A New Technique for Instantaneous Frequency Rate Estimation

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***Abstract*—This letter introduces a two-dimensional bilinear mapping operator referred to as the cubic phase (CP) function. For first-, second-, or third-order polynomial phase signals, the energy of the CP function is concentrated along the frequency rate law of the signal. The function, thus, has an interpretation as a time-frequency rate representation. The peaks of the CP function yield unbiased estimates of the instantaneous (angular) frequency rate (IFR) and, hence, can be used as the basis for an IFR estimation algorithm. The letter defines an IFR estimation algorithm and theoretically analyzes it. The estimation is seen to be asymptotically optimal at the center of the data record for high signal-to-noise ratios. Simulations are provided to verify the theoretical claims.**

***Index Terms*—Cramer–Rao bound, cubic phase, estimation, instantaneous frequency rate, time-frequency rate representation.**

I. Introduction

CONSIDER THE CUBIC phase (CP) signal

$$z_s(t) = b_0 e^{j\phi(t)} = b_0 e^{j(a_0 + a_1 t + a_2 t^2 + a_3 t^3)} \tag{1}$$

where the $\{a_0, a_1, a_2, a_3, b_0\}$ are arbitrary parameters, and $\phi(t)$ is the signal phase. The signal's instantaneous (angular) frequency rate (IFR) is

$$\text{IFR} = \frac{d^2\phi(t)}{dt^2} = 2(a_2 + 3a_3 t). \tag{2}$$

Consider now the CP function, defined by

$$\text{CP}(t, \Omega) = \int_0^{+\infty} z_s(t+\tau) z_s(t-\tau) e^{-j\Omega\tau^2} d\tau. \tag{3}$$

Substituting (1) into (3) yields

$$\text{CP}(t, \Omega) = b_0^2 e^{j2\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3\right)} \cdot \int_0^{+\infty} e^{j[2(a_2 + 3a_3 t) - \Omega]\tau^2} d\tau. \tag{4}$$

It is not hard to see that $\text{CP}(t, \Omega)$ peaks along the curve $\Omega = 2(a_2 + 3a_3 t)$. Thus, the peaks of the CP function are along the IFR law of the signal and can be used for IFR estimation. It will be assumed in practice that the observed signals are "noisy" and discrete-time. The model for such signals is

$$z_r(n) = z_s(n) + z_w(n), \qquad \begin{cases} |n| \le \frac{(N-1)}{2}, \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

where $z_s(n)$ is a noiseless CP signal, and $z_w(n)$ is complex white Gaussian noise of power σ^2. The sampling rate is unity, and n is an odd integer. The discrete-time CP function is defined by

$$\text{CP}(n, \Omega) = \sum_{m=0}^{(N-1)/2} z_r(n+m) z_r(n-m) e^{-j\Omega m^2}. \tag{6}$$

II. IFR Estimation Algorithm

The IFR estimate at time n is

$$\hat{\text{IFR}}(n) = \underset{\Omega}{\text{argmax}}[\text{CP}(n, \Omega)]. \tag{7}$$

The above IFR estimator is analyzed in the Appendix and is seen to be asymptotically optimal at the center of the data record for high signal-to-noise ratios (SNRs). The algorithm requires a one-dimensional maximization, in contrast to maximum-likelihood estimation which requires a three-dimensional maximization. By estimating the IFR at several time positions, one could, for example, find estimates for the $\{a_0, a_1, a_2, a_3\}$ parameters.

The maximization in (7) can be implemented with a "coarse" search followed by a "fine" search. The coarse search can be implemented directly according to (7), requiring $O(N^2)$ operations, or with a subband decomposition approach in the *frequency rate domain*, requiring $O(N \log N)$ computations. The subsequent fine search can be implemented with a Newton algorithm and requires $O(N)$ operations.

III. Simulations

A signal with parameter values $b_0 = 1$, $a_0 = 1$, $a_1 = \pi/8$, $a_2 = 0.005$, $a_3 = 0.000\,01$, and $N = 515$ was immersed in various different levels of noise. The IFR at $n = 0$ was estimated according to (7), with 1000 simulations being run for each SNR value. The observed mean-square error (MSE) for the IFR estimate was plotted in Fig. 1 as a function of SNR. The CR bound (full line) and the theoretically predicted MSE (dotted line) are also shown in the plot. The observed MSE values are seen to be close to the predicted values and to the CR bound at high SNR. The algorithm was found to threshold at -4 dB for a 515-point signal. This is significantly lower than the thresholds for the rival algorithms in [1] and [3], which are approximately 2 dB.

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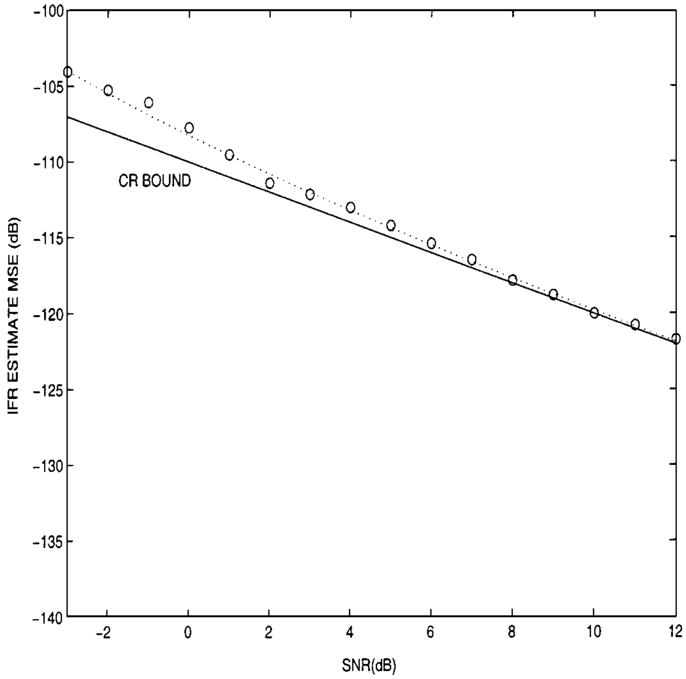

Fig. 1. IFR estimate mean square error (in dB) versus SNR.

APPENDIX
ASYMPTOTIC MSE OF THE IFR ESTIMATE

It is assumed, initially, that the IFR estimate is required at $n = 0$, i.e., at the center of the data record. For the noise-free CP signal $z_s(n)$, the CP function is

$$\mathrm{CP}(n,\Omega)|_{n=0} = \mathrm{CP}(\Omega) = \sum_{m=0}^{(N-1)/2} z_s(m)z_s(-m)e^{-j\Omega m^2} \tag{8}$$

and there is a global maximum at $\Omega = 2a_2$, corresponding to the IFR. If $\mathrm{CP}(\Omega)$ is "perturbed" by noise $z_w(n)$, then $\mathrm{CP}(\Omega)$ is perturbed by

$$\delta\mathrm{CP}(\Omega) = \sum_{m=0}^{(N-1)/2} z_{ws}(m)e^{-j\Omega m^2}. \tag{9}$$

where

$$z_{ws}(m) = [z_s(m)z_w(-m)+z_s(-m)z_w(m)+z_w(m)z_w(-m)].$$

Now with this perturbation, the peak of $\mathrm{CP}(\Omega)$ moves to $\Omega = 2a_2 + \delta\Omega$. This Appendix follows the approach in [2] for deriving an expression for the MSE of $\delta\Omega$. The following formula for the asymptotic mean-square fluctuations of the maximum of a random function is used [2]:

$$E\left\{\delta\Omega^2\right\} = \frac{E\{\alpha^2\}}{\beta^2} \tag{10}$$

where $E\{\cdot\}$ denotes expected value and where

$$\alpha = 2Re\left\{\mathrm{CP}(2a_2)\frac{\partial\delta\mathrm{CP}^*(2a_2)}{\partial\Omega} + \frac{\partial\mathrm{CP}(2a_2)}{\partial\Omega}\delta\mathrm{CP}^*(2a_2)\right\} \tag{11}$$

$$\beta = 2Re\left\{\mathrm{CP}(2a_2)\frac{\partial^2\mathrm{CP}^*(2a_2)}{\partial\Omega^2} + \frac{\partial\mathrm{CP}(2a_2)}{\partial\Omega}\frac{\partial\mathrm{CP}^*(2a_2)}{\partial\Omega}\right\}. \tag{12}$$

Using (10)–(12), the following results can be deduced:

$$\mathrm{CP}(2a_2) \approx b_0^2 e^{ja_0}\frac{N}{2} \tag{13}$$

$$\frac{\partial\mathrm{CP}(2a_2)}{\partial\Omega} \approx -jb_0^2 e^{ja_0}\frac{N^3}{24} \tag{14}$$

$$\frac{\partial^2\mathrm{CP}(2a_2)}{\partial\Omega^2} \approx -b_0^2 e^{ja_0}\frac{N^5}{160} \tag{15}$$

$$\delta\mathrm{CP}(2a_2) = \sum_{m=0}^{(N-1)/2} z_{ws}(m)e^{-j2a_2m^2} \tag{16}$$

$$\frac{\partial\delta\mathrm{CP}(2a_2)}{\partial\Omega} = -j\sum_{m=0}^{(N-1)/2} m^2 z_{ws}(m)e^{-j2a_2m^2} \tag{17}$$

$$\alpha \approx -b_0^2\frac{N}{2}\cdot Im\left\{e^{ja_0}\sum_{m=0}^{(N-1)/2}\left(2m^2-\frac{N^2}{6}\right) \times z_{ws}^*(m)e^{-j2a_2m^2}\right\} \tag{18}$$

$$E\{\alpha^2\} \approx \frac{b_0^4\left(\sigma^4+2b_0^2\sigma^2\right)N^7}{720} \tag{19}$$

$$E\{\delta\Omega^2\} \approx \left[\frac{b_0^4\left(\sigma^4+2b_0^2\sigma^2\right)N^7}{720}\right]\left[\frac{-b_0^4N^6}{360}\right]^{-2} \approx \frac{360\left(1+\frac{1}{2\mathrm{SNR}}\right)}{N^5\pi^2\mathrm{SNR}} \tag{20}$$

where $\mathrm{SNR} = b_0^2/\sigma^2$. By comparison, the Cramer–Rao (CR) bound for estimating the IFR is approximately $360/(\mathrm{SNR}\cdot N^5\pi^2)$. Thus, the MSE of the IFR estimate at $n = 0$ approaches the CR bound asymptotically at high SNR. If an expression is required for the IFR estimate at $n \neq 0$, the analysis proceeds similarly. The resulting asymptotic MSE expression differs only in that N is replaced by $M = 2\cdot\min\{|n-N/2|, |n+N/2|\}$.

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