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Quantum Probability Theory

Miklós Rédei [a,*] and Stephen J. Summers [b]

[a] Department of History and Philosophy of Science,
Loránd Eötvös University,
P.O. Box 32, H-1518 Budapest 112, Hungary

[b] Department of Mathematics, University of Florida,
Gainesville FL 32611, USA

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Abstract

The mathematics of classical probability theory was subsumed into classical measure theory by Kolmogorov in 1933. Quantum theory as nonclassical probability theory was incorporated into the beginnings of noncommutative measure theory by von Neumann in the early thirties, as well. To precisely this end, von Neumann initiated the study of what are now called von Neumann algebras and, with Murray, made a first classification of such algebras into three types. The nonrelativistic quantum theory of systems with finitely many degrees of freedom deals exclusively with type I algebras. However, for the description of further quantum systems, the other types of von Neumann algebras are indispensable. The paper reviews quantum probability theory in terms of general von Neumann algebras, stressing the similarity of the conceptual structure of classical and noncommutative probability theories and emphasizing the correspondence between the classical and quantum concepts, though also indicating the nonclassical nature of quantum probabilistic predictions. In addition, differences between the probability theories in the type I, II and III settings are explained. A brief description is given of quantum systems for which probability theory based on type I algebras is known to be insufficient. These illustrate the physical significance of the previously mentioned differences.

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Contents

1 Introduction

In 1933, probability theory found its modern form in the classic work of N.S. Kolmogorov [37], where it was treated axiomatically as a branch of classical measure theory. There was, of course, significant prior work in that direction — in his own (translated) words [37], "While a conception of probability theory based on the above general viewpoints has been current for some time among certain mathematicians, there was lacking a complete exposition of the whole system, free of extraneous complications." The historical development leading to Kolmogorov's work was lengthy and involved mathematical, physical and philosophical considerations (some stages of this evolution are discussed in [79, 55]). A notable event in this development was D. Hilbert's famous 1900 lecture in Paris on open problems in mathematics, wherein he called for an axiomatic treatment of probability theory (Hilbert's Sixth Problem) [31].

In the same problem, Hilbert also called for an axiomatization of physics, and he himself made important contributions to the subject (cf. [84] for an overview). In the winter term of 1926–1927, he gave a series of lectures on the newly emergent quantum mechanics, which were prepared in collaboration with his assistants, L. Nordheim and J. von Neumann. These were published in a joint paper [32], which was followed up by further papers of von Neumann [47, 48, 49]. This approach culminated in von Neumann's axiomatization of nonrelativistic quantum mechanics in Hilbert space [50].[1]

Motivated by this development, F.J. Murray and von Neumann commenced a study of algebras of bounded operators on Hilbert space [46]. Over time, it slowly became clear that the same mathematical tools were of direct relevance to the quantum theories of more complicated systems, such as quantum statistical mechanics and relativistic quantum field theory. As both the physical theories and the mathematical ideas were refined and

[1] To quote from [84]: "I do not know whether Hilbert regarded von Neumann's book as the fulfillment of the axiomatic method applied to quantum mechanics, but, viewed from afar, that is the way it looks to me. In fact, in my opinion, it is the most important axiomatization of a physical theory up to this time."

generalized, a probability theory emerged which included both classical probability theory and quantum theory as special cases.

In this paper there will be no attempt to describe this historical development. Rather, we shall present an overview of this unification, as well as some of the similarities and differences which this unification both relies on and reveals. We shall also clarify some of the corresponding differences and similarities in the probability theories appropriate for the treatment of quantum theories involving only finitely many degrees of freedom and those involving infinitely many degrees of freedom. Given page constraints, we shall be obliged to treat these matters rather summarily, but further references will be provided for the interested reader. Nothing will be said about the branch of noncommutative probability theory known as free probability theory, but see [82]. Nor shall various extensions of noncommutative probability theory to more general algebraic or functional analytic structures be further mentioned.

We address this paper to readers who have a familiarity with elementary functional analysis (Hilbert spaces and Banach spaces), probability theory and quantum mechanics, but who are not experts in operator algebra theory, noncommutative probability theory or the mathematical physics of quantum systems with infinitely many degrees of freedom.

We shall begin with a brief introduction to the mathematical framework of operator algebra theory, within which this unification has been accomplished. In Section 3, we describe classical probability theory from the point of view of operator algebra theory. This should help the reader in Section 4 to recognize more readily the probability theory inherent in the theory of normal states on von Neumann algebras, which is the setting of noncommutative probability theory. Classical probability theory finds its place therein as the special case where the von Neumann algebra is abelian. Nonrelativistic quantum mechanics is then understood in Section 5 as the special case where the von Neumann algebra is a nonabelian type I algebra.

Although the analogies and differences between classical probability theory and nonrelativistic quantum mechanics are fairly well known, the same cannot be said about the mathematical physics of quantum systems with infinitely many degrees of freedom. We therefore indicate the *necessity* of going beyond the type I case in Section 6, where we discuss quantum statistical mechanics and relativistic quantum field theory, showing how non-type–I algebras arise in situations of immediate physical interest. Some of the physically relevant differences between abelian (classical) probability theory, nonabelian type I probability theory and non-type–I probability theory will be indicated in Section 7.

2 Algebras of Bounded Operators

In this section we shall briefly describe the aspects of operator algebra theory which are most relevant to our topic. For further details, see the texts [61, 77, 35].

Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded operators on a complex Hilbert space[2] $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a complex vector space under pointwise addition and scalar multiplication and a complex algebra under the additional operation of composition. Adding the involution $A \mapsto A^*$ of taking adjoints, $\mathcal{B}(\mathcal{H})$ is a *-algebra. Under the operator norm topology $\mathcal{B}(\mathcal{H})$ is a Banach space. A subalgebra $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ is called a (concrete) *C^*-algebra* if it is closed with respect to the adjoint operation and is closed in the operator norm topology.[3] The

[2]For convenience, we shall always assume our Hilbert spaces to be separable.

[3]There is a useful notion of abstract C^*-algebra which does not refer to a Hilbert space: an involutive

latter requirement means that if $\{A_n\}$ is a sequence of operators from $\mathcal{C}$ which converges in norm to some $A \in \mathcal{B}(\mathcal{H})$, then $A \in \mathcal{C}$. A C^*-algebra is thus a Banach space with respect to this topology, since $\mathcal{B}(\mathcal{H})$ is such a Banach space. We shall always assume that the C^*-algebras $\mathcal{C}$ we employ are unital, *i.e.* $I \in \mathcal{C}$, where I is the identity transformation on $\mathcal{H}$. Note that $\mathcal{B}(\mathcal{H})$ is itself a C^*-algebra. Note further that if X is a compact Hausdorff space, then the algebra $C(X)$ of all continuous complex-valued functions on X supplied with the norm

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

is an abelian C^*-algebra. It is noteworthy that every abelian C^*-algebra is isomorphic to $C(X)$ for some (up to homeomorphism unique) compact Hausdorff space X.

Since every C^*-algebra $\mathcal{C}$ is a Banach space, its topological dual $\mathcal{C}^*$, consisting of continuous linear maps from $\mathcal{C}$ into the complex numbers $\mathbb{C}$, is also a Banach space. A *state* ϕ on a C^*-algebra $\mathcal{C}$ is such a map $\phi \in \mathcal{C}^*$ which is also positive, *i.e.* $\phi(A^*A) \geq 0$ for all $A \in \mathcal{C}$, and normalized, *i.e.* $\phi(I) = 1$. Given a state ϕ on $\mathcal{C}$, one can construct a Hilbert space $\mathcal{H}_\phi$, a distinguished unit vector $\Omega_\phi \in \mathcal{H}_\phi$ and a C^*-algebra homomorphism $\pi_\phi \colon \mathcal{C} \to \mathcal{B}(\mathcal{H}_\phi)$, so that $\pi_\phi(\mathcal{C})$ is a C^*-algebra acting on the Hilbert space $\mathcal{H}_\phi$, the set of vectors $\pi_\phi(\mathcal{C})\Omega_\phi = \{\pi_\phi(A)\Omega : A \in \mathcal{C}\}$ is dense in $\mathcal{H}_\phi$ and

$$\phi(A) = \langle \Omega_\phi, \pi_\phi(A)\Omega_\phi \rangle \,, \; A \in \mathcal{C} \,.$$

The triple $(\mathcal{H}_\phi, \Omega_\phi, \pi_\phi)$ is uniquely determined up to unitary equivalence by these properties, and π_ϕ is called the GNS representation of $\mathcal{C}$ determined by ϕ.

A *von Neumann algebra* $\mathcal{M}$ is a C^*-algebra which is also closed in the strong operator topology. The latter requirement means that if $\{A_n\}$ is a sequence of operators from $\mathcal{M}$ such that there exists an $A \in \mathcal{B}(\mathcal{H})$ so that for all $\Phi \in \mathcal{H}$ $A_n\Phi$ converges to $A\Phi$, then $A \in \mathcal{M}$. In particular, $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra. The operator norm topology is strictly stronger than the strong operator topology when $\mathcal{H}$ is infinite dimensional. Hence, every von Neumann algebra is a C^*-algebra, but the converse is false. When $\mathcal{H}$ is finite dimensional, these two topologies coincide and there is no distinction between C^*- and von Neumann algebras.

A remarkable fact is that a C^*-algebra $\mathcal{M}$ is a von Neumann algebra if and only if there exists a Banach space $\mathcal{B}$ such that $\mathcal{M}$ is (isomorphic to) the Banach dual of $\mathcal{B}$. If $\mathcal{B}$ exists, then it is unique and is called the predual $\mathcal{M}_*$ of $\mathcal{M}$. From Banach space theory, there is a canonical isometric embedding of $\mathcal{M}_*$ into $\mathcal{M}^*$ (using $\phi(A) = A(\phi)$, for all $\phi \in \mathcal{M}_*, A \in \mathcal{M}$). A *normal state* ϕ on a von Neumann algebra $\mathcal{M}$ is a state which lies in (the embedded image of) $\mathcal{M}_*$. Normal states are characterized by an additional continuity property: $\phi(\sup_\alpha A_\alpha) = \sup_\alpha \phi(A_\alpha)$, for any uniformly bounded increasing net $\{A_\alpha\}$ of positive elements of $\mathcal{M}$. This continuity property is equivalent to the following property:

$$\phi(\sum_{n\in\mathbb{N}} P_n) = \sum_{n\in\mathbb{N}} \phi(P_n) \,, \tag{1}$$

for any countable family $\{P_n\}_{n\in\mathbb{N}}$ of mutually orthogonal projections in $\mathcal{M}$. One can therefore define normal states on $\mathcal{M}$ to be states satisfying (1).

Every normal state on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is given by $\phi(A) = \mathrm{Tr}(\rho A)$, $A \in \mathcal{B}(\mathcal{H})$, for some unique *density matrix* ρ acting on $\mathcal{H}$, *i.e.* $0 \leq \rho = \rho^* \in \mathcal{B}(\mathcal{H})$ such

Banach algebra $\mathcal{A}$ which has the property $\|AA^*\| = \|A\|\,\|A^*\|$ for every $A \in \mathcal{A}$ is called an abstract C^*-algebra. However, it is known that every abstract C^*-algebra is isomorphic to a concrete C^*-algebra, so there is no loss of generality to restrict our attention here to the concrete case and to drop the qualifying adjective henceforth.

that $\mathrm{Tr}(\rho) = 1$. In other words, the predual of $\mathcal{B}(\mathcal{H})$ is (isometrically isomorphic to) the Banach space $\mathcal{T}(\mathcal{H})$ of all trace-class operators on $\mathcal{H}$ with the trace norm. A special case of such normal states is constituted by the vector states: if $\Phi \in \mathcal{H}$ is a unit vector and $P_\Phi \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto the one dimensional subspace of $\mathcal{H}$ spanned by Φ, the corresponding vector state is given by

$$\phi(A) = \langle \Phi, A\Phi \rangle = \mathrm{Tr}(P_\Phi A)\,,\ A \in \mathcal{B}(\mathcal{H})\,.$$

It is a fairly straightforward application of the Hahn–Banach Theorem to see that if $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, every state on $\mathcal{M}$ is the restriction to $\mathcal{M}$ of a state on $\mathcal{B}(\mathcal{H})$. But it is much less obvious, though equally true, that every normal state on $\mathcal{M}$ is the restriction of a normal state on $\mathcal{B}(\mathcal{H})$. Hence, for any normal state $\phi \in \mathcal{M}^*$ there exists a (no longer necessarily unique) density matrix $\rho \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = \mathrm{Tr}(\rho A)$, for all $A \in \mathcal{M}$.

A state ϕ on a C^*-algebra $\mathcal{C}$ is *faithful* if $0 \leq A \in \mathcal{C}$ and $\phi(A) = 0$ entail $A = 0$, and ϕ is *tracial* if $\phi(AB) = \phi(BA)$, for all $A, B \in \mathcal{C}$. If $\mathcal{H}$ has dimension n, then $\widehat{\mathrm{Tr}}(A) = \frac{1}{n}\mathrm{Tr}(A)$, $A \in \mathcal{B}(\mathcal{H})$, is called the *normalized trace* on $\mathcal{B}(\mathcal{H})$ and is a faithful normal tracial state. But if $\mathcal{H}$ is infinite dimensional, then there exists no faithful normal tracial state on $\mathcal{B}(\mathcal{H})$.

Let $(X, \mathcal{S}, \mu)$ be a finite measure space, where X is a set, $\mathcal{S}$ is a σ-algebra of subsets of X and μ is a finite σ-additive measure on $\mathcal{S}$. Let $L^\infty(X, \mathcal{S}, \mu)$ denote the Banach space of all essentially bounded complex-valued functions on X supplied with the (essential) supremum norm, and let $L^1(X, \mathcal{S}, \mu)$ be the Banach space of μ-integrable complex-valued functions on X. Then the dual of $L^1(X, \mathcal{S}, \mu)$ is $L^\infty(X, \mathcal{S}, \mu)$. For our purposes, the latter is viewed as an algebra of multiplication operators acting on the Hilbert space $L^2(X, \mathcal{S}, \mu)$ of all square integrable (with respect to μ) complex-valued functions on X. That is to say

$$(fg)(x) = f(x)\,g(x)\,,\ x \in X\,,\ f \in L^\infty(X, \mathcal{S}, \mu)\,,\ g \in L^2(X, \mathcal{S}, \mu)$$

defines for each $f \in L^\infty(X, \mathcal{S}, \mu)$ a linear mapping $M_f \in \mathcal{B}(L^2(X, \mathcal{S}, \mu))$ by $M_f(g) = fg$, $g \in L^2(X, \mathcal{S}, \mu)$. Note that $\|M_f\| = \|f\|_\infty$, so that $L^\infty(X, \mathcal{S}, \mu) \ni f \mapsto M_f \in \mathcal{B}(L^2(X, \mathcal{S}, \mu))$ is an isometry. Hence, $L^\infty(X, \mathcal{S}, \mu)$ is a von Neumann algebra with predual $L^1(X, \mathcal{S}, \mu)$. Conversely, every abelian von Neumann algebra is isomorphic to $L^\infty(X, \mathcal{S}, \mu)$ for some (up to isomorphism unique) localizable measure space (*i.e.* a direct sum of finite measure spaces). Moreover, since von Neumann algebras are C^*-algebras, an abelian von Neumann algebra $\mathcal{M}$ is also C^*-isomorphic to $C(Y)$, for some compact Hausdorff space Y. The normal states on $\mathcal{M}$ are determined precisely by the Radon probability measures on Y.

If S is any subset of $\mathcal{B}(\mathcal{H})$, then its commutant S' is the set of bounded operators which commute with every element in S, *i.e.*

$$S' \equiv \{B \in \mathcal{B}(\mathcal{H}) : AB = BA,\ \text{for all } A \in S\}\,.$$

The operation of taking the commutant can be iterated, $S'' \equiv (S')'$, and it is clear that $S \subset S''$. Von Neumann's *double commutant theorem* asserts if S is a subalgebra containing I and closed under taking adjoints, then S'' is the closure of S in the strong operator topology. The double commutant theorem implies that a subset $S \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $S = S''$. This then is a purely algebraic characterization of von Neumann algebras. It also follows that S' is a von Neumann algebra. A von Neumann algebra $\mathcal{M}$ is called a *factor* if the only elements in $\mathcal{M}$ which commute with every other element in $\mathcal{M}$ are the constant multiples of the identity, *i.e.* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$.

Hence, $\mathcal{B}(\mathcal{H})$ (viewed as acting on $\mathcal{H}$) is a factor (since $\mathcal{B}(\mathcal{H})' = \mathbb{C}I$), and the only abelian factor is $\mathbb{C}I$. Note that a von Neumann algebra $\mathcal{M}$ is abelian if and only if $\mathcal{M} \subset \mathcal{M}'$. $\mathcal{M}$ is said to be *maximally abelian* (in $\mathcal{B}(\mathcal{H})$) if $\mathcal{M} = \mathcal{M}'$. In this case, the only abelian von Neuman algebra (in $\mathcal{B}(\mathcal{H})$) containing $\mathcal{M}$ is $\mathcal{M}$ itself. Note that $L^\infty(X, \mathcal{S}, \mu)$ is maximally abelian when acting on $L^2(X, \mathcal{S}, \mu)$.

An immediate corollary of the double commutant theorem is that the set of projections $\mathcal{P}(\mathcal{M})$ in a von Neumann algebra $\mathcal{M}$ is a complete (orthomodular) lattice and that it determines $\mathcal{M}$ completely in the sense $\mathcal{M} = \mathcal{P}(\mathcal{M})''$. This fact suggests that by investigating the lattice $\mathcal{P}(\mathcal{M})$ one acquires insight into the structure of the algebra itself. Indeed, Murray and von Neumann used this lattice structure to begin a classification of von Neumann algebras. The key concept in the classification is the equivalence of projections: two projections A and B in $\mathcal{M}$ are called *equivalent* ($A \sim B$) with respect to the algebra $\mathcal{M}$ if there is an operator (partial isometry) $W \in \mathcal{M}$ which maps the range of $I - A$ onto $\{0\}$ and is an isometry between the ranges of A and B, *i.e.* $W^*W = A$ and $WW^* = B$. The relation $\sim$ is an equivalence relation on $\mathcal{P}(\mathcal{M})$. Let $\mathcal{P}(\mathcal{M})_\sim$ be the resultant set of equivalence classes. With the help of $\sim$ one can introduce a partial ordering $\preceq$ on $\mathcal{P}(\mathcal{M})$: $A \preceq B$ if there exists a $B' \in \mathcal{P}(\mathcal{M})$ whose range is contained in that of B and which is equivalent to A, *i.e.* $A \sim B' \leq B$. A projection A is called *finite* if it does not contain any projection which is equivalent to A, *i.e.* if $B \leq A$ and $B \sim A$ imply $A = B$. A projection is *infinite* if it is not finite.

From the point of view of Murray and von Neumann's classification of von Neumann algebras, the important fact concerning $\preceq$ is the *Comparison Theorem*: for any two $A, B \in \mathcal{P}(\mathcal{M})$ there exists a projection $Z \in \mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M})'$ such that

$$ZAZ \preceq ZBZ \quad \text{and} \quad (I - Z)B(I - Z) \preceq (I - Z)A(I - Z)\,.$$

It follows that if $\mathcal{M}$ is a factor, then $\mathcal{P}(\mathcal{M})_\sim$ is totally ordered with respect to $\preceq$, *i.e.* either $A \preceq B$, or $B \preceq A$ holds for any $A, B \in \mathcal{P}(\mathcal{M})$. Two factors cannot be isomorphic as von Neumann algebras if the corresponding $\mathcal{P}(\mathcal{M})_\sim$ are not isomorphic as ordered spaces.

Proposition 2.1 *[46] If $\mathcal{M}$ is a factor, then there exists a map $d : \mathcal{P}(\mathcal{M}) \to [0, \infty]$, which is unique up to multiplication by a constant and has the following properties:*

(i) $d(A) = 0$ *if and only if* $A = 0$.

(ii) *If* $A \perp B$, *then* $d(A + B) = d(A) + d(B)$.

(iii) $d(A) \leq d(B)$ *if and only if* $A \preceq B$.

(iv) $d(A) < \infty$ *if and only if A is a finite projection.*

(v) $d(A) = d(B)$ *if and only if* $A \sim B$.

(vi) $d(A) + d(B) = d(A \wedge B) + d(A \vee B)$.

The map d is called the *dimension function* on $\mathcal{P}(\mathcal{M})$. This proposition implies that the order type of $\mathcal{P}(\mathcal{M})_\sim$ can be read off of the order type of the range of the function d. Murray and von Neumann determined the possible ranges of d in [46]. The result is shown in the table below (by choosing suitable normalization of the function d).

range of d	type of factor $\mathcal{M}$	
$\{0, 1, 2, \dots n\}$	I_n	discrete, finite
$\{0, 1, 2, \dots \infty\}$	I_∞	discrete, infinite
$[0, 1]$	II_1	continuous, finite
$\{x \quad \vert 0 \leq x \leq \infty\}$	II_∞	continuous, infinite
$\{0, \infty\}$	III	purely infinite

This thus results in a classification of von Neumann algebras into type I_n, I_∞, II_1, II_∞ and III. Any von Neumann algebra can be decomposed into a direct sum of algebras of these types. And every factor is exactly one of these types. As tensor products play an important role in quantum theory, it is useful to know that if $\mathcal{M}$ is type I_n and $\mathcal{N}$ is type I_m, then their tensor product[4] $\mathcal{M} \otimes \mathcal{N}$ is type I_{nm}. If $\mathcal{M}$ and $\mathcal{N}$ have no direct summand of type III and one of them is type II (*i.e.* has only direct summands of type II_1 or type II_∞), then $\mathcal{M} \otimes \mathcal{N}$ is type II. And if $\mathcal{M}$ is type III, then so is $\mathcal{M} \otimes \mathcal{N}$, for any von Neumann algebra $\mathcal{N}$. Note further that $\mathcal{M}$ is type I (resp. II, III) if and only if $\mathcal{M}'$ is type I (resp. II, III).

The algebra $\mathcal{B}(\mathcal{H})$ is of type I_n if the dimension of $\mathcal{H}$ is n and is of type I_∞ if $\mathcal{H}$ is infinite dimensional. Moreover, a von Neumann algebra $\mathcal{M}$ is of type I if and only if it is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}$, for some Hilbert space $\mathcal{H}$, where $\mathcal{A}$ is some abelian von Neumann algebra. Hence, all abelian von Neumann algebras are of type I. But there are other types of von Neumann algebras, and these other types have properties *radically* different from the properties of $\mathcal{B}(\mathcal{H})$. For instance:

1. If $\mathcal{M}$ is finite, *i.e.* it has only direct summands of type I_n or II_1, then its projection lattice $\mathcal{P}(\mathcal{M})$ is modular; whereas if $\mathcal{M}$ is infinite, $\mathcal{P}(\mathcal{M})$ is orthomodular but not modular.

2. $\mathcal{M}$ is of type I (*i.e.* it has only direct summands of type I_n or I_∞) if and only if it has nonzero minimal projections. (These are atoms in the projection lattice $\mathcal{P}(\mathcal{M})$.)

3. There exists a faithful normal tracial state on a factor $\mathcal{M}$ if and only if $\mathcal{M}$ is finite.

4. If $\mathcal{M}$ is of type III, then all of its nonzero projections are infinite. This implies that for any projection P in a type III algebra there exist countably infinitely many mutually orthogonal projections $P_i \in \mathcal{M}$ such that $P = \vee P_i$.

After the breakthrough of the Tomita–Takesaki modular theory at the end of the 1960's [75], it became possible for A. Connes [12] to further refine the classification of type III algebras into an uncountably infinite family of type III_λ algebras, $\lambda \in [0, 1]$. Particularly the type III_1 case is of physical interest, as will be seen in Sections 6 and 7.

The distinction between types of von Neumann algebras will be seen to have consequences for probability theory, but first we must explain where the probability theory is to be found in this structure. To this end, it will be useful to review Kolmogorov's formulation of classical probability theory from the vantage point of operator algebra theory.

[4]the von Neumann algebra on $\mathcal{H} \otimes \mathcal{H}$ generated by operators of the form $M \otimes N$, $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

3 Classical Probability Theory

A classical *probability space* is a triplet $(X, \mathcal{S}, p)$, where X is a set, $\mathcal{S}$ is a σ-algebra of subsets of X, and $p\colon \mathcal{S} \to [0,1]$ is a σ-additive measure, *i.e.* for every countable collection $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ of mutually disjoint measurable sets, one has

$$p(\bigcup_{n\in\mathbb{N}} S_n) = \sum_{n\in\mathbb{N}} p(S_n)\,. \tag{2}$$

The elements $S \in \mathcal{S}$ are interpreted as possible events and $p(S)$ as the probability that event S takes place. The probabilities for a suitable subclass of events S are the primary data, and the measure p on a suitably generated σ-algebra $\mathcal{S}$ is generally a derived quantity.[5]

Another crucial concept needed in physical applications of probability theory is the concept of random variable, which is used to represent the observable physical quantities in concrete applications. A map $f\colon X \to \mathbb{R}$ is a (real-valued Borel measurable) *random variable* if $f^{-1}(B) \in \mathcal{S}$, for all $B \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$. A distinguished subclass of such random variables is constituted by the essentially bounded measurable functions $f \in L^\infty_{\mathbb{R}}(X, \mathcal{S}, p)$. It is often convenient to view $L^\infty_{\mathbb{R}}(X, \mathcal{S}, p)$ as a subset of the space of complex-valued essentially bounded measurable functions $L^\infty(X, \mathcal{S}, p)$. Within this subset is the subclass $\mathcal{P}(\mathcal{S}) = \{\chi_S : S \in \mathcal{S}\}$ of characteristic functions:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}\,,$$

for which $p(S) = \int_X \chi_S(x)\, dp(x)$. Viewed as multiplication operators on $L^2(X, \mathcal{S}, p)$, each characteristic function χ_S is an orthogonal projection, and the linear span of $\mathcal{P}(\mathcal{S})$ is dense in the von Neumann algebra $L^\infty(X, \mathcal{S}, p)$, *i.e.* $\mathcal{P}(\mathcal{S})$ generates $L^\infty(X, \mathcal{S}, p)$. Indeed, $\mathcal{P}(\mathcal{S})$ coincides with the set of all projections in the von Neumann algebra $L^\infty(X, \mathcal{S}, p)$.

Therefore, classical probability theory yields a distinguished Hilbert space $L^2(X, \mathcal{S}, p)$ on which acts a distinguished abelian von Neumann algebra $L^\infty(X, \mathcal{S}, p)$ generated by the set $\mathcal{P}(\mathcal{S})$ of its projections, each of which has significance in the given probabilistic model. The probability measure p determines uniquely a state ϕ on $L^\infty(X, \mathcal{S}, p)$ by

$$\phi(f) = \int_X f(x)\, dp(x)\,,$$

for all $f \in L^\infty(X, \mathcal{S}, p)$. And because the measure is σ-additive, this state is normal on $L^\infty(X, \mathcal{S}, p)$ — cf. (1) and (2), since $p(S) = \phi(\chi_S)$, for all $S \in \mathcal{S}$, and $\chi_{S_1} \cdot \chi_{S_2} = 0$ if and only if $S_1 \cap S_2 = \emptyset$, modulo sets of p-measure zero. The probabilistically fundamental data, $p(S)$ with $S \in \mathcal{S}$, are reproduced by the expectations, $\phi(P)$ with $P \in \mathcal{P}(L^\infty(X, \mathcal{S}, p)) = \mathcal{P}(\mathcal{S})$, of the projections from the von Neumann algebra in the state ϕ.

Of course, there are further, derived structures in classical probability theory, and some of these are discussed in Section 7.

4 Noncommutative Probability Theory

We have seen in the preceding section that classical probability theory yields an abelian von Neumann algebra with a specified normal state. On the other hand, we have also

[5]This may be seen in standard books on measure theory such as [26].

seen in Section 2 that an abelian von Neumann algebra with a specified normal state yields a measure space with a probability measure on it, which is precisely the point of departure in Kolmogorov's formulation of probability theory. Although these observations are remarkable, why do operator algebraists repeat the phrase "von Neumann algebra theory is noncommutative probability theory"[6] like a mantra? We cannot hope to explain here the full scope of noncommutative probability theory[7]. Instead, we must content ourselves with highlighting some of those aspects which are of particular importance to quantum theory.

From the beginning of quantum theory, orthogonal projections have played a central role, whether it be logical, operational or mathematical. They are used in the description of what are called "yes–no experiments" — is the spin of the electron pointed in this spatial direction? — is the particle to be found in this subset of space? — is the atom in its ground state? Moreover, through von Neumann's spectral theorem, general observables could be constructed out of these particularly elementary observables: even for an unbounded self-adjoint operator A on $\mathcal{H}$, there exists a measure ν on the spectrum $\sigma(A)$ of A taking values in $\mathcal{P}(\mathcal{B}(\mathcal{H}))$ such that

$$A = \int_{\sigma(A)} \lambda \, d\nu(\lambda) \,, \tag{3}$$

where the convergence is in the strong operator topology on the domain of A. Later, it was proven that if $\mathcal{M}$ is any von Neumann algebra and $A = A^* \in \mathcal{M}$, then there exists a measure ν on the spectrum $\sigma(A)$ of A taking values in $\mathcal{P}(\mathcal{M})$ such that equation (3) holds, thus establishing another sense in which $\mathcal{M}$ is generated by $\mathcal{P}(\mathcal{M})$.

Indeed, one has a Borel functional calculus in arbitrary von Neumann algebras.

Proposition 4.1 *[35, Theorem 5.2.8] Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{F}$ be the *-algebra of bounded complex-valued Borel measurable functions on $\mathbb{C}$. Let $A \in \mathcal{M}$ be self-adjoint*[8] *and $f \in \mathcal{F}$. Then*

$$f(A) \equiv \int_{\sigma(A)} f(\lambda) \, d\nu(\lambda) \,,$$

*is an element of $\mathcal{M}$, and the map $f \mapsto f(A)$ is a normal *-homomorphism from $\mathcal{F}$ into $\mathcal{M}$. The image of $\mathcal{F}$ under this map is the abelian subalgebra $\{A\}''$ of $\mathcal{M}$. If f vanishes identically on $\sigma(A)$, then $f(A) = 0$. One has $\overline{f}(A) = f(A)^*$ and $(f \circ g)(A) = f(g(A))$, for any $f, g \in \mathcal{F}$. Moreover, the mapping $S \mapsto \chi_S(A) \in \mathcal{P}(\mathcal{M})$ yields a projection-valued σ-additive measure on the Borel subsets of $\mathbb{C}$ such that A's spectral resolution $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is given by $E_\lambda = I - \chi_{(\lambda,\infty)}(A)$.*[9]

Computations one is already familiar with in $\mathcal{B}(\mathcal{H})$ can therefore be performed in arbitrary von Neumann algebras. And self-adjoint elements $A \in \mathcal{M}$ yield natural σ-algebra homomorphisms $A : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{M})$ in analogy to random variables.

[6] or noncommutative measure theory

[7] which already subsumes noncommutative generalizations of many basic and advanced concepts of classical probability theory, from conditional expectations and central limit theorems to the theory of stochastic processes and stochastic integration, as well as notions from measure theory such as Radon-Nikodym derivatives, Lebesgue decomposition and L^p spaces — cf. [54, 44, 65, 14, 33, 29] for further reading

[8] It actually suffices that A be normal.

[9] Recall that ν can be recovered from $\{E_\lambda\}$.

The projections in $\mathcal{M}$ have also been interpreted as yes–no propositions in a propositional calculus with a view towards establishing a quantum logic or finding another foundation for quantum theory. This idea first appeared in the seminal paper by G.D. Birkhoff and von Neumann [4], where it was assumed that the type II_1 algebras play a privileged role in quantum theory (see [59] for an analysis of this early concept of quantum logic). Subsequent developments have shown, however, that the type II_1 assumption is too restrictive — cf. [42, 34, 58]. From this point of view, a normal state ϕ on $\mathcal{M}$ provides an interpretation (in the sense of logic) of the quantum propositional calculus, an interpretation which satisfies (1).

For whichever reason one accepts the basic nature of the projections in a von Neumann algebra $\mathcal{M}$, any normal state ϕ on $\mathcal{M}$ provides a map

$$\mathcal{P}(\mathcal{M}) \ni P \mapsto \phi(P) \in [0,1]\,,$$

which is σ-additive in the sense of (1). From the discussion in Section 3, it can now be seen that (1) is the noncommutative generalization of (2). Thus, every normal state on $\mathcal{M}$ determines a σ-additive probability measure[10] on the lattice $\mathcal{P}(\mathcal{M})$. For mathematical, operational and logical reasons, the converse of this relation became pressing: given a map $\mu : \mathcal{P}(\mathcal{M}) \to [0,1]$ satisfying (1),[11] does there exist a normal state ϕ on $\mathcal{M}$ such that $\phi(P) = \mu(P)$ for all $P \in \mathcal{P}(\mathcal{M})$? Gleason [23] showed that if $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and the dimension of $\mathcal{H}$ is strictly greater than 2, then this converse is indeed true. And in a lengthy effort, to which there were many contributors (see [43, 29] for an overview of this development, as well as proofs), it was shown that Gleason's Theorem could be generalized to (nearly) arbitrary von Neumann algebras.

Proposition 4.2 *Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type I_2. Then every map $\mu : \mathcal{P}(\mathcal{M}) \to [0,1]$ satisfying (1) extends uniquely to a normal state on $\mathcal{M}$. Moreover, every finitely additive[12] map $\mu : \mathcal{P}(\mathcal{M}) \to [0,1]$ extends uniquely to a state on $\mathcal{M}$.*

This theorem makes clear that any probability theory based upon suitable lattices of projections in Hilbert spaces is subsumed in the framework of normal states on von Neumann algebras.

In noncommutative probability theory, a probability space is a triple $(\mathcal{M}, \mathcal{P}(\mathcal{M}), \phi)$ consisting of a von Neumann algebra, its lattice of orthogonal projections and a normal state on the algebra. As is now clear, the classical starting point is regained precisely when $\mathcal{M}$ is abelian. Examples of derived notions in probability theory with generalization in the noncommutative theory — independence and conditional expectations — are discussed in Section 7.

There are many significant differences between (truly) noncommutative probability theory and classical probability theory. A state ϕ on a von Neumann algebra $\mathcal{M}$ is said to be *dispersion free* if $\phi(A^2) - \phi(A)^2 = 0$, for all $A = A^* \in \mathcal{M}$. Of course, if $\mathcal{M}$ is abelian, then it admits many dispersion free states (pure states, which are multiplicative on abelian C^*-algebras, *i.e.* $\phi(AB) = \phi(A)\phi(B)$, for all $A, B \in \mathcal{M}$). But a nonabelian factor admits no dispersion free states [45, Cor. 2]. That $\mathcal{S}$ is a Boolean algebra and

[10] note: $\phi(I) = 1$

[11] so, necessarily, $\mu(I) = 1$

[12] *i.e.* (1) holds for finite families of mutually orthogonal projections

$\mathcal{P}(\mathcal{M})$ is not when $\mathcal{M}$ is nonabelian is another crucial difference. The consequences of these two differences alone have generated a voluminous literature. This is not the place to discuss these matters in further detail. However, some other important differences *are* discussed in Section 7.

5 Quantum Mechanics: Type I Noncommutative Probability Theory

In light of what has been presented above, one now sees that the basic components of the noncommutative probability theory are inherent in nonrelativistic quantum mechanics. The unit vectors in and the density matrices on $\mathcal{H}$, which are used to model the preparation of the quantum system, induce normal states on $\mathcal{B}(\mathcal{H})$. The observables of the system are modelled by self-adjoint elements of $\mathcal{B}(\mathcal{H})$. And the expectation of the observable A of a system prepared in the state ϕ is given by $\phi(A)$. Of course, quantum theory supplements this basic framework with further structures.

In nonrelativistic quantum mechanics, the operator algebras which arise in modelling algebras of observables are exclusively type I algebras. Typically, they are either $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, or they are abelian subalgebras of $\mathcal{B}(\mathcal{H})$ generated by a commuting family of observables (see below) or perhaps by a single observable (cf. Prop. 4.1). Characteristic of nonrelativistic quantum mechanics is the restriction to systems involving only finitely many degrees of freedom. The Hilbert space $\mathcal{H}$ may be finite or infinite dimensional, depending on which observables are being employed. For example, the use of position and momentum observables entails that $\mathcal{H}$ be infinite dimensional (since there is no representation of the canonical commutation relations in a finite dimensional Hilbert space), while finite dimensional Hilbert spaces suffice when considering only observables associated with spin. In the former case, an irreducible representation of the canonical commutation relations is normally used (cf. [73]), in which case the von Neumann algebra generated by the spectral projections of the position and momentum operators is $\mathcal{B}(\mathcal{H})$. In the latter case, the components of the spin generate the full matrix algebra $\mathcal{B}(\mathcal{H})$.

But abelian algebras are also frequently employed in quantum mechanics. Two observables A, B are said to be compatible (or commensurable) if they commute: $[A, B] = AB - BA = 0$. Any family $\{A_\alpha\}$ of mutually commuting (bounded) self-adjoint operators acting on $\mathcal{H}$ generates an abelian von Neumann algebra $\{A_\alpha\}''$, which is thus isomorphic to a suitable algebra of bounded Borel functions. It is then possible to construct a self-adjoint $A \in \mathcal{B}(\mathcal{H})$ and bounded Borel functions f_α such that $A_\alpha = f_\alpha(A)$, for all α. Hence, when dealing only with compatible observables, classical probability suffices. Note that the complete commuting families of observables, which play an important role in some parts of quantum theory, are precisely those families $\{A_\alpha\}$ for which $\{A_\alpha\}''$ is maximally abelian (viewed as a subalgebra of $\mathcal{B}(\mathcal{H})$).

6 The Necessity of Non-Type–I Probability Spaces in Physics

The different types of noncommutative probability spaces described in Section 2 are not mere mathematical curiosities — they are indispensable in physical applications, because certain quantum systems *cannot* be described using only type I algebras. Such quantum systems typically have infinitely many degrees of freedom and arise in quantum statistical mechanics and in relativistic quantum field theory. We shall briefly indicate how this occurs.

6.1 Quantum Statistical Mechanics

Although the physical systems described by quantum statistical mechanics actually have only finitely many degrees of freedom, the number of degrees of freedom is so large that physicists have found it to be more convenient to work with the idealization known as the thermodynamic limit (or infinite volume limit) of such systems. In this limit, the number of degrees of freedom is indeed infinite. The simplest nontrivial examples of such systems include the so-called lattice gases, which are mathematically precise models of discrete quantum statistical mechanical systems in the thermodynamic limit.

6.1.1 The quasilocal structure of lattice gases

The simplest example of a lattice gas is the one dimensional lattice gas. A quantum system (typically representing an atom or molecule) is located at each point i on a one dimensional lattice $\mathcal{Z}$ (taken to be the additive group of integers) infinite in both directions:

$$\begin{array}{ccccccc} & \mathcal{B}(\mathcal{H}_{i-2}) & \mathcal{B}(\mathcal{H}_{i-1}) & \mathcal{B}(\mathcal{H}_{i}) & \mathcal{B}(\mathcal{H}_{i+1}) & \mathcal{B}(\mathcal{H}_{i+2}) & \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \dots & i-2 & i-1 & i & i+1 & i+2 & \dots \end{array}$$

where the $\mathcal{H}_i$ are copies of a fixed finite dimensional Hilbert space $\mathcal{K}$. The self-adjoint elements of $\mathcal{B}(\mathcal{H}_i)$ represent the observables of the system at site i. For a *finite* subset $\Lambda \subset \mathcal{Z}$, the tensor product[13]

$$\mathcal{A}(\Lambda) = \otimes_{i\in\Lambda}\mathcal{B}(\mathcal{H}_i) \tag{4}$$

represents a quantum system localized in region Λ. If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}(\Lambda_1)$ is isomorphic to and is identified with the algebra $\otimes_{i\in\Lambda_2}\mathcal{B}_i \subset \mathcal{A}(\Lambda_2)$, where $\mathcal{B}_i = \mathbb{C}I_i$, for $i \notin \Lambda_1$, and $\mathcal{B}_i = \mathcal{B}(\mathcal{H}_i)$, for $i \in \Lambda_1$. Then, the collection $\{\mathcal{A}(\Lambda) : \Lambda \subset \mathcal{Z} \text{ finite}\}$ is a directed set under inclusion, and the union $\mathcal{A}_0 = \cup_\Lambda \mathcal{A}(\Lambda)$ is an incomplete normed algebra with involution, which contains all observables which can be localized in some finite region. The minimal norm completion $\mathcal{A}$ of $\mathcal{A}_0$ is then a C^*-algebra [61]. The self-adjoint elements of this C^*-algebra represent the observable quantities of the infinitely extended quantum system.

The one dimensional lattice gas also possesses a natural spatial symmetry — translations along the lattice. Since all Hilbert spaces $\mathcal{H}_i$ are identical copies of the same Hilbert space $\mathcal{K}$, there is a unitary operator $U_i(j)$ which takes the Hilbert space $\mathcal{H}_i$ from site i to site $i+j$:

$$U_i(j)\colon \mathcal{H}_i \to \mathcal{H}_{i+j} \tag{5}$$

These unitaries may be chosen to possess the group property

$$U_i(j+l) = U_{i+l}(j)U_i(l)\,, \tag{6}$$

[13]As the C^*-algebras $\mathcal{B}(\mathcal{H}_i)$ are von Neumann algebras, the tensor product here may be taken to be the (unique) spatial tensor product, as is done throughout this paper.

and they transform observables localized at individual sites:

$$\mathcal{B}(\mathcal{H}_i) \ni A \mapsto U_i(j)AU_i(j)^{-1} \in \mathcal{B}(\mathcal{H}_{i+j})\,. \tag{7}$$

Taking the products $U_\Lambda(j) = \otimes_{i\in\Lambda} U_i(j)$, one obtains unitaries $U_\Lambda(j)$ which take $\mathcal{H}_\Lambda$ onto $\mathcal{H}_{\Lambda+j}$[14] and which act upon the local observables:

$$\mathcal{B}(\mathcal{H}_\Lambda) \ni A \mapsto U_\Lambda(j)AU_\Lambda(j)^{-1} \in \mathcal{B}(\mathcal{H}_{\Lambda+j})\,. \tag{8}$$

Defining

$$\alpha_j(A) = U_\Lambda(j)AU_\Lambda(j)^{-1}\,,\ A \in \mathcal{A}(\Lambda)\,,\ \text{finite}\,\Lambda \subset \mathcal{Z}\,,$$

yields an automorphism on $\mathcal{A}_0$ which extends to an automorphism α_j of $\mathcal{A}$. Thus one obtains a representation $\mathcal{Z} \ni i \mapsto \alpha_i \in \mathrm{Aut}(\mathcal{A})$ of the translation symmetry group of the infinite lattice gas by automorphisms on $\mathcal{A}$.

Also a time evolution of the infinite lattice gas can be constructed from the time evolutions of its local systems. If H_Λ is the generator of the unitary group U_t, $t \in \mathbb{R}$, which gives the time evolution of the quantum system localized in Λ (so that H_Λ carries the interpretation of the total energy operator for the subsystem in Λ), then (under suitable assumptions [5]) the adjoint action of U_t can be extended in a similar manner to an automorphism α_t of $\mathcal{A}$. The dynamical behavior of the infinite lattice gas is then encoded in the C^**-dynamical system* $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$, where $t \mapsto \alpha_t$ is a continuous representation of $(\mathbb{R}, +)$ by automorphisms of $\mathcal{A}$.[15]

This mathematical model of the one dimensional lattice gas can thus be compactly summarized as follows: There exists a net of local algebras of observables

$$\mathcal{Z} \supset \Lambda \mapsto \mathcal{A}(\Lambda) = \mathcal{B}(\mathcal{H}_\Lambda)\,,$$

indexed by the finite subsets Λ of the lattice $\mathcal{Z}$, with these properties:

1. **Isotony**: If $\Lambda_1 \subseteq \Lambda_2$, then $\mathcal{A}(\Lambda_1)$ is a subalgebra of $\mathcal{A}(\Lambda_2)$. This then enables one to define the *quasilocal algebra*:

$$\mathcal{A} = \overline{\cup_{\Lambda\subset\mathcal{Z},\text{finite}}\, \mathcal{A}(\Lambda)}^{norm}\,.$$

2. **Local commutativity**: If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $[A_1, A_2] = 0$, for all $A_1 \in \mathcal{A}(\Lambda_1)$ and $A_2 \in \mathcal{A}(\Lambda_2)$.

3. **Covariance**: There is a representation of the symmetry group $(\mathcal{Z}, +)$ of the space $\mathcal{Z}$ by automorphisms α on $\mathcal{A}$ such that

$$\alpha_a\mathcal{A}(\Lambda) = \mathcal{A}(\Lambda + a)\,,$$

for all $a \in \mathcal{Z}$ and finite $\Lambda \subset \mathcal{Z}$.

4. **Time evolution**: The dynamical behavior of the system is given by a continuous representation of $(\mathbb{R}, +)$ by automorphisms of $\mathcal{A}$, yielding a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$.

[14] $\Lambda + j = \{i + j : i \in \Lambda\}$

[15] Note that only if the local Hamiltonian operators, H_Λ, are bounded will the resultant representation be continuous in the sense required by the term C^*-dynamical system: the map $\mathbb{R} \ni t \mapsto \alpha_t(A) \in \mathcal{A}$ must be continuous for all $A \in \mathcal{A}$. For unbounded energy operators, the representation is continuous in a weaker sense. The results stated below can be extended in a suitable manner to the latter case, as well, but these technicalities will be suppressed here.

6.1.2 States on the Quasilocal Algebra of the Lattice Gas

A fundamental task of investigating quantum statistical systems is the determination of their equilibrium states. For a limited class of quantum systems modelled by a C^*-dynamical system with a total energy operator H, the usual Gibbs equilibrium state at inverse temperature β,

$$\phi(A) = \frac{\mathrm{Tr}(e^{-\beta H}A)}{\mathrm{Tr}(e^{-\beta H})}\,,\ A \in \mathcal{A}\,,$$

is entirely satisfactory. However, in most applications to systems involving infinitely many degrees of freedom, the operator $e^{-\beta H}$ is not of trace class, and the Gibbs state is not defined. Through the efforts of a number of leading mathematical physicists, it has been well established that the appropriate notion of equilibrium state of a C^*-dynamical system is that of a KMS state — cf. [5]:

Definition 6.1 *A state ϕ on the C^*-algebra $\mathcal{A}$ of the C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ is an (α,β)-KMS state at inverse temperature $\beta \in \mathbb{R}$, if for every $A, B \in \mathcal{A}$ there exists a complex-valued function $f_{A,B}$ which is analytic in the strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \beta\}$ and continuous on the closure of this strip such that*

$$f_{A,B}(t + i0) = \phi(\alpha_t(A)B)$$

and

$$f_{A,B}(t + i\beta) = \phi(B\alpha_t(A))\,,$$

for all $t \in \mathbb{R}$.

Note that Gibbs states are KMS states. Moreover, any KMS state ϕ on a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ is *α-invariant*, *i.e.* $\phi \circ \alpha_t = \phi$, for all $t \in \mathbb{R}$. Another characteristic feature of KMS states, which is of particular relevance to our considerations here, is spelled out in the next proposition.

Proposition 6.1 *[56][5, Corollary 5.3.36] If ϕ is a KMS state of the C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ with $\beta > 0$ (and e.g. ϕ is weakly clustering), then $\pi_\phi(\mathcal{A})''$ is a type III factor.*

Hence, one cannot describe phenomena such as equilibrium (thus also phase transition) in quantum physics without going beyond the type I von Neumann algebras.

A special case of (α,β)-KMS states is the infinite temperature ($\beta = 0$) KMS state. Such states are called chaotic, and an $(\alpha, 0)$-KMS state is an α-invariant tracial state (and *vice versa*). In the case of the one dimensional lattice gas, tracial states can be constructed explicitly. If $\widehat{\mathrm{Tr}}_i$ is the normalized trace on $\mathcal{B}(\mathcal{H}_i)$ and τ_Λ is defined on $\otimes_{i\in\Lambda}\mathcal{B}(\mathcal{H}_i)$ by

$$\tau_\Lambda(A_{i_1} \otimes A_{i_2} \ldots \otimes A_{i_k}) = \widehat{\mathrm{Tr}}_{i_1}(A_{i_1})\,\widehat{\mathrm{Tr}}_{i_2}(A_{i_2}) \cdots \widehat{\mathrm{Tr}}_{i_k}(A_{i_k})\,, \tag{9}$$

with $\Lambda = \{i_1, i_2, \ldots, i_k\}$, then τ_0 defined by $\tau_0(A) = \tau_\Lambda(A)$, for $A \in \mathcal{A}(\Lambda)$ and finite $\Lambda \subset \mathcal{Z}$, yields a norm-densely defined linear functional on $\mathcal{A}$. The extension τ of τ_0 from $\mathcal{A}_0$ to $\mathcal{A}$ is a tracial state. As $\mathcal{A}$ is a simple algebra, it may be identified with its GNS representation associated with the state τ. Let $\mathcal{H}$ denote the corresponding Hilbert space.

It follows that the algebra $\mathcal{A}$ representing the bounded observables of the infinitely extended lattice gas cannot be $\mathcal{B}(\mathcal{H})$, because there is no tracial state on $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ is infinite dimensional, since $\mathcal{Z}$ is infinite). The state τ induces a tracial state on the von Neumann algebra $\mathcal{A}''$; so $\mathcal{A}''$ cannot be type I, either. In fact, the von Neumann algebra $\mathcal{A}''$ is a type II_1 von Neumann algebra (see Section XIV.1 in [77]). This is typical of GNS representations associated with chaotic states.

Another type of physically relevant state is the ground state, which formally is an (α, ∞)-KMS state. If a state ϕ on a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ is α-invariant, then in the corresponding GNS space there exists a self-adoint operator H_ϕ such that

$$e^{itH_\phi}\pi_\phi(A)e^{-itH_\phi} = \pi_\phi(\alpha_t(A))\,,\; A \in \mathcal{A}\,,\; t \in \mathbb{R}\,. \tag{10}$$

A state ϕ on a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$ is a *ground state*, if ϕ is α-invariant and $H_\phi \geq 0$. For ground states one has the following result.

Proposition 6.2 *[5, Theorem 5.3.37] If ϕ is an (extremal) ground state of a C^*-dynamical system $(\mathcal{A}, \{\alpha_t\}_{t\in\mathbb{R}})$, then $\pi_\phi(\mathcal{A})''$ is a type I factor.*

In sum, the behavior of general quantum systems modelled by C^*-dynamical systems $(\mathcal{A}, \alpha)$ cannot be described using solely type I algebras, which typically only arise (for the quasilocal algebra $\mathcal{A}$) in the GNS representation corresponding to a ground state. Note, however, that in the simple lattice gas models discussed above, the local algebras of observables, $\mathcal{A}(\Lambda)$, *are* type I algebras. This is not typical of quantum statistical models, as shall be seen below.

6.2 Brief Return to General Quantum Statistical Mechanics

If one wishes to model quantum gases on suitable lattices in two or three dimensional space, then one can repeat the steps described above and arrive at the same sort of quasilocal structure. And more general models on such lattices can be constructed which still manifest the structure properties 1-4 emphasized at the end of Section 6.1.1. One also can drop the assumption that the physical systems are restricted to lattices and consider an assignment of suitable algebras $\mathcal{A}(\Lambda)$ of observables to bounded regions Λ in the three dimensional Euclidean space $\mathbb{R}^3$ – the resulting structure describes the thermodynamical limit of quantum systems in three dimensional space. The Euclidean group replaces the lattice symmetry group in such cases. This also leads to structures entirely analogous to those manifested by the lattice gas models — cf. [20, 5, 64]. The theorems discussed in Section 6.1.2 are therefore applicable to such models, as well. Note that there are models of this kind in which even the local observable algebras $\mathcal{A}(\Lambda)$ are of type III or type II (see *e.g.* [1, 2]).

Local quantum physics is the name given to the branch of mathematical physics which investigates the mathematical models of quantum systems wherein taking account of the localization of observables leads to structures with properties analogous to those isolated at the end of Section 6.1.1. This approach has also proven to be fruitful in the study of relativistic quantum fields on general space–times.

6.3 Local Relativistic Quantum Field Theory

Here we restrict our attention to quantum fields on four dimensional Minkowski space M, so the label Λ (replaced by convention with $\mathcal{O}$ in quantum field theory) indicates the spatio*temporal* localization of the algebra $\mathcal{A}(\mathcal{O})$ of observables in M. The algebra $\mathcal{A}(\mathcal{O})$

is interpreted as the algebra generated by all the observables measurable in the spacetime region $\mathcal{O}$. The net of local algebras of observables

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \tag{11}$$

indexed by the open bounded spacetime regions $\mathcal{O}$ of the Minkowski space–time is assumed to satisfy a number of physically motivated conditions (cf. [25, 3]), which closely resemble the structural properties of the net describing the one dimensional lattice gas and which are natural in light of the mentioned interpretation.

1. **Isotony**: If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1)$ is a subalgebra of $\mathcal{A}(\mathcal{O}_2)$. This enables the definition of the quasilocal algebra as the inductive limit of the net, *i.e.* the smallest C^*-algebra $\mathcal{A}$ containing all the local algebras $\mathcal{A}(\mathcal{O})$.
2. **Local commutativity (Einstein causality)**: If $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then $[A_1, A_2] = 0$, for all $A_1 \in \mathcal{A}(\mathcal{O}_1)$ and $A_2 \in \mathcal{A}(\mathcal{O}_2)$.
3. **Relativistic covariance**: There exists a continuous representation α of the identity-connected component $\mathcal{P}_+^\uparrow$ of the Poincaré group by automorphisms on $\mathcal{A}$ such that $\alpha_\lambda(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\lambda\mathcal{O})$, for all $\mathcal{O}$ and $\lambda \in \mathcal{P}_+^\uparrow$.

   Though there are many kinds of physically relevant representations, one of the most completely studied is the *vacuum representation*.
4. **Irreducible vacuum representation**: For each $\mathcal{O}$, $\mathcal{A}(\mathcal{O})$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ in which $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$, in which there is a distinguished unit vector Ω, and on which there is a strongly continuous unitary representation $U(\mathcal{P}_+^\uparrow)$ satisfying $U(\lambda)\Omega = \Omega$, for all $\lambda \in \mathcal{P}_+^\uparrow$, and

   $$\alpha_\lambda(A) = U(\lambda)AU(\lambda)^{-1} \quad , \quad \text{for all} \quad A \in \mathcal{A} \quad ,$$

   as well as the spectrum condition: the spectrum of the self-adjoint generators of the strongly continuous unitary representation $U(\mathbb{R}^4)$ of the translation subgroup of $\mathcal{P}_+^\uparrow$ (which has the physical interpretation of the global energy–momentum spectrum of the theory) must lie in the closed forward light cone.

   A common assumption made when dealing with vacuum representations is given next.
5. **Weak additivity**: For each $\mathcal{O}$,

   $$\{U(x)\mathcal{A}(\mathcal{O})U^{-1}(x) : x \in \mathbb{R}^4\}'' = \mathcal{A}'' .$$

These assumptions entail the Reeh–Schlieder Theorem (cf. [25, 3]), which permits the use of Tomita–Takesaki modular theory in quantum field theory [25, 3].

Proposition 6.3 *For every $\mathcal{O}$, the vector Ω is cyclic and separating for $\mathcal{A}(\mathcal{O})$, i.e. the set of vectors $\mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathcal{H}$, and $A \in \mathcal{A}(\mathcal{O})$ and $A\Omega = 0$ entail $A = 0$.*

Thus, no local observable can annihilate Ω, *i.e.* $\langle \Omega, P\Omega \rangle \neq 0$, for all projections $P \in \mathcal{A}(\mathcal{O})$.

Many concrete models satisfying these conditions have been constructed, though none of them is an interacting quantum field in four spacetime dimensions. Of course, no such model has *ever* been constructed,[16] so one can hardly attribute the source of the

[16] In two and three spacetime dimensions, models of interacting quantum fields satisfying conditions 1–5 have been constructed — see *e.g.* [24, 62, 66].

problem to the set of "axioms" above. On the contrary, we are convinced that the above conditions are operationally natural and express the minimal conditions to be satisfied by any local relativistic quantum field theory in the vacuum on Minkowski space. So we view consequences of these assumptions to be generic properties in the stated context.

It will be convenient to concentrate attention on two special classes of spacetime regions in M. A double cone is a (nonempty) intersection of some open forward light cone with an open backward light cone. Such regions are bounded, and the set $\mathcal{D}$ of all double cones is left invariant by the natural action of $\mathcal{P}_+^\uparrow$ upon it. An important class of unbounded regions is specified as follows. After choosing a coordinatization of M, one defines the right wedge to be the set $W_R = \{x = (x_0, x_1, x_2, x_3) \in M : x_1 > |x_0|\}$ and the set of wedges to be $\mathcal{W} = \{\lambda W_R : \lambda \in \mathcal{P}_+^\uparrow\}$. The set of wedges is thus independent of the choice of coordinatization; only which wedge is called the right wedge is coordinate-dependent. Because $\mathcal{D}$ is a base and $\mathcal{W}$ is (nearly [78]) a subbase for the topology on M, one can construct a net indexed by all open subsets of M in an natural manner from a net indexed by $\mathcal{D}$, or even $\mathcal{W}$, alone.

Tomita–Takesaki modular theory is used to prove the following results.

Proposition 6.4 *(i) [18, 41] Under the above conditions, $\mathcal{A}(W)$ is a type III_1 factor, for every wedge $W \in \mathcal{W}$.*

(ii) [22, 9] Under the above conditions and with a mild additional assumption (existence of a scaling limit), also the double cone algebras, $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{D}$, are type III_1 (though not necessarily factors).

The fact that local algebras in relativistic quantum field theory are typically type III algebras is neither restricted to vacuum representations nor even to Minkowski space theories. Moreover, although in a vacuum representation the quasilocal von Neumann algebra $\mathcal{A}''$ is type I, as seen in Section 6.1.2 $\mathcal{A}''$ will not be type I in GNS representations corresponding to temperature equilibrium states. Indeed, there even exists a class of physically relevant representations in which *any* properly infinite hyperfinite factor can be realized as $\mathcal{A}''$ [16, 6]. It is therefore evident that not only are type I algebras insufficient for the description of models in quantum statistical mechanics, they are even less suitable for relativistic quantum field theory. That is not to say that they are irrelevant in quantum field theory, for they play important auxiliary roles, which cannot be discussed here (but see [7, 8, 15, 17, 25, 66, 71, 85]).

7 Some Differences of Note

Some of the notable differences between the structure and predictions of classical (abelian), nonabelian type I, and non-type–I probability theories will be adumbrated in this section.

7.1 Entanglement and Bell's Inequalities

Consider a composite system consisting of two subsystems whose observables are given by the self-adjoint elements of the von Neumann algebras $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, respectively. If these two subsystems are in a certain sense independent, then the algebras mutually commute, *i.e.* $\mathcal{M} \subset \mathcal{N}'$. The algebra of observables of the composite system would be $\mathcal{M} \bigvee \mathcal{N} = (\mathcal{M} \cup \mathcal{N})''$. A state ϕ on $\mathcal{M} \bigvee \mathcal{N}$ is a *product state* if

$$\phi(MN) = \phi(M)\,\phi(N)\,,\; M \in \mathcal{M}\,,\; N \in \mathcal{N}\,. \tag{12}$$

In classical probability, where $\mathcal{M} = L^\infty(X_1, \mathcal{S}_1, p_1)$ and $\mathcal{N} = L^\infty(X_2, \mathcal{S}_2, p_2)$, if a state ϕ induced by a probability measure p on $X_1 \times X_2$ satisfied (12), one would say that the random variables of the two subsystems are mutually independent (with respect to p). Recall that any (normal) state on $\mathcal{M} \bigvee \mathcal{N}$ can be nonuniquely extended to a (normal) state on $\mathcal{B}(\mathcal{H})$, so it is often convenient to view ϕ in (12) as a state on $\mathcal{B}(\mathcal{H})$.

In many applications of quantum theory (though certainly not all[17]), the algebra of observers of the composite system can be taken to be the tensor product $\mathcal{M} \otimes \mathcal{N} \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \simeq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, where $\mathcal{M}$ is identified with $\mathcal{M} \otimes I$ and $\mathcal{N}$ with $I \otimes \mathcal{N}$. A normal state ϕ on $\mathcal{M} \otimes \mathcal{N}$ is *separable*[18] if it is in the norm closure of the convex hull of the normal product states on $\mathcal{M} \otimes \mathcal{N}$, *i.e.* it is a mixture of normal product states. Otherwise, ϕ is said to be *entangled*. From the point of view of what is now called quantum information theory, the primary difference between classical and noncommutative probability theory is the existence of entangled states in the latter case. In fact, one has the following result:

Proposition 7.1 *[57] Every state on $\mathcal{M} \otimes \mathcal{N}$ is separable if and only if either $\mathcal{M}$ or $\mathcal{N}$ is abelian.*

Hence, if both systems are quantum, *i.e.* both algebras are noncommutative, then there exist entangled states on the composite system. Although not understood at that time in this manner, some of the founders of quantum theory realized as early as 1935 [19, 63] that such entangled states were the source of "paradoxical" behavior of quantum theory, as viewed from the vantage point of classical physics. Today, entangled states are viewed as a resource to be employed to carry out tasks which cannot be done classically, *i.e.* only with separable states — cf. [36, 83].

A primary task of any probability theory is to describe and estimate the strength of observed correlations. In this connection, a profound glimpse into the differences between classical, nonabelian type I and type III probability theories is provided by Bell's inequalities. We shall only discuss those aspects of Bell's inequalities which are of immediate relevance to our purposes and refer the reader to [68, 83] for background and further references.

The following definition was made in [67].

Definition 7.1 *Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be von Neumann algebras such that $\mathcal{M} \subset \mathcal{N}'$. The maximal Bell correlation of the pair $(\mathcal{M}, \mathcal{N})$ in the state $\phi \in \mathcal{B}(\mathcal{H})^*$ is*

$$\beta(\phi, \mathcal{M}, \mathcal{N}) \equiv \sup \frac{1}{2}\, \phi(M_1(N_1 + N_2) + M_2(N_1 - N_2)) \,,$$

where the supremum is taken over all self-adjoint $M_i \in \mathcal{M}, N_j \in \mathcal{N}$ with norm less than or equal to 1.

As explained in *e.g.* [68], the CHSH version of Bell's inequalities can be formulated in algebraic quantum theory as

$$\beta(\phi, \mathcal{M}, \mathcal{N}) \leq 1 \,. \tag{13}$$

This inequality places a bound on the strength of a certain family of correlations of observables in $\mathcal{M}$, $\mathcal{N}$ in the state ϕ. This bound is satisfied in *every* state, if at least one of the systems is classical.

[17] See Section 7.3 for a brief discussion of when this assumption is justifiable.

[18] also termed decomposable, classically correlated, or unentangled by various authors

Proposition 7.2 *[68] Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras. If either $\mathcal{M}$ or $\mathcal{N}$ is abelian, then $\beta(\phi, \mathcal{M}, \mathcal{N}) = 1$ for all states $\phi \in \mathcal{B}(\mathcal{H})^*$.*

If, on the other hand, both algebras are nonabelian, then there always exists a state in which the inequality (13) is violated. Note that it is known [11, 68] that $1 \leq \beta(\phi, \mathcal{M}, \mathcal{N}) \leq \sqrt{2}$, for all states ϕ on $\mathcal{B}(\mathcal{H})$. For this reason, one says that if $\beta(\phi, \mathcal{M}, \mathcal{N}) = \sqrt{2}$, then the pair $(\mathcal{M}, \mathcal{N})$ maximally violates Bell's inequalities in the state ϕ.

Proposition 7.3 *[39] If $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ are nonabelian, mutually commuting von Neumann algebras satisfying the Schlieder property, i.e. $0 = MN$ for $M \in \mathcal{M}$ and $N \in \mathcal{N}$ entails either $M = 0$ or $N = 0$, then there exists a normal state $\phi \in \mathcal{B}(\mathcal{H})^*$ such that $\beta(\phi, \mathcal{M}, \mathcal{N}) = \sqrt{2}$.*

Hence, when $\mathcal{M}$ and $\mathcal{N}$ are nonabelian, there even exists a normal state in which Bell's inequalities are maximally violated. However, the genericity of the states in which (13) is violated, as well as the degree to which it is violated, depends on finer structure properties of the algebras. We shall only mention enough results of this type to clearly indicate that there is an important difference between type I and non-type–I behavior. For further discussion and references concerning the violation of Bell's inequalities in algebraic quantum theory, see [72, 58, 27].

Proposition 7.4 *Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras.*

(i) [71] If the algebras $\mathcal{M}, \mathcal{N}$ are type I factors (or are contained in mutually commuting type I factors), then there exist infinitely many normal states $\phi \in \mathcal{B}(\mathcal{H})^*$ such that $\beta(\phi, \mathcal{M}, \mathcal{N}) = 1$.

(ii) [69] If $\mathcal{N} = \mathcal{M}'$, then $\beta(\phi, \mathcal{M}, \mathcal{N}) = \sqrt{2}$ for every normal state $\phi \in \mathcal{B}(\mathcal{H})^*$ if and only if $\mathcal{M} \simeq \mathcal{M} \otimes \mathcal{R}_1$, where $\mathcal{R}_1$ is the (up to isomorphism) unique hyperfinite type II_1 factor.

Note that $\mathcal{M} \otimes \mathcal{R}_1$ is never a type I algebra. Maximal violation of Bell's inequalities in every normal state can only occur in the non-type–I case. In [70] it is shown under quite general physical assumptions in relativistic quantum field theory that there exist local observable algebras $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{A}(\mathcal{O}_2)$ such that $\beta(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)) = \sqrt{2}$, for every normal state ϕ, and that the circumstances described in Prop. 7.4 (ii) actually obtain. We remark further that under another set of general physical assumptions [7], the local algebras $\mathcal{A}(\mathcal{O})$ appearing in relativistic quantum field theory are isomorphic to $\mathcal{R} \otimes \mathcal{Z}$, where $\mathcal{Z}$ is an abelian von Neumann algebra and $\mathcal{R}$ is the (up to isomorphism) unique hyperfinite type III_1 factor. Since $\mathcal{R} \simeq \mathcal{R} \otimes \mathcal{R}_1$, the relevance of Prop. 7.4 (ii) is reinforced.

To return briefly to the starting point of this section, states which violate Bell's inequalities are necessarily entangled. The converse is not true (cf. [83] for a discussion and references). In the now quite extensive quantum information theory literature, there are various attempts to quantify the degree of entanglement of a given state (cf. [36]), but all agree that maximal violation of inequality (13) entails maximal entanglement. Only in the non-type–I case is it possible for every normal state to be maximally entangled.

7.2 Independence

Another standard topic in probability theory is independence, which has already been briefly touched upon above. This is an extensively studied subject, and we shall only mention a few salient points. For further discussion and references, see [71, 58, 29]. As is typical of noncommutative generalizations of abelian concepts, there are many notions which are distinct for nonabelian algebras but equivalent in the abelian special case. We shall discuss only three here.

Definition 7.2 *Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be von Neumann algebras. The pair $(\mathcal{M}, \mathcal{N})$ is C^*-independent if for every state ϕ_1 on $\mathcal{M}$ and every state ϕ_2 on $\mathcal{N}$ there exists a state ϕ on $\mathcal{B}(\mathcal{H})$ such that $\phi(M) = \phi_1(M)$, for every $M \in \mathcal{M}$ and $\phi(N) = \phi_2(N)$, for every $N \in \mathcal{N}$.*

So $(\mathcal{M}, \mathcal{N})$ is C^*-independent if every pair of states on the subsystems represented by the algebras $\mathcal{M}, \mathcal{N}$ has a joint extension to the composite system — in operational terms, this means that no preparation of one subsystem excludes any preparation of the other. If every pair of *normal* states on the subsystems has a joint extension to a *normal* state on the composite system, then $(\mathcal{M}, \mathcal{N})$ is said to be W^*-independent. W^*-independence implies C^*-independence [71, 21]. If $\mathcal{M} \subset \mathcal{N}'$, then these notions are equivalent [21], but W^*-independence is strictly stronger when the algebras do not mutually commute [28]. If $\mathcal{M} \subset \mathcal{N}'$ and $(\mathcal{M}, \mathcal{N})$ is C^*-independent, then the (not necessarily normal) joint extension can be chosen to be a product state [60]:

$$\phi(MN) = \phi_1(M)\, \phi_2(N)\, , \; M \in \mathcal{M}\, , \; N \in \mathcal{N}\, .$$

We reemphasize that the independence expressed by a product state is the most directly analogous to the notion of independence familiar from classical probability theory.

Another of the distinctions between type I and non-type–I probability theory is enunciated in the following theorem.

Proposition 7.5 *(1) If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a type I factor and $\mathcal{N} = \mathcal{M}'$, then $(\mathcal{M}, \mathcal{N})$ is W^*-independent, and given any normal state ϕ_1 on $\mathcal{M}$ and ϕ_2 on $\mathcal{N}$, the normal joint extension ϕ can be chosen to be a product state.*

(2) If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a type III (or II) factor and $\mathcal{N} = \mathcal{M}'$, then $(\mathcal{M}, \mathcal{N})$ is W^*-independent. But given any normal states ϕ_1 on $\mathcal{M}$ and ϕ_2 on $\mathcal{N}$, the normal joint extension cannot be chosen to be a product state. ϕ_1 and ϕ_2 have a joint extension to a product state ϕ, but ϕ cannot be normal. Hence, ϕ is only finitely additive on $\mathcal{P}(\mathcal{H})$ and not σ-additive.

Proof. (1) is well known, and (2) is discussed in [71, 21] for the type III case, but for the reader's benefit a proof will be sketched here. Since $\mathcal{M}$ and $\mathcal{N}$ are commuting factors, they satisfy the Schlieder property and thus are C^*-independent [60]. This entails that the states ϕ_1 and ϕ_2 have a joint extension to a product state on $\mathcal{M} \bigvee \mathcal{N}$ [60] and that $(\mathcal{M}, \mathcal{N})$ is W^*-independent [21]. If there did exist a normal product state across $(\mathcal{M}, \mathcal{N})$, then [74] $\mathcal{M} \bigvee \mathcal{N}$ is isomorphic to $\mathcal{M} \otimes \mathcal{N}$, so that there exists a type I factor $\mathcal{L}$ such that [15, Th. 1 and Cor. 1] $\mathcal{M} \subset \mathcal{L} \subset \mathcal{N}' = \mathcal{M}$, *i.e.* $\mathcal{M} = \mathcal{L}$, a contradiction unless $\mathcal{M}$ is type I. If $\mathcal{M}$ is type I, then $\mathcal{M} \bigvee \mathcal{N} \simeq \mathcal{M} \otimes \mathcal{N}$ (see *e.g.* [15]), and the state $\phi_1 \times \phi_2$ on $\mathcal{M} \otimes \mathcal{N}$ (defined by $(\phi_1 \times \phi_2)(\sum_i M_i N_i) = \sum_i \phi_1(M_i)\, \phi_2(N_i)$) precomposed with the isomorphism implementing $\mathcal{M} \bigvee \mathcal{N} \to \mathcal{M} \otimes \mathcal{N}$ is a normal joint extension of ϕ_1 and ϕ_2 on $\mathcal{M} \bigvee \mathcal{N}$. $\square$

Hence, there are many normal product states on $(\mathcal{M}, \mathcal{M}')$ in the type I case, but there are none when $\mathcal{M}$ is type II or III. It is perhaps worthwhile in this connection to mention that in a physical setting with a specified dynamics implemented unitarily as in (10), long experience has indicated that the following folklore is correct: it takes an infinite amount of energy to create a nonnormal state from a normal state. Thus, the nonnormal product state in the type III case is not likely to be physically realizable, whereas the normal product state in the type I case is prepared in actual laboratories every day. This supports the physical relevance of the distinction drawn in the previous theorem.

A further notion of statistical independence was proposed in [40, 38].

Definition 7.3 *A pair $(\mathcal{M}, \mathcal{N})$ of von Neumann algebras on $\mathcal{H}$ is strictly local (in the sense of vector states) if for any $P \in \mathcal{P}(\mathcal{M})$ and any unit vector $\Phi \in \mathcal{H}$ there exists a unit vector $\Psi \in P\mathcal{H}$ such that $\langle \Phi, N\Phi \rangle = \langle \Psi, N\Psi \rangle$ for all $N \in \mathcal{N}$.*

As shown in [71], this condition implies a more transparent condition of independence called *strict locality*: for every $P \in \mathcal{P}(\mathcal{M})$ and every normal state ϕ_2 on $\mathcal{N}$, there exists a normal state $\phi \in \mathcal{B}(\mathcal{H})^*$ such that $\phi(P) = 1$ and $\phi(N) = \phi_2(N)$, for all $N \in \mathcal{N}$. Hence, no preparation on the subsystem represented by $\mathcal{N}$ can exclude the truth of any proposition in $\mathcal{M}$ (cf. [71] for further discussion of the relation between these properties). Both of these properties imply C^*-independence [21], and W^*-independence implies strict locality [71]. The following result thus associates the type III structure property with an independence property having physical significance.

Proposition 7.6 *[40] Let $\mathcal{H}$ have dimension greater than 1, $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann factor and $\mathcal{N} = \mathcal{M}'$. The pair $(\mathcal{M}, \mathcal{N})$ is strictly local in the sense of vector states if and only if $\mathcal{M}$ is type III.*

7.3 Conditional Expectations

A concept of central importance in probability theory is that of conditional expectation. We present the classical notion from the point of view of operator algebra theory in order to motivate the general definition in the noncommutative setting. Although the concept extends to certain classes of unbounded random variables and corresponding classes of unbounded operators, we shall only discuss the bounded case here.

Let $(X, \mathcal{S}, p)$ be a probability space and $\mathcal{T}$ be a sub-σ-algebra of $\mathcal{S}$. Then for any $f \in L^\infty(X, \mathcal{S}, p)$ there exists a unique $E_\mathcal{T}(f) \in L^\infty(X, \mathcal{T}, p)$ such that

$$\int \chi_T f \, dp = \int \chi_T E_\mathcal{T}(f) \, dp \, , \; T \in \mathcal{T} \, . \tag{14}$$

$E_\mathcal{T}(f)$ is called the *conditional expectation* of f with respect to $\mathcal{T}$. Its existence is assured by the Radon–Nikodym Theorem. This then yields a linear map $E_\mathcal{T} : L^\infty(X, \mathcal{S}, p) \to L^\infty(X, \mathcal{T}, p)$ with the following properties: $E_\mathcal{T}(f) = f$, for all $f \in L^\infty(X, \mathcal{T}, p)$ and $\|E_\mathcal{T}(f)\| \leq \|f\|$, for all $f \in L^\infty(X, \mathcal{S}, p)$. So $E_\mathcal{T} : L^\infty(X, \mathcal{S}, p) \to L^\infty(X, \mathcal{T}, p)$ is a projection of norm 1. Moreover, $E_\mathcal{T}(\sup_\alpha f_\alpha) = \sup_\alpha E_\mathcal{T}(f_\alpha)$, for any uniformly bounded increasing net $\{f_\alpha\}$ of positive elements of $L^\infty(X, \mathcal{S}, p)$.

The noncommutative generalization can therefore be formulated as follows. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\mathcal{N} \subset \mathcal{M}$ be a subalgebra. A *conditional expectation* (on $\mathcal{M}$ relative to $\mathcal{N}$) is a linear map $E_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ with norm 1 whose restriction to $\mathcal{N}$ is the identity. If, in addition, $E_\mathcal{N}$ satisfies the continuity condition $E_\mathcal{N}(\sup_\alpha A_\alpha) = \sup_\alpha E_\mathcal{N}(A_\alpha)$, for any uniformly bounded increasing net $\{A_\alpha\}$ of positive elements of $\mathcal{M}$,

$E_{\mathcal{N}}$ is a *normal conditional expectation*. Given $M \in \mathcal{M}$, $E_{\mathcal{N}}(M)$ is called the conditional expectation of M with respect to $\mathcal{N}$.

Of course, this definition does not yet reproduce the essential (for probability theory) condition (14), whose generalization requires specifying a normal state ϕ on $\mathcal{M}$:

$$\phi(PM) = \phi(PE_{\mathcal{N}}(M)) \, , \; M \in \mathcal{M} \, , \; P \in \mathcal{P}(\mathcal{N}) \, . \tag{15}$$

However, Tomiyama proved the following result.

Proposition 7.7 *[80] If $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ is a conditional expectation, then for any $0 \leq M \in \mathcal{M}$ one has $E_{\mathcal{N}}(M) \geq 0$. Moreover, $E_{\mathcal{N}}(NM) = NE_{\mathcal{N}}(M)$, for all $N \in \mathcal{N}$, $M \in \mathcal{M}$. Consequently, $E_{\mathcal{N}}(M)^* = E_{\mathcal{N}}(M^*)$ and $E_{\mathcal{N}}(MN) = E_{\mathcal{N}}(M)N$, for all $N \in \mathcal{N}$, $M \in \mathcal{M}$.*

One therefore sees that condition (15) is equivalent to $\phi = \phi \circ E_{\mathcal{N}}$. One says that a conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ is *faithful* if $0 \leq M \in \mathcal{M}$ and $E_{\mathcal{N}}(M) = 0$ entail $M = 0$. In the classical case, the conditional expectations *are* faithful. Thus, given a normal state ϕ on $\mathcal{M}$, a faithful normal conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ such that $\phi \circ E_{\mathcal{N}} = \phi$ is the proper generalization of the classical concept.

As indicated above, in the abelian case, for any subalgebra $\mathcal{N} \subset \mathcal{M}$ and any normal state ϕ on $\mathcal{M}$ there exists a unique faithful normal conditional expectation on $\mathcal{M}$ relative to $\mathcal{N}$ leaving ϕ invariant. The same is not true in general. Once again, we can only discuss certain aspects of the matter.

First, let us appreciate the significance of the normality of the conditional expectation in the general case. A factor $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is said to be *injective* if there exists a (not necessarily normal) conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$. It is certainly not the case that every factor is injective, but type I factors are injective, as are the non-type–I factors which typically arise in quantum statistical mechanics [1, 2] and in quantum field theory [7], since hyperfinite factors are injective. Requiring that a conditional expectation be normal imposes serious constraints on the strucure, as the following result makes clear.

Proposition 7.8 *[81] Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{N}$ be a subalgebra of $\mathcal{M}$.*

(i) Let $\mathcal{M}$ be a factor and $\mathcal{N}$ be type III. If there exists a normal conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, then $\mathcal{M}$ is type III.

(ii) Let $\mathcal{M}$ be semifinite, i.e. have no direct summand of type III. If there exists a normal conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, then $\mathcal{N}$ is semifinite.

(iii) If $\mathcal{M}$ is type I and there exists a normal conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, then $\mathcal{N}$ is type I.

Adding the condition that the normal conditional expectation leaves a distinguished state invariant is even more restrictive. Takesaki has given a characterization of this situation when the state is faithful, but to state the result properly, a few preparations must be made. Given a faithful normal state ϕ on $\mathcal{M}$, there is uniquely associated a one-parameter group of automorphisms, σ_t, $t \in \mathbb{R}$, of $\mathcal{M}$ called the *modular automorphism group* corresponding to ϕ, and $(\mathcal{M}, \{\sigma_t\}_{t \in \mathbb{R}})$ forms a (W^*-)dynamical system for which ϕ is a KMS state at inverse temperature $\beta = 1$ [75, 5, 35, 77].

Proposition 7.9 *[76] Let ϕ be a faithful normal state on the von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra. There exists a faithful normal conditional expectation $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ such that $\phi \circ E_{\mathcal{N}} = \phi$ if and only if $\sigma_t(\mathcal{N}) \subset \mathcal{N}$, for all $t \in \mathbb{R}$.*

Under these circumstances, if $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{M}$ is the relative commutant of $\mathcal{N}$ in $\mathcal{M}$ and $\mathcal{N}$ is a factor, then $\mathcal{N} \bigvee \mathcal{N}^c \simeq \mathcal{N} \otimes \mathcal{N}^c$ and ϕ is a product state on $\mathcal{N} \bigvee \mathcal{N}^c$. Moreover, $E_{\mathcal{N}}$ is unique.

It should now be clear that the existence of such a conditional expectation in truly noncommutative probability theory is more the exception than the rule. Nonetheless, they do exist in physically interesting circumstances (cf. [17, 7, 54, 14]) and are quite useful.

As mentioned in Section 7.1, it is of physical relevance to know under which conditions the algebra of observables of the composite system $\mathcal{M} \bigvee \mathcal{N}$ is isomorphic to $\mathcal{M} \otimes \mathcal{N}$, and the matter has received a lot of attention from mathematical physicists and operator algebraists. It is fitting to give here a characterization in terms of the existence of a normal conditional expectation.[19]

Proposition 7.10 *[74] Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting factors. Then $\mathcal{M} \bigvee \mathcal{N}$ is isomorphic to $\mathcal{M} \otimes \mathcal{N}$ if and only if there exists a normal conditional expectation $E : \mathcal{M} \bigvee \mathcal{N} \to \mathcal{M}$.*

For operationally motivated sufficient conditions entailing $\mathcal{M} \bigvee \mathcal{N} \simeq \mathcal{M} \otimes \mathcal{N}$, see [10].

It may be instructive to see simple examples of such conditional expectations. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ have purely discrete spectrum consisting of simple eigenvalues and P_i, $i \in \mathbb{N}$, denote the projections onto the corresponding one dimensional eigenspaces. In quantum measurement theory, the following map is associated with the so–called *projection postulate* (cf. [50]):

$$T_A(B) = \sum_i P_i B P_i \, , \; B \in \mathcal{B}(\mathcal{H}) \, .$$

Note that $AT_A(B) = T_A(B)A$, for all $B \in \mathcal{B}(\mathcal{H})$, and that for any $B \in \{A\}'$, $T_A(B) = B$ (see Prop. 4.1). Hence, $T_A : \mathcal{B}(\mathcal{H}) \to \{A\}'$ is a normal conditional expectation of a type I algebra onto a type I algebra which preserves states with density matrix P_i.

A situation commonly arising in nonrelativistic quantum theory (and quantum information theory) is a composite system consisting of two subsystems, which is modelled by $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, prepared in a state ϕ determined by the density matrix ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ given by $\rho = \rho_1 \otimes \rho_2$, where ρ_i is a density matrix on $\mathcal{H}_i$, $i = 1, 2$. Let Tr_1 represent the trace on $\mathcal{B}(\mathcal{H}_1)$. Then the map $E_2 : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to I_1 \otimes \mathcal{B}(\mathcal{H}_2)$ determined on special elements of the form $A_1 \otimes A_2$, $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$, by

$$E_2(A_1 \otimes A_2) = (\mathrm{Tr}_1(\rho_1 A_1)\, I_1) \otimes A_2 = \mathrm{Tr}_1(\rho_1 A_1)(I_1 \otimes A_2)$$

and extended to $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ by linearity and continuity, is a faithful normal conditional expectation. Moreover, it leaves the state ϕ invariant, since

$$\begin{aligned} \phi(E_2(A_1 \otimes A_2)) &= \phi(\mathrm{Tr}_1(\rho_1 A_1)(I_1 \otimes A_2)) = \mathrm{Tr}_1(\rho_1 A_1) \cdot \phi(I_1 \otimes A_2) \\ &= \mathrm{Tr}_1(\rho_1 A_1) \cdot \mathrm{Tr}_1(\rho_1 I_1) \cdot \mathrm{Tr}_2(\rho_2 A_2) = \mathrm{Tr}_1(\rho_1 A_1) \cdot \mathrm{Tr}_2(\rho_2 A_2) \\ &= \phi(A_1 \otimes A_2) \, . \end{aligned}$$

This is a map from a type I algebra to a type I algebra.

A related, but less elementary example is provided by the one dimensional lattice gas discussed above. Referring to the tracial state τ on $\mathcal{A}''$ constructed in Section 6.1.2,

[19] For a generalization to the nonfactor case, see [81].

let $\Lambda \subset \mathcal{Z}$ be finite and consider the algebra $\mathcal{A}(\Lambda)$ defined in Section 6.1.1, which is the subalgebra of $\mathcal{A}$ generated by the observables in Λ. Then define the map $E_\Lambda : \mathcal{A}'' \to \mathcal{A}(\Lambda)''$ by considering an arbitrary element $\otimes_{i\in\Lambda_0} A_i \in \mathcal{A}_0$ (for some finite $\Lambda_0 \subset \mathcal{Z}$), setting

$$E_\Lambda(\otimes_{i\in\Lambda_0} A_i) = \otimes_{i\in\Lambda_0} \widetilde{A_i}\,,$$

where $\widetilde{A_i} = \widehat{\mathrm{Tr}}_i(A_i)I_i$, if $i \notin \Lambda$, and $\widetilde{A_i} = A_i$, if $i \in \Lambda \cap \Lambda_0$, and extending by linearity and continuity. E_Λ is then a faithful normal conditional expectation leaving the state τ invariant. This is a map from a type II algebra to a type I algebra.

7.4 Further Comments on Type I versus Type III

In this section, we briefly discuss some further distinctions between the type I and type III cases, which, although perhaps not strictly probabilistic in nature, are of direct physical relevance.

As previously mentioned, every nonzero projection P in a type III algebra $\mathcal{M} \subset \mathcal{H}$ is infinite. This entails that $P\mathcal{H}$ is an infinite dimensional subspace of $\mathcal{H}$. So, in particular, if $A = A^* \in \mathcal{M}$ and P is the projection onto an eigenspace of A, then since $P \in \mathcal{M}$, it follows that every eigenvalue of A is infinitely degenerate. Thus, the many papers in the physics literature which restrict their considerations to observables with simple eigenvalues tacitly exclude all the physical situations in which type III algebras arise.

Typically in relativistic quantum field theory, both the algebra of observables localized in a region $\mathcal{O}$ and that localized in its causal complement $\mathcal{O}'$ are type III, and they are each other's commutant: $\mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}')$. In such a circumstance, given any orthogonal projection $P \in \mathcal{A}(\mathcal{O})$ and any state ϕ on the quasilocal algebra $\mathcal{A}$, it is possible to change ϕ into an eigenstate of P with an operation strictly localized in $\mathcal{O}$, *i.e.* an operation which does not disturb the expectation of any observable localized in $\mathcal{O}'$ [40]. Namely, since P is equivalent to I in $\mathcal{A}(\mathcal{O})$, there exists a partial isometry $W \in \mathcal{A}(\mathcal{O})$ such that $P = WW^*$ and $I = W^*W$. The state $\phi_W(\cdot) = \phi(W^* \cdot W)$ then satisfies

$$\phi_W(P) = \phi(W^*PW) = \phi(W^*WW^*W) = \phi(I) = 1\,,$$

on the one hand, and, for all $A \in \mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$,

$$\phi_W(A) = \phi(W^*AW) = \phi(AW^*W) = \phi(A)\,,$$

on the other. Moreover, since the algebra $\mathcal{A}(\mathcal{O})$ is usually a type III_1 factor, the transitivity of the action of the group of unitaries on the normal state space of such a factor [13] entails that given any two normal states ϕ, ω on $\mathcal{A}(\mathcal{O})$ and an $\epsilon > 0$, there exists a unitary $W \in \mathcal{A}(\mathcal{O})$ such that

$$|\omega(A) - \phi_W(A)| \le \epsilon \|A\|\,,$$

for all $A \in \mathcal{A}(\mathcal{O})$. In other words, every normal state can be prepared locally with arbitrary precision from any other normal state. These facts rely upon properties of type III algebras which do not obtain for type I algebras.

These and other distinctions are sources of errors found in the literature which essentially amount to applying reasoning valid for type I quantum theory to type III quantum theory. A notable example of this is the argument given in [30], which purports to demonstrate that relativistic quantum field theory violates causality. In an immediate retort [8] (see [85] for a possibly more accessible explanation), it was pointed out that the argument employed in [30] rested upon an inadmissible use of type I reasoning.

8 Closing Words

So, what is "quantum probability theory"? Some authors use the term synonymously with noncommutative probability theory. And others use it to mean noncommutative probability theory with (some of) the additional structures which physical considerations add to basic von Neumann algebra theory. We regard this as a matter of personal taste. In either case, we have endeavored to make clear at least two main points: (1) both classical probability theory and quantum theory are special cases of noncommutative probability theory, and (2) there are significant differences between the type I and non-type–I quantum theories. The former point emphasizes the existence of an elegant, unifying framework within which the latter can be studied and better understood. These probability theories form a spectrum with the abelian case located at one extreme, the type III case at the other, and the standard type I quantum theory located squarely between them.

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