THE COLLEGE OF AERONAUTICS

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SKIN FRICTION IN THE LAMINAR BOUNDARY LAYER IN COMPRESSIBLE FLOW

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- SUMMARY -

From an analysis of the work of Crocco and others, semi-empirical formulae are derived for the skin friction on a flat plate at zero incidence with a laminar boundary layer. These formulae are

$$c_f \sqrt{R_x} = 0.664 \left[0.45 + 0.55 i(o) + 0.09(\gamma - 1) M_1^2 \sigma^{\frac{1}{2}} \right]^{\frac{\omega-1}{2}}$$

for the general case of heat transfer, and

$$c_f \sqrt{R_x} = 0.664 \left[1 + 0.365 (\gamma - 1) M_1^2 \sigma^{\frac{1}{2}} \right]^{\frac{\omega-1}{2}}$$

when there is no heat transfer.

The problems of heat transfer, dissipation and the effect of radiation are discussed in the light of these formulae. The second formula is then utilised in the development of an approximate method for solving the momentum equation of the boundary layer on a cylinder without heat transfer. The method indicates with increase of Mach number a marked forward movement of separation from a flat plate in the presence of a constant adverse velocity gradient.

/ Notation ...

x	distance measured along surface
y	distance measured normal to surface
u	velocity component in x direction
v	velocity component in y direction
L	standard length (e.g. length of plate)
M	Mach number
T	temperature (absolute)
J	mechanical equivalent of heat
I	enthalpy ($= Jc_pT$, for a perfect gas with constant specific heats)
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
γ	c_p/c_v
τ	viscous stress
μ	coefficient of viscosity
ν	μ/ρ
k	coefficient of heat conduction
σ	Prandtl number $= \mu c_p/k$
ω	defined by $\mu \propto T^{\omega}$

Suffix 1 refers to quantities measured at outer edge of boundary layer, suffix w to quantities measured at the wall, suffix o to some standard condition, e.g. main stream at leading edge of plate, suffix i refers to incompressible flow.

c_f	$2\tau_w/\rho_1 u_1^2$
R_x	$u_1 x/\nu_1$
R_L	$u_1 L/\nu_1$
i	I/I_1
η	u/u_1
r	ρ/ρ_1
m	μ/μ_1

/Q ...

Q rate of transfer of heat from surface of unit breadth and length L to gas

q local rate of transfer per unit area

S_F total frictional force on surface of unit breadth and length L

D rate of dissipation of mechanical energy by viscosity on a plate of unit breadth and length L

C_B Boltzmann's radiation constant

θ momentum thickness $= \int_0^\infty \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u}{u_1}\right) dy$

δ^+ displacement thickness $= \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$

H δ^+/θ

Y $\int_0^y \frac{\mu_0 dy}{\mu}$

ρ^+ ρ/ρ_0

μ^+ μ/μ_0

θ^+ θ/L

u^+ u/u_0

x^+ x/L

/1. Introduction ...

1. INTRODUCTION

The general analysis of the flow in the laminar boundary layer at high speeds is of formidable complexity. This complexity derives from the variation of viscosity, heat conductivity and density with temperature and the consequent inter-dependence of the equations of motion and energy. The simplest problem, viz uniform flow past a flat plate at zero incidence, has naturally attracted most attention, nevertheless results for particular cases have only been evaluated after long and laborious calculations. We are indebted for a store of such results to the work of Busemann [1], Karman and Tsien[2], Hantzsche and Wendt [3], Emmons and Brainerd [4], Cope and Hartree [5], and Crocco [6]. The work of the latter is particularly important since in addition to results for particular cases, it reveals results of a general character. The more general problem of the boundary layer on a cylinder is still in large measure unsolved, valuable pioneer work on lines analogous to Howarth's series solution for incompressible flow [7] has been begun by Cope and Hartree, and Howarth [8] has similarly developed solutions for the case when $\sigma = 1.0$, and $\omega = 1.0$. In addition approximate methods have been developed by Frankl[9], Oswatitsch and Wieghardt [10], Young and Winterbottom [11], Illingworth [12], and Howarth [8] in which solutions are obtained of the momentum and energy equations on lines analogous to Pohlhausen's method [13], or in which other assumptions or approximations suggested by incompressible flow theory are made.

In the following a general semi-empirical formula is developed for the skin friction on a flat plate at zero incidence, both with and without heat transfer in a uniform flow of a perfect gas with constant specific heats. This formula is derived from an approximate solution of Crocco's integral equation for the viscous stress. The form of this solution is retained but the constants are adjusted to give the best overall agreement with the available calculated results for particular cases. The problems of heat transfer dissipation and the effect of radiation on the temperature measured by a thermometer are briefly discussed in the light of this formula.

The formula is then applied to the development of an approximate method for solving the momentum equation of the boundary layer on a cylinder with zero heat transfer. The accuracy of the method cannot as yet be gauged, as there are no accurate results available with which to compare it. It has, however, two features to commend it, firstly, it gives the correct result for the case of zero external pressure gradient, secondly, it is relatively simple and quick to apply, the main operation in any given case being a graphical operation.

The method has been applied to estimate the distance to separate from a flat plate in the presence of a constant negative velocity gradient when $\sigma = 1.0$, and $\omega = 1.0$. This is the case considered by Howarth [2] by a method closely analogous to Pohlhausen's method. It is doubtful whether either method can be relied on to give the separation distance accurately, and their results differ considerably for high Mach numbers, but both methods agree in predicting a rapid forward movement of separation with increase of Mach number at the leading edge for constant ratio of velocity fall per unit length of plate to initial velocity.

2. SKIN FRICTION ON A FLAT PLATE AT ZERO INCIDENCE.

2.1. Boundary Layer Equations

The equations of motion for the boundary layer are

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \\ \frac{\partial p}{\partial y} &= 0, \end{aligned}\right\} \qquad \ldots\ldots (1)$$

the equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 , \qquad \ldots\ldots (2)$$

and the equation of energy is

$$\rho u \frac{\partial I}{\partial x} + \rho v \frac{\partial I}{\partial y} = \frac{1}{\sigma} \frac{\partial}{\partial y}\left(\mu \frac{\partial I}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad \ldots\ldots (3)$$

when $I \equiv Jc_pT$ (enthalpy), and σ is the Prandtl number $= \mu \frac{c_p}{k}$ and is assumed constant.

The equation of state for a perfect gas is

$$p = \rho \bar{R}\, T/\bar{m} \qquad \ldots\ldots (4)$$

where $\bar{R}$ is the gas constant and $\bar{m}$ is the molecular weight of the gas. Since p is constant for the flat plate at zero incidence $\rho = \rho(I)$.

The viscosity coefficient μ is a function of T only, the relation being given by Sutherland's formula

$$\mu \propto \frac{T^{1.5}}{T + c}, \quad \text{where } c = 110^{\circ}K, \text{ for air.}$$

It is convenient to approximate to this relation by writing

$$\mu \propto T^{\omega}, \qquad \ldots\ldots (5)$$

where ω varies according to the range of T in which we are interested, but can be taken equal to $8/9$ for T between $75^{\circ}K$ and $300^{\circ}K$ [14].

/ 2.2 ...

2.2. Crocco's Transformation.[6]

The case of the flat plate at zero incidence is the only one which permits the reduction of the boundary layer equations to ordinary equations in terms of a single independent variable. This variable is of the form $\xi = \text{const.}y/\sqrt{2x}$. In this case we can write $I = I(u)$, $\rho = \rho(u)$, $\mu = \mu(u)$ and

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right) = \mu\left(\frac{du}{d\xi}\right)\cdot\frac{\text{const.}}{\sqrt{2x}} = \frac{f(u)}{\sqrt{2x}} \text{ , say} \qquad \ldots. (6)$$

If we transform from ξ to u as independent variable, the equations (1), (2) and (3) reduce to 2 equations in f and I , viz

$$ff'' + u\rho\mu = 0, \qquad \ldots. (7)$$

$$(I'' + \sigma)f + (1-\sigma)I'f' = 0, \qquad \ldots. (8)$$

where dashes denote differentiation with respect to u.

These equations can be put into a non-dimensional form by writing

$$\frac{u}{u_1} = \eta \text{ , } \frac{I}{I_1} = i \text{ , } \frac{\rho}{\rho_1} = r \text{ , } \frac{\mu}{\mu_1} = m \text{ , } F(\eta) = \sqrt{\frac{2}{\rho_1\mu_1 u_1^3}}.f(u) \quad \ldots(9)$$

when suffix $_1$ refers to quantities in the main stream.

Equations (7) and (8) then become

$$FF'' + 2\eta rm = 0 \text{ ,} \qquad \ldots.(10)$$

$$\left(i'' + \sigma\frac{u_1^2}{I_1}\right) F + (1-\sigma)i'F' = 0, \qquad \ldots.(11)$$

and we note that $\frac{u_1^2}{I_1} = (\gamma - 1)M_1^2$,where dashes now indicate differentiation with respect to η .

The boundary conditions are

$$\eta = 0, \; F'=0, \; i = I_W/I_1 \; ;$$

$$\eta = 1, \; F=0, \; i = 1 \text{ .}$$

suffix w denotes quantities measured at the wall.

The equation of state (4) gives

$$r = \frac{\rho}{\rho_1} = \frac{T_1}{T} = \frac{I_1}{I} = \frac{1}{i} \qquad \ldots.(12)$$

and the viscosity - temperature relation (5) becomes

$$m = i^{\omega} \qquad \ldots.(13)$$

Using (12) and (13) equation (10) can be written

$$FF'' + 2\eta\, i^{-(1-\omega)} = 0 \qquad \ldots.(14)$$

/ From ...

From (6) and (9) it is easy to deduce that

$$c_f \sqrt{R_x} = F(o) \qquad ....(15)$$

where c_f is the local skin friction coefficient ($= 2\tau_w/\rho_1 u_1^2$) at a point distant x from the leading edge, and $R_x = u_1 x/\nu_1$.

Once $F(\eta)$ is determined, the relation between y and u is given by (6) and hence

$$y = \int \frac{\mu \, du}{\tau(x,u)} ,$$

which reduces to

$$\frac{u_1 y}{\nu_1} = 2\sqrt{R_x} \int^{\eta} \frac{m}{F(\eta)} \cdot d\eta \qquad ....(16)$$

2.3 Hantzsche and Wendt's Transformation.[3]

Hantzsche and Wendt arrived at different forms of these equations by using different non-dimensional expressions involving I and $f(u)$. Thus, they wrote

$$j = I/u_1^2, \text{ and } G(\eta) = \sqrt{\frac{2}{\mu_w \rho_w u_1^3}} \cdot f(u), \qquad ....(17)$$

and their resulting equations which correspond to (14) and (11) were

$$G\,G'' + 2\eta\left(\frac{j_w}{j}\right)^{(1-\omega)} = 0 , \qquad ....(18)$$

and

$$G(j'' + \sigma) + (1 - \sigma)\, G' j' = 0. \qquad ....(19)$$

In this case the skin friction coefficient is given by

$$c_f \sqrt{R_x} = G(o)\left(\frac{j_1}{j_w}\right)^{\frac{1-\omega}{2}} = G(o)\left(\frac{T_1}{T_w}\right)^{\frac{1-\omega}{2}}. \qquad ....(20)$$

2.4. Some important results of Crocco's analysis.

It will be immediately apparent from the form of equations (14) and (11) (or from equations (18) and (19)) that considerable simplifications result when $\rho\mu$ = const. and hence $\omega = 1.0$, or when $\sigma = 1.0$.

Thus, when $\sigma = 1.0$, equation (11) reduces to

$$i'' + \sigma \frac{u_1^2}{I_1} = 0, \qquad ....(21)$$

and hence i is a quadratic function of η , independent of F , and taking account of the boundary conditions this function may be written

$$\left.\begin{array}{l} i = 1 - i'(o)(1-\eta) + \dfrac{u_1^2}{2I_1}(1-\eta^2), \\ \text{or} \\ i = i(o) + \left[1 - i(o) + \dfrac{u_1^2}{2I_1}\right]\eta - \dfrac{u_1^2}{2I_1}\eta^2 \end{array}\right\} \qquad ....(22)$$

/ Having ...

Having i as a function of u, equation (14) for F takes the form

$$FF'' + k(\eta) = 0$$

where $k(\eta)$ is known. For details of an iterative process that can be used to solve this equation for F see Reference 6.

When $\rho\mu$ = constant or $\omega = 1.0$, equation (14) simplifies to

$$FF'' + 2\eta = 0, \qquad \ldots.(23)$$

which can be transformed to the Blasius equation for incompressible flow, viz,

$$\frac{d^3\zeta}{d\xi^3} + \zeta\frac{d^2\zeta}{d\xi^2} = 0,$$

by writing $F = \frac{1}{2}\frac{d^2\zeta}{d\xi^2}$, $\eta = \frac{1}{2}\frac{d\zeta}{d\xi}$.

The relation between ξ and y_i, the lateral ordinate in incompressible flow is

$$\xi = \frac{1}{2}\sqrt{\frac{u_1}{\nu_1}} \cdot \frac{y_i}{\sqrt{x}} .$$

From equation (16), the lateral ordinate in compressible flow corresponding to a given value of η, is

$$y = \frac{2x}{\sqrt{R_x}}\int_0^{\eta}\frac{m\,d\eta}{F(\eta)} = \frac{2x}{\sqrt{R_x}}\int_0^{\xi} m\,d\xi = \int_0^{y_i} m\,dy_i$$

$$= \int_0^{y_i}\frac{\mu}{\mu_1}\cdot dy_i = \int_0^{y_i}\frac{i}{i_1}\,dy_i . \qquad \ldots.(24)$$

Hence for this case all solutions are simple transformations of the incompressible flow solution, the relation between corresponding values of y for given values of u (or η) being given by equation (24).

We note immediately that since $\eta = 0$ when $\xi = 0$, $F(o)$ is the same for compressible as for incompressible flow and hence in all cases when $\rho\mu$ = const. (or $\omega = 1.0$)

$$c_f\sqrt{R_x} = F(o) = 0.664 \qquad \ldots.(25)$$

/ In ...

In the general case, when neither ω nor σ are necessarily unity, let us suppose we have solved for F as a function of η. Then equation (11) which is linear in i can be integrated twice to give

$$\left.\begin{aligned} i(\eta) &= i(o) + i'(o).A(\eta,\sigma) - \sigma\frac{u_1^2}{I_1}B(\eta,\sigma), \\ \text{where } A(\eta,\sigma) &= \int_o^{\eta}\left[\frac{F(\eta_1)}{F(o)}\right]^{(\sigma-1)} d\eta_1, \\ \text{and } B(\eta,\sigma) &= \int_o^{\eta}\left[\frac{F(\eta_1)}{F(o)}\right]^{\sigma-1}\left\{\int_0^{\eta_1}\left[\frac{F(\eta_2)}{F(o)}\right]^{1-\sigma} d\eta_2\right\} d\eta_1 . \end{aligned}\right\} \quad \ldots.(26)$$

Alternatively, taking account of the fact that $i(1) = 1$, we have

$$\left.\begin{aligned} i(\eta) &= 1 - i'(o)\,C(\eta,\sigma) + \sigma\frac{u_1^2}{I_1}D(\eta,\sigma), \\ \text{where } C(\eta,\sigma) &= \int_{\eta}^{1}\left[\frac{F(\eta_1)}{F(o)}\right]^{\sigma-1} . \, d\eta_1, \\ \text{and } D(\eta,\sigma) &= \int_{\eta}^{1}\left[\frac{F(\eta_1)}{F(o)}\right]^{\sigma-1}\left\{\int_o^{\eta_1}\left[\frac{F(\eta_2)}{F(o)}\right]^{1-\sigma} . \, d\eta_2\right\} . \, d\eta_1 . \end{aligned}\right\} \quad \ldots.(27)$$

Hence, for the case of zero heat transfer when $i'(o) = o$

$$\left.\begin{aligned} \frac{I(\eta) - I(1)}{u_1^2/2} &= 2\sigma D(\eta,\sigma) \\ \text{and in particular} \\ \frac{I(o) - I(1)}{u_1^2/2} &= 2\sigma D(o,\sigma) . \end{aligned}\right\} \quad \ldots.(28)$$

Since $D(o,\sigma)$ does not involve M_1, it will be independent of Mach number for all cases where $F(\eta)$ is independent of Mach number, i.e. for all cases where $\rho\mu$ = const. (or $\omega = 1.0$). Pohlhausen's solution[15] for incompressible flow gave to a very close order of accuracy

$$\frac{I(o) - I(1)}{u_1^2/2} = \sigma^{\frac{1}{2}},$$

hence

$$2\sigma D(o,\sigma) = \sigma^{\frac{1}{2}},$$

and therefore for all cases where $\rho\mu$ = const.

$$\frac{T_W}{T_1} = 1 + \frac{(\gamma - 1)}{2} M_1^2\,\sigma^{\frac{1}{2}} . \quad \ldots.(29)$$

/ However ...

However, by a laborious trial and error method Crocco integrated equations (11) and (14) step by step for the cases $\sigma = 0.725$, a range of Mach numbers up to 5.0 and values of ω of 1.25, 0.75 and 0.5. In every case the results showed the functions A, B, C and D to be practically independent of the particular value of ω assumed and hence independent of the function $F(\eta)$ within the range of values of ω considered. Crocco suggested that the reason for this lay in the fact that whilst the variation of $F(\eta)$ with ω may be considerable, it occurs in the integrals from which A, B, C and D are derived in the form $\left[\frac{F(\eta)}{F(o)}\right]^{1-\sigma}$ or $\left[\frac{F(\eta)}{F(o)}\right]^{\sigma-1}$.

The ratio $\frac{F(\eta)}{F(o)}$ must in every case be unity when $\eta = o$, and be zero when $\eta = 1.0$, and hence its possible variation with ω is limited. Further, as long as σ is not far off unity, the exponents $(1-\sigma)$ or $(\sigma-1)$ are small and reduce still further the sensitivity of the expressions A, B, C and D to variations in ω. The final result is therefore that in general the variation of enthalpy with velocity is practically independent of the viscosity-temperature relation, for values of σ not very different from unity and is given by equations (26) or (27) with the functions A, B, C and D calculated for the case $\omega = 1.0$ when $F(\eta)$ is given immediately by the Blasius incompressible flow solution. In particular, when there is zero heat transfer equation (29) for the wall temperature is of general validity, a fact that was demonstrated by the cases calculated by Brainerd and Emmons [4].

Since $i(\eta)$ can now be regarded as independent of $F(\eta)$ in general, and is readily determined from the case $w = 1.0$, the solution of the main equations (10) and (11) reduce to the solution of the equation

$$FF'' + k(\eta) = 0$$

where $k(\eta)$ is known. This enabled Crocco to develop a more rapid and more accurate process of solution than the first process he had used, but for further details see Ref.6.

2.5. Skin friction distribution. Deduction from Hantzsche and Wendt's results for case of zero heat transfer.

Hantzsche and Wendt's calculations [3] for the case of zero heat transfer showed the interesting result that the quantity $G(o)$ was only a slightly varying function of ω and M_1, for $\sigma = 1.0$, and was independent of σ and M_1 when $\omega = 1.0$. In general therefore, we may expect $G(o)$ to vary little with σ, ω and M_1. The importance of this may be seen from equation (20) which with equation (29) gives us

$$c_f\sqrt{R_x} = G(o) \Big/ \left[1 + \frac{(\gamma-1)}{2}M_1^{\,2}\,\sigma^{\frac{1}{2}}\right]^{\frac{1-\omega}{2}} \quad \ldots.(30)$$

A reasonable approximation to $G(o)$, reproducing its small variation with ω, σ and M_1 should then give us a reliable formula for $c_f\sqrt{R_x}$, for which the major variation with ω, σ and M_1 is described by the denominator of (30). In Fig.1 $G(o)$ is shown as a function of M_1, for various values of ω for the case $\sigma = 1.0$, and this can be

/represented ...

represented fairly closely by

$$G(o) \approx 0.664 \left[\frac{1 + \left(\frac{\gamma-1}{2}\right) M_1^{\,2}}{1 + 0.3(\underline{\gamma-1}) M_1^{\,2}} \right]^{\frac{1-\omega}{2}} .$$

This suggests that in general

$$G(0) \approx 0.664 \left[\frac{1 + \frac{(\gamma-1)}{2} M_1^{\,2} \sigma^{\frac{1}{2}}}{1 + 0.3(\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}}} \right]^{\frac{1-\omega}{2}} ,$$

and hence for the case of zero heat transfer we may expect from (30) that

$$\frac{c_f \sqrt{R_x}}{0.664} = \frac{F(o)}{F(o)_{\omega=1.0}} \approx \left[1 + 0.3(\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}} \right]^{\frac{(1-\omega)}{2}} \qquad \ldots.(31)$$

2.6. Skin friction distribution with or without heat transfer. Deduction from approximate solution of integral equation for F.

From equation (14) and the boundary conditions for F it follows that

$$F = \int_{\eta_1}^{1} \left[\int_0^{\eta_1} \frac{2\eta . i^{\omega-1}}{F} . \, d\eta \right] d\eta_1 \qquad \ldots.(32)$$

From a survey of Crocco's results for F it appears that in all cases a very close fit to F is given by

$$F \approx F(o) \, (1 - \eta^2)^{\frac{1}{2}} .$$

Hence, from (32) we may expect to get an even closer fit from the relation

$$F \approx \int_{\eta_1}^{1} \left[\int_0^{\eta_1} \frac{2\eta . i^{\omega-1}}{F(o)(1-\eta^2)^{\frac{1}{2}}} . \, d\eta \right] . d\eta_1 \quad .$$

In particular,

$$F(o)^2 \approx \int_0^1 \left[\int_0^{\eta_1} \frac{2\eta . i^{\omega-1}}{(1-\eta^2)^{\frac{1}{2}}} . \, d\eta \right] . d\eta_1 \quad . \qquad \ldots.(33)$$

We have i as a function of η and σ given by (26) or (27) so that in theory we can determine $F(o)^2$ from (33) in any given case by integration (which must in general be numerical) and hence from (15) we can obtain the skin friction coefficient. We are interested, however, in obtaining a general explicit formula for F(o) in terms of i(o) (or T_W / T_1), M_1, σ and ω. Analytic integration of (33) appears to be impossible in general, the process adopted therefore has been to develop a crude approximate solution of (33) to suggest the form of the required formula for F(o), the constants in this formula have then been adjusted to give the best fit with the numerical results of Crocco and other workers.

/ Consider ...

Consider the case when $\sigma = 1.0$. Then we can write

$$i = A_1 + A_2\eta + A_3\eta^2 ,$$

where $A_1 = i(o)$, $A_2 = 1 - i(o) + \frac{u_1^2}{2I_1}$, $A_3 = -\frac{u_1^2}{2I_1}$

Write $\eta = \sin\theta$, then (33) can be written

$$F(o)^2 = \int_0^{\frac{\pi}{2}} \left[\int_0^{\theta_1} 2\sin\theta\,(A_1 + A_2\sin\theta + A_3\sin^2\theta)^{\omega-1}\, d\theta \right] d\theta ,$$

Let $A_4 = 1 + i(o) + \frac{u_1^2}{2I_1}$, then $A_4 > A_1 + A_2\sin\theta + A_3\sin^2\theta$.

Hence, we can write

$$F(o)^2 = (A_4)^{\omega-1} \int_0^{\frac{\pi}{2}} \left[\int_0^{\theta_1} 2\sin\theta \left[1 + (\omega-1)\left(\frac{A_2}{A_4}\sin\theta + \frac{A_3}{A_4}\sin^2\theta - \frac{A_4 - A_1}{A_4} \right) + - - \right] d\theta ,\right.$$

$$= (A_4)^{\omega-1} \left\{ (2 - \frac{\pi}{2}) + (\omega-1)\left[\left(\frac{A_1 - A_4}{A_4}\right)(2 - \frac{\pi}{2}) + \frac{A_2}{A_4}\left(\frac{\pi}{2} - \frac{4}{3}\right) + \frac{A_3}{2A_4}\left(\frac{8}{3} - \frac{3\pi}{4}\right) \right] + \text{terms in } (\omega-1)^2 \right\} \quad ..(34)$$

This leads to the result that when

$$\omega = 1.0,\ F(o)^2 = 2 - \frac{\pi}{2} ,$$

and therefore $F(o) = 0.655$ (compare the accurate value of 0.664)

Write equation (34) in the form

$$\frac{F(o)}{F(o)_{\omega=1.0}} = A_4^{\frac{\omega-1}{2}} \left\{ 1 + (\omega-1)\left[\frac{A_1 - A_4}{A_4} + \frac{A_2}{A_4}\left(\frac{3\pi - 8}{12 - 3\pi}\right) + \frac{A_3}{12A_4}\frac{(32 - 9\pi)}{(4 - \pi)} \right] + \text{terms in } (\omega-1)^2 + \right\}^{\frac{1}{2}}$$

/ Then ...

Then assuming ω is not very far from unity (as will be the case in practice), the terms in $(\omega - 1)^2$ can be allowed for to some extent by approximating to the right hand side with

$$\frac{F(o)}{F(o)_{\omega=1.0}} = \left\{ A_4 + (A_1 - A_4) + A_2 \left(\frac{3\pi - 8}{12 - 3\pi}\right) + \frac{A_3}{12} \frac{(32 - 9\pi)}{(4 - \pi)} \right\}^{\frac{\omega - 1}{2}}$$

$$= \left\{ \frac{3\pi - 8}{12 - 3\pi} + i(o) \left[\frac{20 - 6\pi}{12 - 3\pi}\right] + \frac{u_1^2}{2 I_1} \left[\frac{21\pi - 64}{12(4 - \pi)}\right] \right\}^{\frac{\omega - 1}{2}}$$

$$= \left[0.553 + 0.447\, i(o) + 0.096 \frac{u_1^2}{I_1} \right]^{\frac{\omega - 1}{2}} \qquad \ldots.(35)$$

Even before any attempt is made to modify the constants in this formula to improve agreement with calculated results we may note that it gives good agreement with formula (31) for the case of zero heat transfer and $\sigma = 1.0$. Thus, in this case, from (29)

$$i(o) = 1 + \frac{(\gamma - 1)}{2} M_1^2 = 1 + \frac{u_1^2}{2 I_1} ,$$

and hence (35) becomes

$$\frac{F(o)}{F(o)_{\omega=1}} = \left[1 + 0.32 \frac{u_1^2}{I_1} \right]^{\frac{\omega - 1}{2}} = \left[1 + 0.32 (\gamma - 1) M_1^2 \right]^{\frac{\omega - 1}{2}}$$

This comparison indicates that any modification to (35) should retain the following features:-

(1) The sum of the constant term and the coefficient of $i(o)$ should be unity.

(2) Half the coefficient of $i(o)$ plus the coefficient of $\frac{u_1^2}{I_1}$ should be approximately equal to 0.3

The comparison also indicates that to take account of values of σ other than unity the coefficient of $\frac{u_1^2}{I_1}$ should include the term $\sigma^{\frac{1}{2}}$.

Formula (35) with this last modification was in fact found to give results in surprisingly close agreement with the calculations of Ref.2 to 6, bearing in mind the approximate nature of the analysis. The closest overall agreement was, however, found with the formula

$$\frac{F(o)}{F(o)_{\omega=1.0}} = \left[0.45 + 0.55\, i(o) + 0.09 (\gamma - 1) M_1^2 \sigma^{\frac{1}{2}} \right]^{\frac{\omega - 1}{2}} .$$

/ Using ...

Using the correct basic value of $F(o)_{\omega=1.0} = 0.664$, this can be written

$$c_f \sqrt{R_x} = 0.664 \left[0.45 + 0.55\, i(o) + 0.09\, (\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}} \right]^{\frac{\omega-1}{2}} \qquad (36)$$

For the case of zero heat transfer this reduces to

$$c_f \sqrt{R_x} = 0.664 \left[1 + 0.365\, (\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}} \right]^{\frac{\omega-1}{2}} \quad \ldots.(37)$$

Table I compares the results given by equation (36) and Crocco's calculations, and it will be seen that in every case the agreement is within 1% , which is within the accuracy of Crocco's calculations. In Table II Crocco's results for zero heat transfer are compared with the results given by equation (37), and the agreement will be seen to be equally good. Finally, in Fig.2 some of the results of Ref.2, 4 and 5 are compared with equation (37) for the case of zero heat transfer, and in every case close agreement is found. One can conclude that the formulae (36) and (37) are reliable to within about 1% for values of the Mach number up to 10, and for values of ω and σ likely to be of practical interest when air is the working fluid.

3. HEAT TRANSFER AND DISSIPATION - FLAT PLATE AT ZERO INCIDENCE.

3.1. Heat Transfer

If Q is the total heat transferred per unit time from a surface of unit breadth and length L, say, q is the local rate of heat transfer per unit area and S_F is the total frictional force on the surface, then it readily follows from Crocco's analysis that for σ near unity

$$\frac{Q}{S_F} = \frac{q}{\tau_w} = \frac{J\, C_p}{\sigma\, 273} \left[\frac{T_w - T_{th.}}{u_1} \right] \quad \ldots.(38)$$

where $T_{th.}$ is the temperature measured by a thermometer, i.e.

$$T_{th} = T_w \text{ for zero heat transfer}$$
$$= T_1 \left[1 + \frac{(\gamma-1)}{2} M_1^{\,2} \sigma^{\frac{1}{2}} \right].$$

If we now use equation (36) to derive S_F and τ_w, we find

$$S_F = 0.664 \frac{\rho_1 u_1^{\,2} L}{\sqrt{\frac{u_1 L}{\nu_1}}} \left[0.45 + 0.55\, i(o) + 0.09 (\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}} \right]^{-\frac{(1-\omega)}{2}} \quad ..(39)$$

and

$$\tau_w = 0.332 \frac{\rho_1 u_1^{\,2}}{\sqrt{\frac{u_1 x}{\nu_1}}} \left[0.45 + 0.55 i(o) + 0.09 (\gamma-1) M_1^{\,2} \sigma^{\frac{1}{2}} \right]^{-\frac{(1-\omega)}{2}} \quad ..(40)$$

Equations (38), (39) and (40) enable us to calculate q and Q in any case where T_w, T_1, M_1, σ and ω are given.

/3.2 ...

3.2. Dissipation

It has been pointed out by Karman and Tsien[2] that when a hot gas is cooled by a cold surface, the rate of heat transfer from the gas to the wall is increased by increase of Mach number (see equation (38)) but on the other hand so is the mechanical energy which is dissipated by viscosity. The latter increases more rapidly than the former and eventually at some Mach number the gas becomes heated and not cooled. The results of calculations by Karman and Tsien[2] and Hantzsche and Wendt[3] for the cases when $i(o) = \frac{1}{4}$, $\sigma = 1.0$, and $\omega = 1.0$ and 0.76 are shown in Fig.3. In the following a simple formula is derived for the Mach number at which the change from cooling to heating takes place.

The rate of dissipation of mechanical energy by viscosity on a plate of unit breadth and length L is

$$D = \int_0^L \left[\int_0^\infty \mu \left(\frac{\partial u}{\partial y} \right)^2 dy \right] . dx$$

$$= \mu_1 u_1^2 \sqrt{R_L} \int_0^1 F(\eta) . d\eta \; , \qquad \ldots .(41)$$

where $R_L = u_1 L / \nu_1$.

The rate of heat transfer from the gas to the wall is from (38)

$$-Q = \frac{J c_p}{\sigma^{2/3}} \; \frac{T_{th} - T_w}{u_1} . \frac{2F(o)}{\sqrt{R_L}} . \tfrac{1}{2} \rho_1 u_1^2 L \; . \qquad \ldots .(42)$$

Hence the gas is cooled or heated according as

$$\frac{J c_p}{\sigma^{2/3}} \left[\frac{T_{th} - T_w}{u_1^2} \right] . F(o) > \text{ or } < \int_0^1 F(\eta) . d\eta ,$$

or

$$\frac{1 + \frac{(\gamma - 1)}{2} M_1^2 \sigma^{\frac{1}{2}} - i(o)}{\sigma^{2/3} (\gamma - 1) M_1^2} > \text{ or } < \int_0^1 \frac{F(\eta)}{F(o)} . d\eta \qquad \ldots .(43)$$

We now make use of the close approximation

$$F(\eta) = F(o) \, (1 - \eta^2)^{\frac{1}{2}} ,$$

then (43) becomes

$$1 + \frac{(\gamma - 1)}{2} M_1^2 \sigma^{\frac{1}{2}} - i(o) > \text{ or } < \sigma^{2/3} \frac{\pi}{4} (\gamma - 1) M_1^2$$

$$\text{or } M_1^2 < \text{ or } > \frac{2}{(\gamma - 1)} \left[\frac{1 - i(o)}{\sigma^{2/3} \frac{\pi}{2} - \sigma^{\frac{1}{2}}} \right] . \qquad \ldots .(44)$$

/ For ...

For the cases considered by Karman and Hantzsche and Wendt, viz. $i(o) = \frac{1}{4}$ and $\sigma = 1.0$, this relation predicts that heating of the gas begins when

$$M_1^{\,2} = \frac{3}{(\gamma-1)}\,\frac{1}{(\pi-2)} = 6.58$$

$$\text{or} \quad M_1 = 2.56.$$

This value is in good agreement with the value shown in Fig.3, where it will also be seen that in conformity with relation (44) above the value of the Mach number at which heating begins is practically independent of the value of w.

3.3. Effect of radiation on thermometer temperature.

The following treatment is a modification of that due to Hantzsche and Wendt [3], use being made of the explicit formula for $F(o)$, equation (36).

Allowing for the radiation of heat in the thermometer problem, then the rate of heat transfer from gas to wall must equal the rate at which heat is radiated. Hence

$$Q = \int_o^L C_B\,(T_1^{\,4} - T_W^{\,4}).\,dx,$$

where C_B = Stefan-Boltzmann's constant multiplied by the emissivity.

Using equation (38) we have therefore

$$\frac{c_p(T_W - T_{th})}{\sigma^{2/3}\,u_1}\cdot\frac{2F(o)}{\sqrt{R_L}}\cdot\tfrac{1}{2}\,\rho_1\,u_1^{\,2}L = C_B\,L(T_W^{\,4} - T_1^{\,4})\,,$$

or

$$\frac{\left[i(o) - 1 - \frac{(\gamma-1)}{2}M_1^{\,2}\sigma^{\frac{1}{2}}\right]}{\sigma^{2/3}\,(i(o)^4 - 1)}\;F(o).\;c_p\sqrt{\frac{u_1\rho_1\mu_1}{L}} = C_B T_1^{\,3}$$

Write

$$K_R = \frac{C_B T_1^{\,3} L}{c_p R_L^{\frac{1}{2}}\mu_1} \qquad \ldots\ldots(45)$$

Then

$$K_R = \frac{\left[i(o) - 1 - \frac{(\gamma-1)}{2}M_1^{\,2}\sigma^{\frac{1}{2}}\right]}{\sigma^{2/3}\left[i(o)^4 - 1\right]}\,.\,F(o)$$

where, as before, we have

$$F(o) = 0.664\left[0.45 + 0.55\,i(o) + 0.09\,M_1^{\,2}\sigma^{\frac{1}{2}}\right]^{-\frac{(1-\omega)}{2}} \qquad \ldots\ldots(46)$$

/ The ...

The problem in general is, given K_R, what is the value of $i(o)$? With the aid of equation (46) the problem is best treated in reverse, that is for given values of M_1, ω, σ and $i(o)$ the corresponding values of K_R can be calculated and the values of $i(o)$ can be plotted against K_R. We may note immediately that for $K_R = 0$ (as when $L = 0$) $T_W = T_{th}$., for $K_R = \infty$ (as when $L = \infty$), $i(o) = 1$, i.e. $T_W = T_1$. For intermediate values, $i(o)$ decreases steadily with increase of K_R.

4. THE LAMINAR BOUNDARY LAYER ON A CYLINDER.

The method to be developed here is essentially the same as a method previously put forward by the author [11], but with a number of improvements suggested by the results given in paragraph 2.

Suffix o will be used to denote undisturbed stream values, and **suffix** 1 to denote values just outside the boundary layer.

The momentum equation of the boundary layer is readily shown to be [11]

$$\theta' + \left[(H+2)\ \frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1}\right]\theta = \frac{\tau_w}{\rho_1 u_1^{\,2}} \qquad \ldots.(47)$$

where

$$\theta = \int_o^\delta \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right) dy\ , \qquad \delta^+ = \int_o^\delta \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$$

$H = \delta^+/\theta$, and dashes denote differentiation with respect to x the distance along the surface.

We will now follow some of the implications of an approach on the lines of Pohlhausen's method for incompressible flow. Instead of expressing the velocity distribution as a function of y however, it will be expressed as a function of $Y = \int \frac{\mu_o dy}{\mu}$, since we know that on a flat plate, with $\omega = 1.0$, the velocity distribution expressed as a function of Y is independent of Mach number. Thus suppose the velocity distribution across the boundary layer to be given by the quartic

$$u = aY + bY^2 + cY^3 + dY^4 \qquad \ldots.(48)$$

with the boundary conditions that

for $Y = \delta_1$, $u = u_1$, $\frac{\partial u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2} = o$; and

for $Y = 0$, $u = o$, $\frac{\partial T}{\partial Y} = o$, and hence $\frac{\partial \mu}{\partial Y} = o$.

The quantity δ_1 is at present unknown, but is assumed to correspond to the value of Y defining the outer edge of the boundary layer.

/ The ...

The first boundary layer equation yields

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)_w = \frac{\partial p}{\partial x} = -\rho_1 u_1 u_1{}'.$$

But $$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)_w = \mu_w \left(\frac{\partial^2 u}{\partial y^2}\right)_w , \text{ since } \left(\frac{\partial \mu}{\partial y}\right)_w = 0,$$

$$= \frac{\mu_o^2}{\mu_w}\left(\frac{\partial^2 u}{\partial Y^2}\right)_w .$$

Hence $$\left(\frac{\partial^2 u}{\partial Y^2}\right)_w = -\frac{\mu_w}{\mu_o^2} \cdot \rho_1 u_1 u_1{}'.$$

With these boundary conditions we can solve for the coefficients a, b, c and d in terms of δ_1, and we obtain

$$\left.\begin{aligned} a &= \frac{u_1(12+\Lambda)}{6\delta_1}, \quad b = \frac{-u_1\Lambda}{2\delta_1^2}, \quad c = \frac{-u_1(4-\Lambda)}{2\delta_1^3}, \\ d &= \frac{u_1(6-\Lambda)}{6\delta_1^4}, \quad \text{where} \\ \Lambda &= u_1{}'\delta_1^2 \rho_1 \frac{\mu_w}{\mu_o^2} \end{aligned}\right\} \quad \ldots.(49)$$

Further, since $\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w = \mu_o \left(\frac{\partial u}{\partial Y}\right)_w$

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{\mu_o (12+\Lambda)}{6\rho_1 u_1 \delta_1} \quad \ldots.(50)$$

From this point the method departs from the classical Pohlhausen approach. It depends on the solution of the momentum equation, making the following assumptions

(1) Equation (50) is accepted.

(2) The variation of H with local velocity, pressure gradient, ω and σ is neglected, H being treated as a function of M_o, only. The justification for this lies in the fact that H enters the solution of the momentum equation in a form suggesting that the solution is relatively insensitive to small variations of H, and for slim bodies at small incidences the possible variations of H are unlikely to be large.+

+ It is possible to avoid making this assumption, but the computing involved then becomes considerably heavier.

For a flat plate at zero incidence, with ω and σ both unity we know that $\frac{u}{u_o}$ is a unique function of Y, and also the total energy is constant across the boundary layer. Hence

$$\theta = \int_o^\infty \frac{\rho u}{\rho_o u_o} \left(1 - \frac{u}{u_o}\right) dy = \int_o^\infty \frac{u}{u_o} \left(1 - \frac{u}{u_o}\right) dY$$

$$= \theta_i ,$$

where suffix i denotes the value in incompressible flow.
Also

$$\delta^* = \int_o^\infty \left(1 - \frac{\rho u}{\rho_o u_o}\right) dy = \int_o^\infty \left(\frac{\mu}{\mu_o} - \frac{u}{u_o}\right) dY.$$

But $$\frac{\mu}{\mu_o} = \frac{T}{T_o} = 1 + \frac{(\gamma - 1)}{2} M_o^2 \left(1 - \frac{u^2}{u_o^2}\right)$$

Hence $$\delta^+ = \int_o^\infty \left(1 - \frac{u}{u_o}\right) dY + \frac{(\gamma - 1) M_o^2}{2} \int_o^\infty \left(1 - \frac{u^2}{u_o^2}\right) dy$$

$$= \delta_i^+ + \frac{(\gamma - 1)}{2} M_o^2 \left(\delta_i^+ + \theta_i\right)$$

Therefore

$$H = \frac{\delta^+}{\theta} = H_i \left\{1 + \frac{(\gamma - 1)}{2} M_o^2 \left(1 + \frac{1}{H_i}\right)\right\} .$$

The Blasius solution gives

$$H_i = 2.59,$$

hence $$H = 2.59 \left(1 + 0.277 M_o^2\right) \qquad ....(51)$$

This relation is in almost exact agreement with the results of the calculations of Brainerd and Emmons [4].

(3) The ratio δ_1/θ is also assumed to be independent of variations in local velocity and pressure gradient. For the present we write

$$\delta_1/\theta = f\,(M_o, \omega, \sigma) \qquad ....(52)$$

making no further assumptions at this stage about the form of the function, except to note that we expect it to be constant when $\omega = 1.0$.

$\frac{\delta^+}{\delta_i^+} = 1 + 0.277 M_o^2$

/ From...

From (50) and the expression for Λ in (49) we have

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{\mu_o}{6\rho_1 u_1 \delta_1}\left[u_1' \delta_1^2 \rho_1 \frac{\mu_w}{\mu_o^2} + 12 \right]$$

$$= \frac{u_1'}{6\,u_1}\ \delta_1 \frac{\mu_w}{\mu_o} + \frac{2\mu_o}{\rho_1 u_1 \delta_1}$$

$$= \frac{u_1'}{6\,u_1}\ \frac{\mu_w}{\mu_o}\ f\,\theta + \frac{2\mu_o}{\rho_1 u_1 f\,\theta}\ .$$

Hence, the momentum equation (47) becomes

$$\theta' + \left[(H + 2)\ \frac{u_1'}{u_1} + \frac{\rho_1'}{\rho_1} \right]\theta = \frac{u_1'}{6u_1}\ \frac{\mu_w}{\mu_o}\ .\ f\,\theta + \frac{2\mu_o}{\rho_1 u_1 f\theta}\ .$$

Multiply both sides by $\rho_1^2\,\theta$ and we get

$$\frac{d}{dx}\left[\rho_1^2\theta^2\right] + 2\rho_1^2\theta^2\frac{u_1'}{u_1}\left[(H + 2) - \frac{f}{6}\ \frac{\mu_w}{\mu_o}\right] = 4\frac{\mu_o \rho_1}{u_1\ f} \qquad \text{....(53)}$$

At this stage a further assumption is made, viz. the value of $\frac{\mu_w}{\mu_o}$ in the second bracket is taken to be that for a flat plate at zero incidence. This gives

$$\frac{\mu_w}{\mu_o} \doteq \left[1 + \frac{(\gamma - 1)}{2}\ \sigma^{\frac{1}{2}}\ M_o^2 \right]^{\omega} \qquad \text{....(54)}$$

Write $H + 2 - \frac{f}{6}\ .\ \frac{\mu_w}{\mu_o} = g\,(M_o, \omega, \sigma)\ /2$, say. ....(55)

Then (53) can be integrated with respect to x to yield

$$\left[u_1^g\ \rho_1^2\theta^2 \right]_o^{x_1} = \frac{4}{f}\,\mu_o \int_o^{x_1} \rho_1\ u_1^{g-1}\ .\ dx \qquad \text{....(56)}$$

The leading edge is taken at $x = 0$, where either $u_1 = 0$, or u_1 is finite. If u_1 is finite, then $\theta = 0$ there, since we cannot have a finite momentum loss there. Hence

$$\boxed{\left[\rho_1^2\theta^2\right]_{x_1} = \frac{4\mu_o}{f\ u_1(x_1)^g}\int_o^{x_1} \rho_1 u_1^{g-1}\ .\ dx} \qquad \text{....(57)}$$

/ We ...

We may note here that g occurs as an exponent of u_1 inside the integral, and as an exponent of $u_1(x_1)$ outside the integral in the denominator. This suggests that provided the variation of u_1 with x is small, small errors in g will have little effect, and this provides the justification for the neglect of the variation of H and $\frac{\mu_w}{\mu_o}$ with pressure gradient and local velocity.

We have yet to determine the function f.

For the flat plate at zero incidence (57) reduces to

$$\left[\rho_o \theta^2\right]_{x_1} = \frac{4\mu_o x_1}{u_o f}$$

or $$\frac{\theta\sqrt{R_x}}{2x} = \sqrt{\frac{1}{f}} ,$$

and $$\frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{R_x}} \frac{1}{\sqrt{f}} .$$

But $$c_f = \frac{2\tau_w}{\rho_o u_o^2} = 2\frac{\partial\theta}{\partial x}$$

$$= \frac{2}{\sqrt{R_x}} \cdot \frac{1}{\sqrt{f}} .$$

Hence $$\frac{2}{\sqrt{f}} = c_f\sqrt{R_x} = 0.664\left[1 + 0.365(\gamma-1)\sigma^{\frac{1}{2}} M_o^2\right]^{\frac{\omega-1}{2}}$$

from equation (37) .

It follows that

$$f = 9.072\left[1 + 0.365(\gamma-1)\sigma^{\frac{1}{2}} M_o^2\right]^{(1-\omega)} \qquad \dots.(58)$$

The function g is given by

$$g = 2(H+2) - \frac{f}{3}\frac{\mu_w}{\mu_o}$$

$$= 9.18 + 1.436 M_o^2 - \frac{f}{3}\left[1 + \frac{(\gamma-1)}{2}\sigma^{\frac{1}{2}} M_o^2\right]^{\omega} \qquad \dots.(59)$$

It is sometimes convenient to express these relations in non-dimensional forms. Thus, write

$\rho^+ = \frac{\rho}{\rho_o}$, $\mu^+ = \frac{\mu}{\mu_o}$, $u^+ = \frac{u}{u_o}$, $\theta^+ = \frac{\theta}{L}$, $x^+ = \frac{x}{L}$, when

L is some standard length, then (57) becomes

$$\left[\rho_1^{+2}\theta^{*2}\right]^R_{x_1} = \frac{4}{u_1(x_1^+)^g f}\int_0^{x_1}\rho_1^+ u_1^{+\,g-1}\,dx^+ , \qquad \dots.(60)$$

where $R = U_oL/\nu_o$.

/ The ...

The relation between ρ_1^+ and u_1^+ is

$$\rho_1^+ = \left[1 + \frac{(\gamma - 1)}{2} M_o^2 (1 - u_1^{+\,2})\right]^{\frac{1}{\gamma-1}} \qquad \ldots.(61)$$

Having u_1^+ as a function of x^+, we can then obtain θ^+ at any point from equations (58), (59), (60) and (61), the solution of (60) being obtained by graphical or numerical integration.

To obtain the local skin friction coefficient when θ^+ is known, we have

$$\left.\begin{aligned} c_f &= \frac{2\tau_w}{\rho_o u_o^2} = \frac{(\Lambda + 12)\, u_1}{3 R\, \theta^* f} \\ \text{and} \quad \Lambda &= u_1^1\, \delta_1^2\, \rho_1\, \frac{\mu_w}{\mu_o^2} \\ &= R(u_1^{+1}\, \theta^{+2} f^2 \rho_1^+ \mu_w^+) . \end{aligned}\right\} \qquad \ldots.(62)$$

For $\mu_w^+ = \frac{\mu_w}{\mu_o}$, we may use equation (54) but it is probably more accurate at this point to assume that the local relation between T_w, T_1, and u_1, is the same as for the flat plate which leads to the relation

$$\mu_w^+ = \left\{1 + \frac{(\gamma - 1)}{2} M_o^2 \left[1 + \frac{u_1^2}{u_o^2}(\sigma^{\frac{1}{2}} - 1)\right]\right\}^{\omega} \qquad \ldots.(63)$$

The accuracy of the method outlined above cannot as yet be assessed, this must await the development of a method that can be accepted as accurate and its application to a few test cases. In favour of the above method it should be noted that it gives the correct answer for the flat plate at zero incidence and the numerical work involved in its application is relatively small. It may also be noted that when applied to incompressible flow (Ref.11) the method gave results which were in very close agreement with results given by the standard Pohlhausen method.

5. SEPARATION OF THE BOUNDARY LAYER.

It is unlikely that the above method will be any more successful in predicting the separation point of the laminar boundary layer than is the Pohlhausen method in incompressible flow. Nevertheless, it should provide a guide as to the effect of increase of Mach number on tendency to separation.

From (49) separation occurs when $\Lambda = -12$, i.e. when

$$R\, \theta^{+2} \rho_1 = \frac{-12}{u_1^{+1} f^2 \mu_w^+}, \text{ from (62).}$$

/ Hence ...

Hence, from (60) separation occurs at x_1, if

$$\left[\frac{-3\, u_1^{+\,g}\, \rho_1^{+}}{u_1^{+1}\; f\, \mu_w^{+}}\right]_{x_1} = \int_0^{x_1^+} \rho_1^{+}\; u_1^{+\,g-1} .\, dx^{+} . \qquad ....(64)$$

Howarth has considered the case of a uniform adverse velocity gradient with $\omega = 1.0$ and $\sigma = 1.0$, and has estimated the separation distance for different main stream Mach numbers at the beginning of the plate, using a method that is essentially an analogue of Pohlhausen's method. It is of interest to compare his results with the results given by the present method.

With $\omega = 1.0$, and $\sigma = 1.0$,

$$f = 9.072,$$

$$g = 6.156 + 0.831\, M_o^2,$$

and $\mu_w^{+} = 1 + \frac{(\gamma - 1)}{2} M_o^2$.

Suppose $\qquad u_1 = u_o - \beta x$, $\qquad$ where β is a constant.

Then

$$u_1^{+} = 1 - \frac{\beta L}{u_o} .\, x^{+} \quad, \text{ and } \quad u_1^{+\,1} = -\frac{\beta L}{u_o} = \alpha, \text{ say.}$$

Equation (64) becomes

$$\frac{3}{9.072\,(1 + \frac{\gamma-1}{2} M_o^2)\alpha} = \frac{1}{(1 - \alpha x_1^{+})^{g}\, \rho_1^{+}\,(x_1)} \int_0^{x_1^+} \rho_1^{+}\,(1 - \alpha x^{+})^{g-1} .\, dx^{+}$$

and, from (61)

$$\rho_1^{+} = \left[1 + \frac{(\gamma-1)}{2} M_o^2\,(2\alpha x^{+} - \alpha^2 x^{+\,2})\right]^{\frac{1}{\gamma-1}} .$$

Let $\alpha x^{+} = \xi$, then we have

$$0.331 = \frac{\left[1 + \frac{(\gamma-1)}{2} M_o^2\right]}{(1 - \xi_1)^{g} \left[1 + \frac{(\gamma-1)}{2} M_o^2\,(2\xi_1 - \xi_1^2)\right]^{\frac{1}{\gamma-1}}} \times$$

$$\int_0^{\xi_1} \left[1 + \frac{(\gamma-1)}{2} M_o^2\,(2\xi - \xi^2)\right]^{\frac{1}{\gamma-1}} (1 - \xi)^{g-1} .\, d\xi \,, \quad ..(65)$$

with $\quad g = 6.156 + 0.831\, M_o^2$.

The process then is given M_o, to evaluate the R.H.S. numerically for a range of values of ξ_1, and to interpolate to obtain the value of ξ_1 for which equation (65) is satisfied.

/ The ...

The results for the separation distance (x_s) given by this method are compared with those given by Howarth's method in the following table:

M_o	$-\beta x_s/u_o$ Howarth	$-\beta x_s/u_o$ Present Method.
0	0.156	0.165
1	0.148	0.148
3.16	0.107	0.081
10	0.052	0.013

The agreement between the two methods is poor for the higher Mach numbers, but neither method can be claimed a priori as very reliable for predicting the point of separation, and it is impossible at this stage to estimate their relative degrees of reliability for this purpose. It can be inferred from both sets of results, however, that the separation point moves forward with increase of initial Mach number for constant $\frac{\beta}{u_o} = -\frac{du}{dx} / u_o$, i.e. for constant ratio of velocity fall per unit length of plate to initial velocity.

(σ = 0.725)

M_1	ω	i(o)	$\frac{c_f \sqrt{R_x}}{c_f \sqrt{R_x} \; (\omega = 1.0)}$ ※ (Crocco)	$\frac{c_f \sqrt{R_x}}{0.664}$ (Equation 36)
1	1.25	0.25	0.940	0.942
2	1.25	0.25	0.957	0.960
5	1.25	0.25	1.040	1.039
1	1.25	1.0	1.006	1.004
2	1.25	1.0	1.016	1.015
5	1.25	1.0	1.076	1.074
1	1.25	2.0	1.056	1.059
2	1.25	2.0	1.066	1.066
5	1.25	2.0	1.111	1.111
1	0.75	0.25	1.070	1.062
2	0.75	0.25	1.049	1.041
5	0.75	0.25	0.960	0.963
1	0.75	1.0	0.996	0.996
2	0.75	1.0	0.985	0.986
5	0.75	1.0	0.928	0.931
1	0.75	2.0	0.946	0.944
2	0.75	2.0	0.940	0.938
5	0.75	2.0	0.903	0.900
1	0.5	0.25	1.139	1.128
2	0.5	0.25	1.098	1.089
5	0.5	0.25	0.931	0.927
1	0.5	1.0	0.991	0.992
2	0.5	1.0	0.970	0.971
5	0.5	1.0	0.868	0.868
1	0.5	2.0	0.897	0.892
2	0.5	2.0	0.886	0.879
5	0.5	2.0	0.815	0.811

※ To minimise the affects of a possible error of the order of 1% in Crocco's calculations, ratios rather than absolute values of skin friction coefficients are compared.

TABLE II. COMPARISON OF CROCCO'S CALCULATED VALUES OF SKIN FRICTION ON A FLAT PLATE WITH VALUES GIVEN BY EQUATION 37 FOR CASE OF ZERO HEAT TRANSFER.

($\sigma = 0.725$)

M_1	ω	$\dfrac{c_f \sqrt{R_x}}{c_f \sqrt{R_x} \, (\omega = 1.0)}$	$\dfrac{c_f \sqrt{R_x}}{0.664}$ (Equation 37)
1	1.25	1.015	1.015
2	1.25	1.053	1.052
5	1.25	1.194	1.193
1	0.75	0.984	0.986
2	0.75	0.950	0.951
5	0.75	0.842	0.838
1	0.5	0.969	0.971
2	0.5	0.908	0.904
5	0.5	0.707	0.702

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No.	Author	Title
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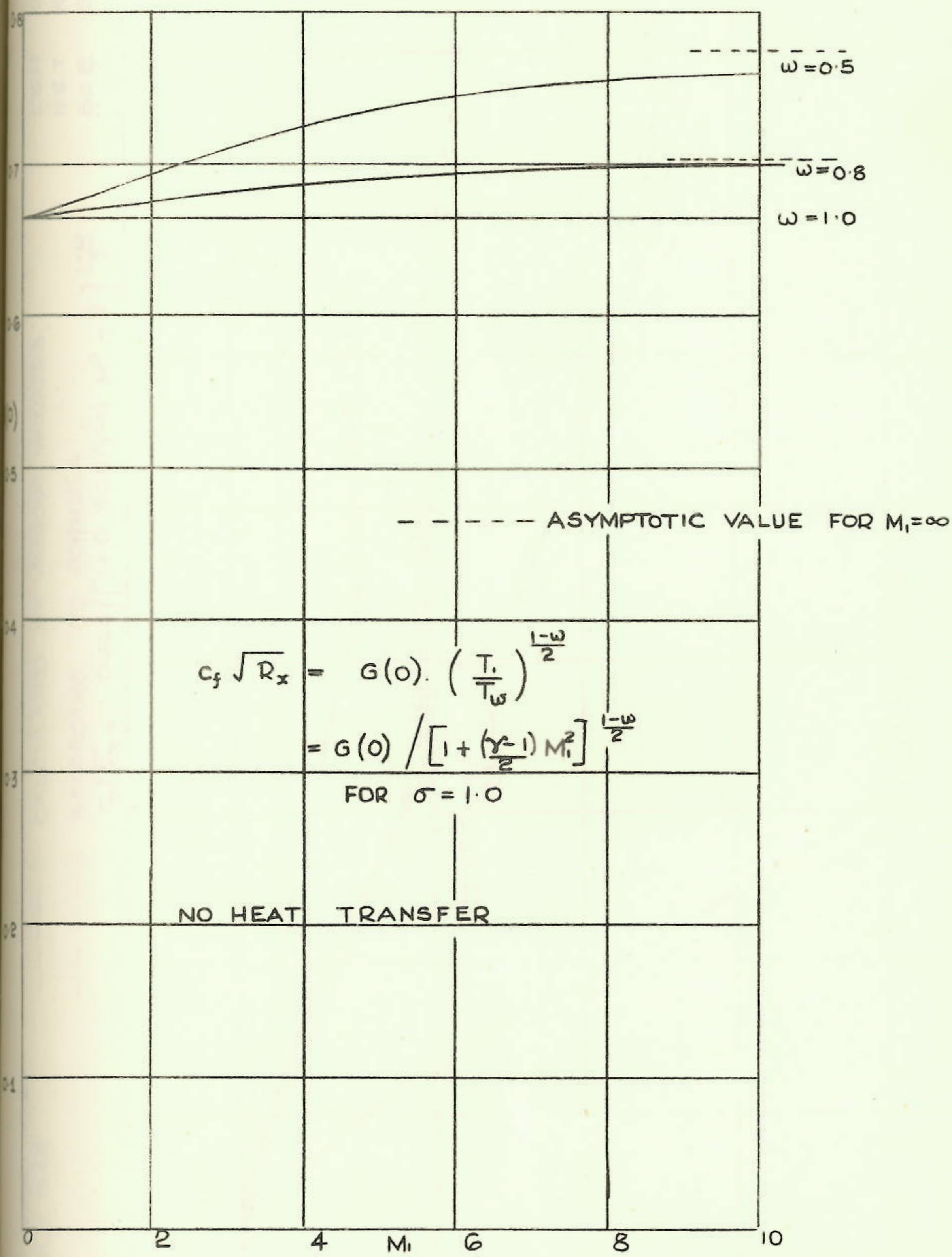
14.	COPE	Note on the law of variation of viscosity and on the Prandtl number of air at low temperatures. (A.R.C. 9287, 1945).
15.	Edited by GOLDSTEIN	Modern Developments in Fluid Dynamics. Vol.II. p.627 - .

FIG 1



SKIN FRICTION OF FLAT PLATE AT ZERO INCIDENCE (NO HEAT TRANSFER)
THE FUNCTION G(0) ACCORDING TO CALCULATIONS
OF HANTZSCHE AND WENDT ($\sigma = 1\cdot 0$)

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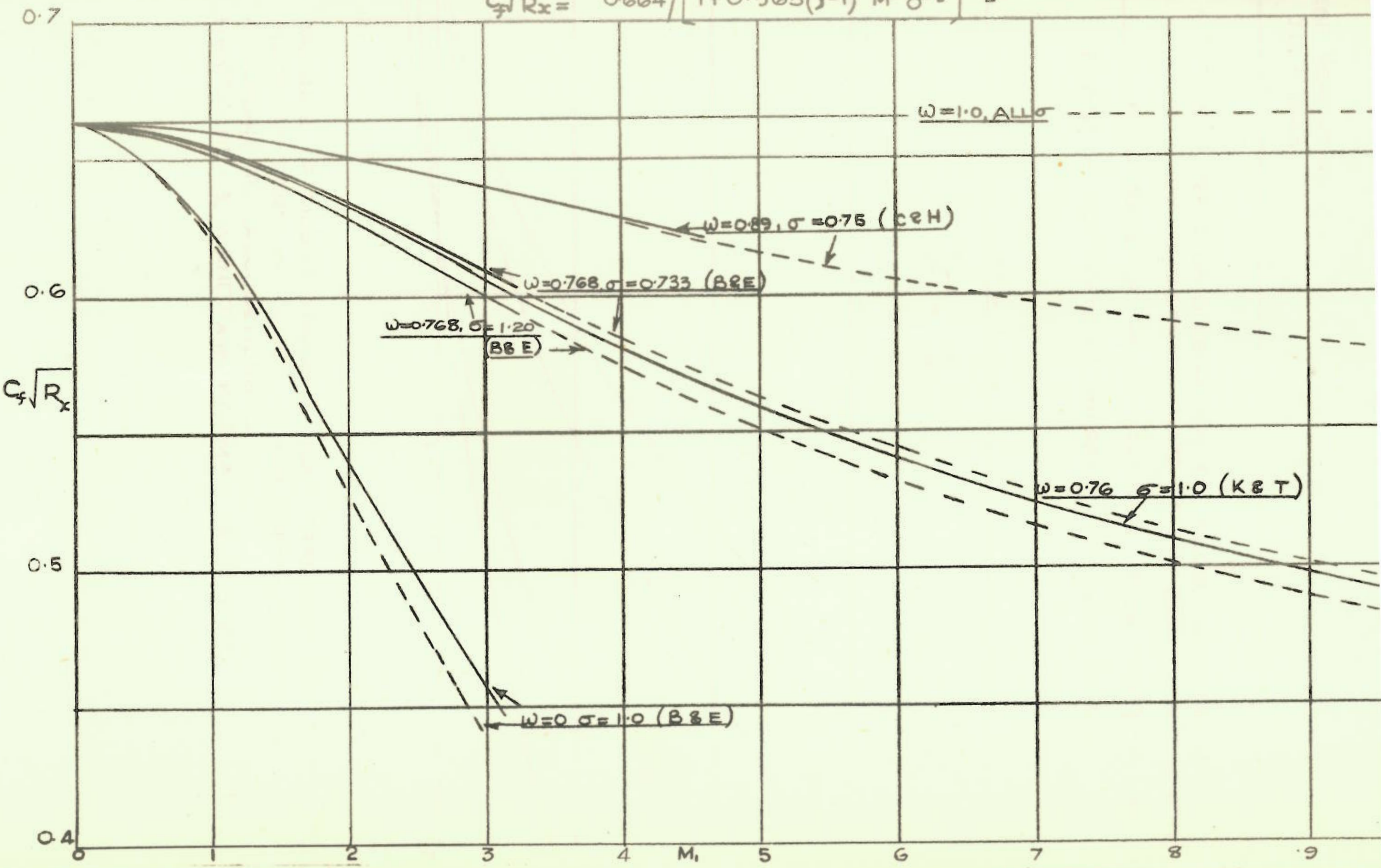
——— CALCULATED BY VARIOUS WORKERS
- - - - ACCORDING TO FORMULA

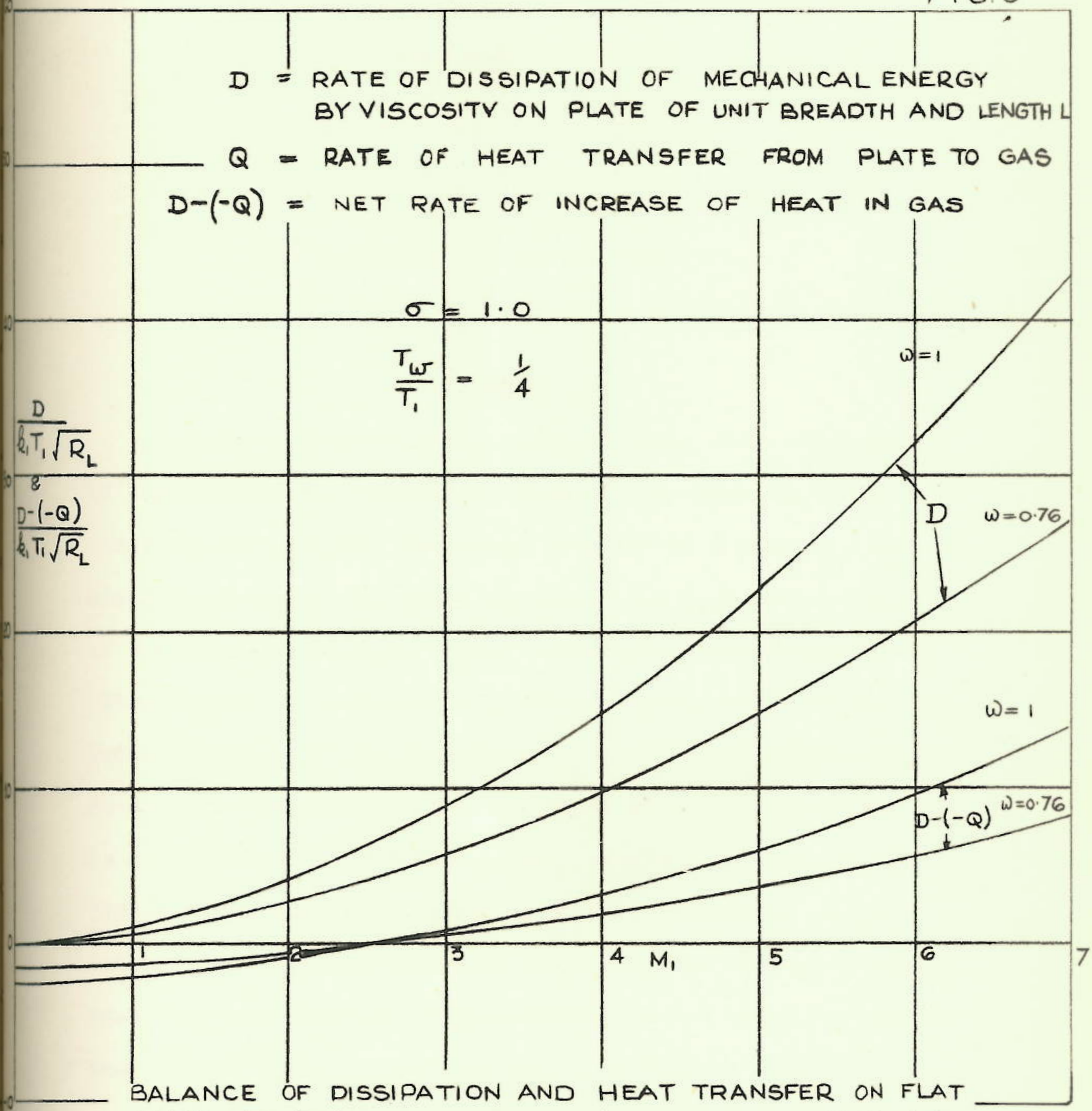
$$C_f\sqrt{R_x} = 0.664 \Big/ \left[1 + 0.365(\gamma - 1)\,M^2\,\sigma^{\frac{1}{2}}\right]^{\frac{1-\omega}{2}}$$

C&H COPE & HART
K&T KARMAN & TSI
B&E BRAINERD& EM



SKIN FRICTION ON A FLAT PLATE AT ZERO INCIDENCE
NO HEAT TRANSFER

FIG. 3



BALANCE OF DISSIPATION AND HEAT TRANSFER ON FLAT PLATE AT ZERO INCIDENCE, WHEN $\frac{T_\omega}{T_1} = \frac{1}{4}$, AND $\sigma = 1{\cdot}0$, ACCORDING TO CALCULATIONS OF HANTZCHE & WENDT AND KARMAN & TSIEN.

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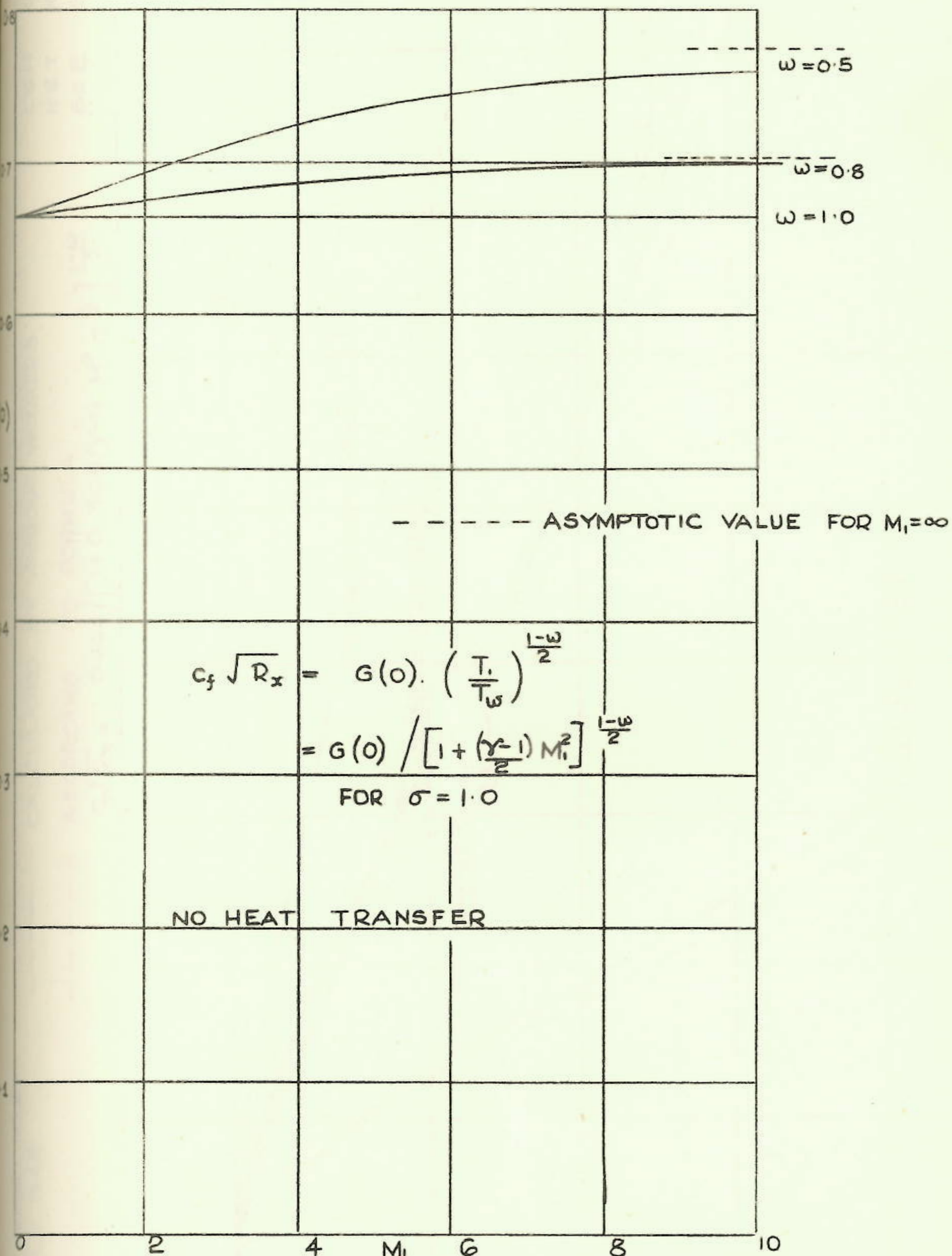
FIG 1



SKIN FRICTION OF FLAT PLATE AT ZERO INCIDENCE (NO HEAT TRANSFER)
THE FUNCTION G(0) ACCORDING TO CALCULATIONS
OF HANTZSCHE AND WENDT ($\sigma = 1 \cdot 0$)

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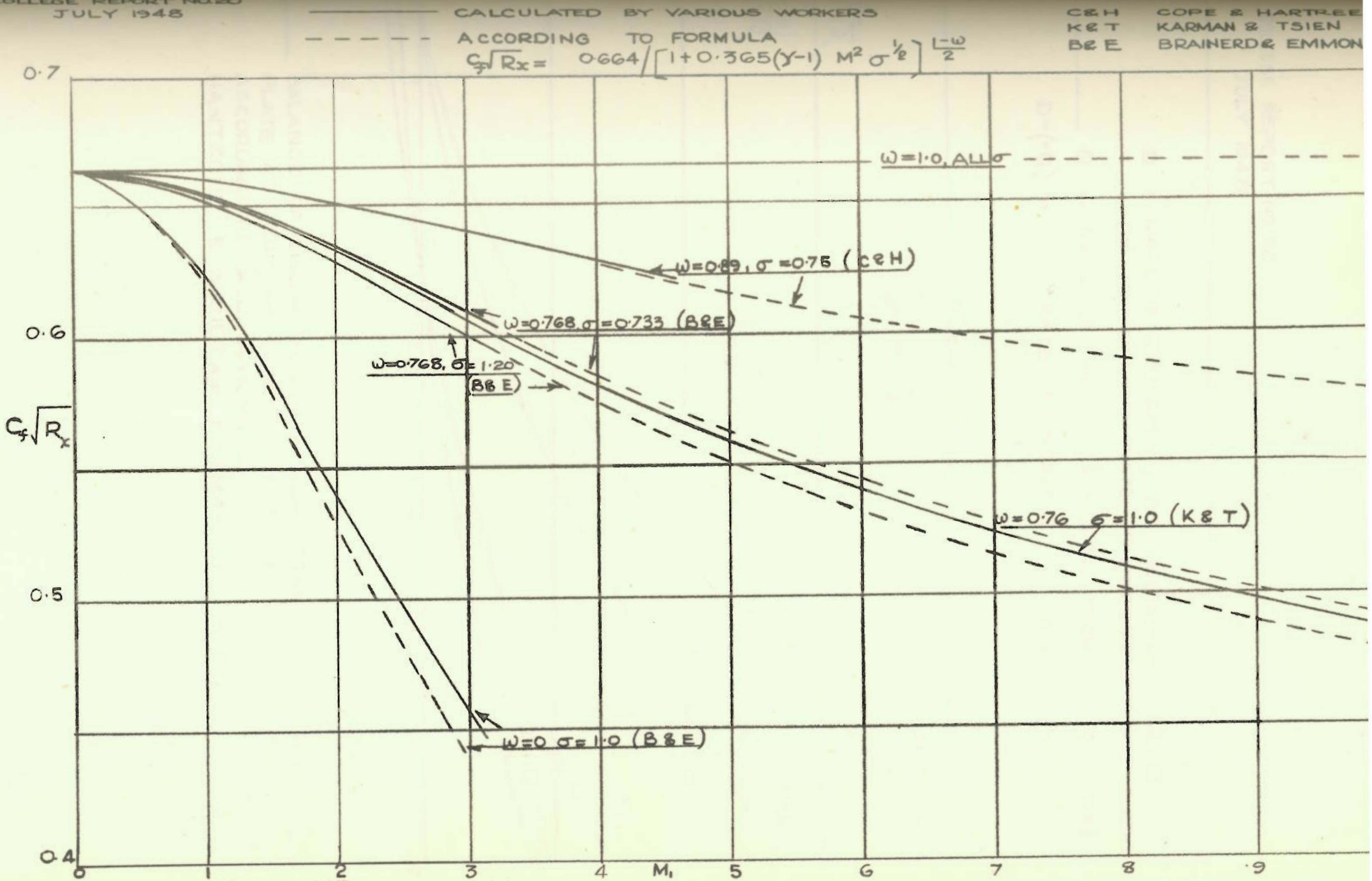

SKIN FRICTION ON A FLAT PLATE AT ZERO INCIDENCE
NO HEAT TRANSFER

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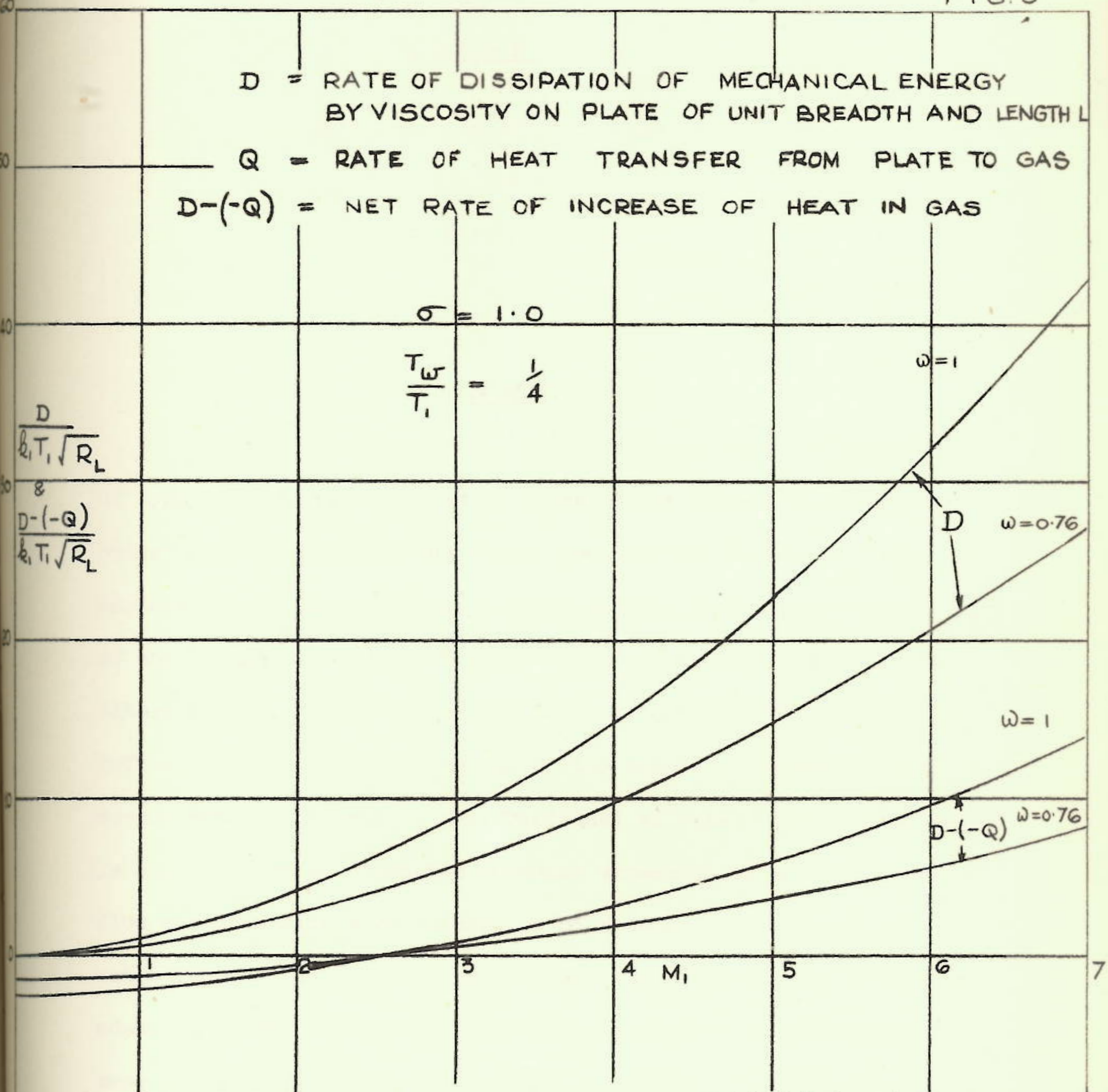
FIG. 3



BALANCE OF DISSIPATION AND HEAT TRANSFER ON FLAT PLATE AT ZERO INCIDENCE, WHEN $\frac{T_\omega}{T_1} = 1/4$, AND $\sigma = 1{\cdot}0$, ACCORDING TO CALCULATIONS OF HANTZCHE & WENDT AND KARMAN & TSIEN.