ldots t_n \neq 0$, then no subword can equal zero, and so $\gamma(A \setminus \{0\}) \subseteq K$.

To show that γ is well defined, we show that γ respects the equations that define an inverse semigroup, as given on page 1061. It is easy to see that if v and w are non-zero words in the elements of S and T with vw non-zero, then $\gamma(vw) = \gamma(v)\gamma(w)$. So γ respects the relations that impose associativity. Since α and β agree on U, the image under γ of a word does not depend on how we regard an element of U as in either S or T.

If w is a word in the non-zero elements of S and T, let w^{-1} be the word in the inverse elements, written in reverse order. This is clearly an involution on such words. It is easy to see that, for w as above, $ww^{-1} = s_1 t_1 \ldots s_n t_n t_n^{-1} s_n^{-1} \ldots s_1^{-1}$. Using $\beta(t_n t_n^{-1}) = 1_K$, the identity of K, and so on, we obtain $\gamma(ww^{-1}) = 1_K$. Thus γ respects the equations that define the inverse semigroup $S *_U T$ and so is a well-defined map.

We have already observed that $\gamma(vw) = \gamma(v)\gamma(w)$ for words v and w with $vw \neq 0$, so γ is a 0-morphism. By the construction of γ, the two squares—one involving S, A, $G(S)^0$, and K^0, and the other involving T, A, $G(T)^0$, and K^0—each commute.

We will show that (K, γ) is the universal group for A. Suppose that $\psi : A \to H^0$ is a 0-morphism, where H is some group. We have 0-morphisms from S to H^0, and from T to H^0, given by composition of ψ with the maps in the pushout diagram, and these maps agree on U. By the universal properties of $G(S)$ and $G(T)$, we have group morphisms $G(S) \to H$ and $G(T) \to H$ that agree on $G(U)$. By the universal property of K, these maps give a map $\tau : K \to H$. Using the commuting triangles and squares, $\tau^0 \circ \gamma$ agrees with ψ when restricted to either S or T. Since A is the amalgam of S and T, it follows that $\psi = \tau^0 \circ \gamma$.

Suppose that $\sigma : K \to H$ is another 0-morphism satisfying $\sigma^0 \circ \gamma = \psi = \tau^0 \circ \gamma$. Since σ and τ agree on $\gamma(A)$ in K^0, they agree on the images of S and T under γ composed with the natural inclusions. By the universal properties of $G(S)$ and $G(T)$, σ and τ agree on the images of $G(S)$ and $G(T)$ in K. But these images determine maps on K, and so $\sigma = \tau$. ◻

The following fact was proved by Bennett.

Proposition 4.3 **(**[3, Corollary 9]**).** *Let U be a full unitary inverse submonoid of the inverse monoid S. Then $S *_U S$ is E-unitary if and only if S is E-unitary.*

If $\varphi : S \to G(S)^0$ above also satisfies $\varphi^{-1}(1_G) = E(S) - \{0_S\}$, then we say that S is *strongly E^*-unitary*. Strongly E^*-unitary inverse semigroups are precisely Rees quotients of E-unitary inverse semigroups; see Section 3 of [25]. Such semigroups are E^*-unitary, but there are E^*-unitary inverse semigroups that are not strongly E^*-unitary [6, p. 22]. We refer the reader to Lawson's book [21] and his paper [23] for more information about these concepts and the important role that they play in the theory of inverse semigroups.

We use Bennett's result to establish the following fact about special amalgams of strongly E^*-unitary inverse semigroups.

Lemma 4.4. *Let S be a strongly E^*-unitary inverse semigroup with semilattice $E = E(S)$. Then $S *_E S$ is strongly E^*-unitary.*

Proof. With $\hat{S}$ as above, it follows that $\hat{S}$ is an E-unitary cover of S (i.e., it is E-unitary, and the natural map that sends (s, g) to s if $s \neq 0$ and $(0, g)$ to 0 is an idempotent-separating map from $\hat{S}$ onto S). From Proposition 4.3, it follows that $\hat{S} *_E \hat{S}$ is E-unitary.

Let $\theta : S \to S$ be the isomorphism in the construction of the special amalgam $S *_E S$. Every non-zero element of $S *_E S$ may be expressed (not uniquely) in the form $s_1\theta(t_1)s_2\theta(t_2)\dots s_n\theta(t_n)$ for some non-zero elements $s_i, t_i \in S$. From Lemma 4.1 it follows that $s_1\theta(t_1)s_2\theta(t_2)\dots s_n\theta(t_n) \neq 0$ in $S *_E S$ if and only if $s_1t_1s_2t_2\dots s_nt_n \neq 0$ in S. Also, any sequence of elementary transitions that transforms a non-zero element $s_1t_1\dots s_nt_n$ to an equivalent element $s_1't_1'\dots s_m't_m'$ in $S *_E S$ may be replaced by an obvious sequence that transforms the corresponding elements in $\hat{S} *_E \hat{S}$. We use J for the set of elements in $\hat{S} *_E \hat{S}$ of the form

$$(s_1, \varphi(s_1))(\theta(t_1), \varphi(\theta(t_1)))\dots(s_n, \varphi(s_n))(\theta(t_n), \varphi(\theta(t_n))), \tag{4.2}$$

where $s_1t_1\dots s_nt_n = 0$ in S. Clearly J is an ideal of $\hat{S} *_E \hat{S}$. Consider the map that projects an element (4.2) of $\hat{S} *_E \hat{S}$ onto its first component $s_1\theta(t_1)\dots s_n\theta(t_n)$ if it is not in J and to 0 if it is in J. By the observations above, this sends $\hat{S} *_E \hat{S}$ onto $S *_E S$, and $S *_E S \cong (\hat{S} *_E \hat{S})/J$. Since $\hat{S} *_E \hat{S}$ is E-unitary, it follows that $S *_E S$ is strongly E^*-unitary [25]. □

5. EXAMPLES

We now use our main result, Theorem 2.3, and the results of the previous section to describe the special amalgams of various C^*-algebras.

The Toeplitz C^*-algebra. The Toeplitz C^*-algebra, which we denote by $\mathcal{T}$, is the C^*-subalgebra of $B(\ell^2)$ generated by the unilateral shift S; see, for example, [11, Section V.1]. It can be identified with C^*-algebra of the bicyclic monoid B, that is, the inverse monoid generated by an element a subject to the relation $aa^{-1} = 1$, with the semigroup homomorphism $B \to \mathcal{T}$ determined by $a \mapsto S$. The semilattice of idempotents $E = E(B)$ is a chain order-isomorphic to the negative integers under the usual ordering. For each element $t = a^{-i}a^j \in B$, there are only finitely many elements of $s \in B$ such that $t \leq s$. From [21, Theorem 5.4.4] it is easy to see that the non-trivial unitary full inverse submonoids of B are $E(B)$ and submonoids of the form $B(n) = \{1\} \cup \{a^{-i}a^j : i + j \equiv 0 \mod n\}$ for $n \geq 2$. The submonoid $E(B)$ has infinitely many $\mathcal{D}$-classes, while the submonoid $B(n)$ has n $\mathcal{D}$-classes.

We write $\mathcal{D}$ for $C^*(E)$, the subalgebra generated by the diagonal matrices in $C^*(B)$. It is isomorphic to the algebra of convergent sequences of complex numbers. Further, $C^*(B(n))$ can be described in several ways. Perhaps the simplest is the C^*-subalgebra of $\mathcal{T}$ generated by S^n and the $n-1$ minimal projections onto the first $n-1$ basis vectors in ℓ^2.

If $\mathcal{E} \subset \mathcal{T}$ is a C^*-subalgebra of $\mathcal{T} = C^*(B)$ generated by a full submonoid U of B, then by Theorem 2.3,

$$\mathcal{T} *_{\mathcal{E}} \mathcal{T} = C^*(B *_U B).$$

To describe this C^*-algebra, we study the inverse semigroup structure of $B *_U B$. By Proposition 4.3, $B *_U B$ is E-unitary. For each full inverse submonoid U of B, the semigroup $B *_U B$ is a Reilly semigroup of the form $B(G, \alpha)$, where G is the maximal subgroup of $B *_U B$ containing 1 and α is some endomorphism of G. The endomorphism α is injective since $B *_U B$ is E-unitary. From the results of [16], G is F_∞, the free group of infinite rank, if $U = E(B)$, and is F_{n-1}, the free group of rank $n-1$, if $U = B(n)$. To see this, note that the graph has two vertices (as each copy of B has one $\mathcal{D}$-class) and either infinitely many edges (if $U = E$) or n edges (if $U = B(n)$); adapting the discussion before Theorem 3.1 to this context gives F_∞ or F_{n-1}, respectively.

See [32, Section II.6] for details of structure of $B(G, \alpha)$. Briefly, elements of $B *_U B$ may be identified with triples (i, g, j), where i, j are positive integers and $g \in G$, with multiplication

$$(i, g, j)(k, h, l) = \begin{cases} (i + k - j, \alpha^{k-j}(g)h, l) & \text{if } k \geq j, \\ (i, g\alpha^{j-k}(h), l + j - k) & \text{if } j \geq k. \end{cases}$$

An element (i, g, j) of $B *_U B$ can only be less than or equal to elements of the form $(i - k, h, j - k)$, where $h \in G$ and $k \in \mathbb{N}$ satisfy $i - k, j - k \geq 0$ and $\alpha^k(h) = g$. Since α is injective, there is at most one such h. Thus each element of $B *_U B$ has only finitely many elements above it in the natural partial order since it is an E-unitary inverse semigroup. We note that $B *_U B$ is F-inverse; that is, each element has a unique maximal element above it in the natural partial order. To see this, suppose that $(i, g, j) \leq (i - k, h, j - k), (i - l, h', j - l)$, where $l < k$. Then $\alpha^k(h) = \alpha^l(h')$. By the injectivity of α, $\alpha^{k-l}(h) = h'$ and so $(i - l, h', j - l) \leq (i - k, h, j - k)$. It follows that $B *_U B$ is F-inverse.

By results of Khoshkam and Skandalis [20] (cf. [35]), $C^*(B *_U B)$ is strongly Moria equivalent to $C^*(E) \times_\mu H$, a crossed product of the abelian C^*-algebra

$C^*(E)$ by H, the maximal group homomorphic image of $B *_U B$. By Proposition 4.2 (or, more precisely, by equation (4.1)), if $U = E(B)$, then $G(E(B)) = \{0\}$ and $H = \mathbb{Z} *_{\{0\}} \mathbb{Z} = F_2$, while if $U = B(n)$, then $G(B(n))$ is $\mathbb{Z}$, which we can identify as $n\mathbb{Z}$ inside $\mathbb{Z} \cong G(B)$, and so $H = \mathbb{Z} *_{n\mathbb{Z}} \mathbb{Z} = \langle a, b \mid a^n = b^n \rangle$. In each case, H is also a semidirect product of G by $\mathbb{Z}$, from [28].

To describe the action μ of H on $C^*(E)$, we start with the Munn representation; that is, $s \in S$ maps the set $\{e \in E : e \le s^*s\}$ onto the set $\{e \in E : e \le ss^*\}$, via $e \mapsto ses^*$.

If $\hat{E}$ is the spectrum of $C^*(E)$, then $C(\hat{E})$, the continuous functions on $\hat{E}$, is isomorphic to $C^*(E)$. Moreover, $\hat{E}$ can be identified with the multiplicative linear functionals on E with the relative weak-$*$ topology. There is a dual action of S on $\hat{E}$, where $s \in S$ maps $\{x \in \hat{E} : x(s^*s) = 1\}$ onto $\{x \in \hat{E} : x(ss^*) = 1\}$ via $x \mapsto s.x$, where $s.x(e) = x(s^*es)$ for all $e \in E$.

This lifts to an action, also called μ, of S/σ on $\hat{E}$, given by $g.x = s.x$ for any $s \in S$ with $\sigma(s) = g$ and $x(s^*s) = 1$. To see that this is well defined, note that if $f \in E$ and $x(f) = 1$, then $s.x = (sf).x$. By [21, Lemma 1.4.12], for $s, t \in S$ with $\sigma(s) = \sigma(t)$, $f = s^*st^*t$ satisfies $sf = tf$, and so $s.x = (sf).x = (tf).x = t.x$.

We summarize this discussion in the following theorem.

Theorem 5.1. *If $\mathcal{D}$ is the diagonal matrices in $\mathcal{T}$ and $\mathcal{E} = C^*(B(n))$, then $\mathcal{T} *_{\mathcal{D}} \mathcal{T}$ and $\mathcal{T} *_{\mathcal{E}} \mathcal{T}$ are strongly Morita equivalent to, respectively,*

$$\mathcal{A} \times_\mu F_2, \quad \mathcal{A} \times_\mu \langle a, b \mid a^n = b^n \rangle,$$

where μ is the action described above and $\mathcal{A}$ is the algebra of convergent sequences of complex numbers.

Toeplitz graph C^*-algebras. Inverse semigroups associated to graphs have been defined independently several times: [1], [22, Section 8.1], and [30]. We think of a (directed) graph Γ as having a set of vertices, Γ^0, a set of edges, Γ^1, and range and source functions, $r, s : \Gamma^1 \to \Gamma^0$, where the edge e goes from $s(e)$ to $r(e)$. Define $I(\Gamma)$, the inverse semigroup associated to Γ, as the inverse semigroup generated by $\Gamma^0 \cup \Gamma^1$ with a zero $z \notin \Gamma^0 \cup \Gamma^1$, subject to certain relations. Here, we use * for the inverse operation. If we extend the source and range maps of Γ^1 to $\{e^* : e \in \Gamma^1\}$ by $s(e^*) = r(e)$ and $r(e^*) = s(e)$ and to Γ^0 by $s(v) = r(v) = v$, then these relations can be conveniently summarized as

(1) $s(e)e = er(s) = e$ for all $e \in \Gamma^1 \cup \{e^* : e \in \Gamma^1\}$;
(2) $ab = z$ if $a, b \in \Gamma^0 \cup \Gamma^1 \cup \{e^* : e \in \Gamma^1\}$ with $r(a) \ne s(b)$, and
(3) $a^*b = z$ if $a, b \in \Gamma^1$ and $a \ne b$;
(4) $b^*b = r(b)$ if $b \in \Gamma^1$.

Define a path in Γ to be either a vertex, v, or a finite sequence of edges $\alpha = e_1e_2 \dots e_n$ with $r(e_i) = s(e_{i+1})$, $1 \le i < n$. For such a path α, we use α^* for $e_n^*e_{n-1}^* \dots e_1^*$. Extending s and r to paths by $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$, there is a natural composition of paths: the product of α and β is $\alpha\beta$ if $r(\alpha) = s(\beta)$ and is z otherwise.

Relations (1) and (2) show that any word in $\Gamma^0 \cup \Gamma^1$ must be a path and any word in $\Gamma^0 \cup \{e^* : e \in \Gamma^1\}$ is p^*, where p is a path. Using relation (3), it follows that each non-zero element of $I(\Gamma)$ has the form pq^*, where p and q are paths with $r(p) = r(q)$; further, the product of pq^* and rs^* is non-zero exactly when either $q = rt$ for a path t or $r = qt$ for a path t. The product is either $(pt)s^*$ or $p(st)^*$, respectively. The idempotents of $I(\Gamma)$ are the elements of the form pp^* for p a path. The natural order in $I(\Gamma)$ is given by $pq^* \leq rs^*$ exactly when $p = rt$ and $q = st$ for a path t.

It is worth observing that if Γ is a vertex with a single edge, then $I(\Gamma)$ is the bicyclic monoid adjoin a (removable) zero, while if Γ is a vertex with n edges, then $I(\Gamma)$ is the *polycyclic monoid*—that is, the monoid generated by n elements $a_1, a_2, \ldots, a_n$ subject to the relations $a_i a_i^{-1} = 1$, $a_i a_j^{-1} = 0$ for $i \neq j$. These monoids were introduced by Nivat and Perrot [29] in the context of formal language theory; they were rediscovered by Renault [33] and are often referred to as *Cuntz semigroups* in the operator algebra literature.

Each graph inverse semigroup is F^*-inverse and strongly E^*-unitary with universal group the free group on the edges of Γ, F_{Γ^1} [23]. As $I(\Gamma)$ is strongly E^*-unitary, [27] shows $C_0^*(I(\Gamma))$ can be described as a partial crossed product of $C_0^*(E(I(\Gamma)))$ by F_{Γ^1}.

This associated contracted C^*-algebra is *not* the C^*-algebra of the graph, but rather the Toeplitz C^*-algebra of the graph, as defined in [14]. (Of course, the C^*-algebra of the graph is a proper quotient of $C_0^*(I(\Gamma))$.) To see this, first let $\pi : I(\Gamma) \to C_0^*(I(\Gamma))$ be the canonical injection of $I(\Gamma)$ in its C^*-algebra and define, for $v \in \Gamma^0$, $P_v = \pi(v)$ and, for $e \in \Gamma^1$, $S_e = \pi(e)$. Then the relations above imply that $(\{P_v : v \in \Gamma^0\}, \{S_e : e \in \Gamma^1\})$ form a Toeplitz-Cuntz-Krieger Γ-family and moreover, $P_v \neq 0$ for each v and, if $s^{-1}(v)$ is finite, $P_v > \sum_{s(e)=v} S_e S_e^*$. Thus, by [14, Corollary 4.2], $C_0^*(I(G)) = C^*(\{P_v, S_e\})$ is the Toeplitz C^*-algebra of Γ.

The C^*-subalgebra of $C_0^*(I(\Gamma))$ generated by the idempotents, call it $\mathcal{D}$, is isomorphic to $C_0(X)$, the continuous functions vanishing at infinity on a suitable locally compact, totally disconnected, Hausdorff space X. The simplest way to describe X is as the space of all finite or infinite paths on Γ, with the following topology. A finite path α is closed and open if $s^{-1}(r(\alpha))$ is finite and otherwise has a neighborhood basis, $D_{\alpha,F}$, indexed by finite subsets $F \subset s^{-1}(r(\alpha))$. Each $D_{\alpha,F}$ consists of paths $\alpha\beta$, where β is a finite or infinite path with $s(\beta) = r(\alpha)$ and the first edge of β is not in F. An infinite path α has a neighborhood base indexed by natural numbers n, $D_{\alpha,n}$ consisting of paths β whose first n edges agree with α and whose other edges can be any edges consistent with β being a path.

Invoking Theorem 2.3,

$$C_0^*(I(\Gamma)) *_{\mathcal{D}} C_0^*(I(\Gamma)) \cong C_0^*(I(\Gamma) *_E I(\Gamma)).$$

By Lemma 4.4, $I(\Gamma) *_E I(\Gamma)$ is strongly E^*-unitary, and by Proposition 4.2, its universal group is $F_{\Gamma^1} * F_{\Gamma^1}$. Applying Milan's Theorem [27, Theorem 3.3.3] again, we have the following result.

Theorem 5.2. *Let Γ be a directed graph. If $\mathcal{D}$ is the diagonal subalgebra of $C^*(I(\Gamma))$, then*

$$C_0^*(I(\Gamma)) *_{\mathcal{D}} C_0^*(I(\Gamma)) \cong \mathcal{D} \times_\mu H,$$

where $H = F_{\Gamma^1} * F_{\Gamma^1}$ and μ is the partial action of H on $\mathcal{D}$ lifted from the Munn representation.

In particular, this result applies to the bicyclic monoid, so we have two descriptions of the amalgam of the Toeplitz algebra with itself, either as a crossed product (up to strong Morita equivalence) or as a partial crossed product (up to $*$-isomorphism). The theorem also applies to the Cuntz-Toeplitz algebra, when Γ is a vertex with n loops, describing the amalgam of this algebra with itself as a partial crossed product by F_{2n}, the free group of rank $2n$.

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