Fields Institute Communications
Volume 41, 2004

Amicable Pairs, a Survey

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This paper is amicably dedicated to Hugh Williams on the occasion of his sixtieth birthday.

Abstract. In 1750, Euler [20, 21] published an extensive paper on amicable pairs, by which he added fifty-nine new amicable pairs to the three amicable pairs known thus far. In 1972, Lee and Madachy [45] published a historical survey of amicable pairs, with a list of the 1108 amicable pairs then known. In 1995, Pedersen [48] started to create and maintain an Internet site with lists of all the known amicable pairs. The current (February 2003) number of amicable pairs in these lists exceeds four million.

The purpose of this paper is to update the 1972 paper of Lee and Madachy, in order to document the developments which have led to the explosion of known amicable pairs in the past thirty years. We hope that this may stimulate research in the direction of finding a *proof* that the number of amicable pairs is infinite.

1 Introduction

Definition 1.1 The pair of numbers (m, n), with $m, n \in \mathbb{N}$ and $m < n$, is called *amicable* if each of m and n is the sum of the proper divisors of the other

2000 *Mathematics Subject Classification.* Primary 11A25; Secondary 11Y70.
Key words and phrases. Amicable numbers.
The research of Herman te Riele was carried out under project MAS2.2 "Computational number theory and data security".

in the above amicable pair by these six values, we obtain six other amicable pairs, isotopic to the first one. □

Now and then, we will abbreviate "amicable pair" to: "AP". When we write: "a 100D AP", we mean an amicable pair (m, n) where m (and very often also n) has 100 decimal digits.

3 Theoretical results

3.1 The number of amicable pairs. Let $A(x)$ be the number of amicable pairs (m, n) with $m \leq x$. Despite the fact that the number of currently known amicable pairs exceeds four million, it is not known whether $A(x)$ is unbounded. On the other hand, Kanold [41] showed that the density of the amicable pairs is less than 0.204, i.e., $\lim_{x\to\infty} A(x)/x < 0.204$, and Erdős [18] proved that the density of the amicable pairs is zero. The best result to date is from Pomerance [49] who showed that

$$A(x) \leq x \cdot \exp(-\log^{1/3} x) \tag{3.1}$$

for large x. For $x = 10^{10},\ 10^{11},\ 10^{12},\ 10^{13}$, we have

$$A(x) = 1427,\ 3340,\ 7642,\ 17519,$$

while the right hand side of (3.1) yields 5.8×10^{8}, 5.3×10^{9}, 4.9×10^{10}, and 4.5×10^{11}, respectively. This illustrates how far the best theoretical estimates are still away from the actual amicable pair counts.

Borho proved [6] that if w is the total number of prime factors of an amicable pair (m, n) (taking into account *multiple* prime factors), then $m\cdot n < w^{2^w}$. It follows that for a given positive bound S there are only finitely many amicable pairs (m, n) with less than S prime divisors (in $m \cdot n$). This result was improved by Borho [7] as follows: if we fix the number of *different* prime factors of one member of an amicable pair and the *total* number of *divisors* of the other member, then there are only finitely many amicable pairs satisfying these conditions. If we fix the number of *different* prime factors of *both* members of an amicable pair, then there are only finitely many *relatively prime* amicable pairs which satisfy these conditions.

3.2 Relatively prime amicable pairs. Inspection of the lists of known amicable pairs [48] shows that all known amicable pairs have a common divisor > 1. It is not known whether amicable pairs exist whose members are relatively prime. Lee and Madachy [45, p. 84] report that Hagis determined that there are no relatively prime amicable pairs (m, n) with $m < 10^{60}$. In [37] Hagis proved that the product mn of the members of a relatively prime amicable pair has at least twenty-two different prime factors. Concerning relatively prime amicable pairs of *opposite parity*, i.e., one member is even and the other is odd, Hagis proved that $mn > 10^{121}$ and both m and n exceed 10^{60} [36].

3.3 Amicable pairs of a given form. In all known amicable pairs, one member has at least two and the other has at least three different prime factors.

Concerning the question of the existence of amicable pairs where one member is a pure prime power, Kanold proved that if one member is of the form p^α and the other of the form $q_1^{\beta_1} q_2^{\beta_2} \ldots q_j^{\beta_j}$, where $p, q_1, q_2, \ldots, q_j$ are distinct primes and $\alpha, \beta_1, \beta_2, \ldots, \beta_j$ are positive integers, then both members are odd, α is odd, $\alpha > 1400$, $j > 300$, $n = p^\alpha (> m)$, and $m > 10^{1500}$ [39].

Concerning pairs where one member has precisely two distinct prime factors, Kanold [40] proved that $m = p_1^{\alpha_1} p_2^{\alpha_2}$, $n = q_1^{\beta_1} q_2^{\beta_2}$ cannot be an amicable pair, and in view of the known pair $(2^2 \cdot 5 \cdot 11, 2^2 \cdot 71)$, this result is best possible.

For even-odd (not necessarily relatively prime) amicable pairs it is known that one member is of the form $2^\alpha M^2$ and the other is of the form N^2, both M and N being odd. Kanold [40] proved that if $\alpha > 1$ then $m = 2^\alpha M^2 (< n)$, and that if $\alpha = 1$ then $m = N^2 (< n)$ and N must contain at least five distinct prime factors. For more results on even-odd amicable pairs, we refer to [40, 35].

For even-even amicable pairs, Lee [44] showed that neither member of an even-even amicable pair is divisible by three. Gardner [34] observed that most known even-even amicable pairs have sums divisible by nine. Lee [44] characterized the exceptions to this observation and Pedersen [48] has listed all the (557 currently) known exceptions. Concerning the ratio m/n of an even-even amicable pair (m, n), it is known [45] that $m/n > 1/2$. The smallest known m/n ratio for even-even amicable pairs is 0.6468, for the following irregular AP (11D) of type (5, 3):

$$2^{11} \left\{ \begin{array}{l} 5^3 \cdot 7^2 \cdot 23 \cdot 43 \cdot 263 \\ 191 \cdot 967 \cdot 13337 \end{array} \right. ,$$

found in 1992 by David and Paul Moews [47].

For odd-odd amicable pairs, Bratley and McKay [12] conjectured that both members of an odd-odd AP are divisible by 3. This was disproved by Battiato and Borho in 1988 [3] who gave fifteen odd-odd APs (between 36D and 73D), with members coprime to 6. Many more such APs were published in 1992 by García [26]. The currently smallest known AP (16D) with members coprime to 6 was found by Walker and Einstein in 2001 [48]:

$$\left\{ \begin{array}{l} 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 17 \cdot 23 \cdot 103 \cdot 1319 \\ 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 809 \end{array} \right. .$$

In 1997, Yasutoshi Kohmoto [48] found many odd-odd APs with smallest prime divisor 7, the smallest being 193D. The smallest known m/n ratio for odd-odd amicable pairs is 0.5983, for the following irregular AP (14D) of type (6, 3):

$$3^2 \cdot 13^2 \left\{ \begin{array}{l} 5^2 \cdot 7^2 \cdot 11 \cdot 23 \cdot 29^2 \cdot 233 \\ 53 \cdot 337 \cdot 5682671 \end{array} \right. ,$$

found in 1997 by David Einstein. The pairs with ratio closest to 1/2 are all irregular.

4 Thabit-rules to generate amicable pairs

The three amicable pairs mentioned in the Introduction are the oldest known APs and, according to the classification given in Section 2, they all are of type (2, 1), and regular. This is no accident, since these APs are instances of a "rule" to find regular amicable pairs of type (2, 1), known as the

Rule of Thābit ibn Qurra (9th century) *$2^k pq$ and $2^k r$ form an amicable pair if $p = 3 \cdot 2^{k-1} - 1$, $q = 3 \cdot 2^k - 1$ and $r = 9 \cdot 2^{2k-1} - 1$ are all primes and $k > 1$.*

For $k = 2, 4, 7$ this rule yields the three amicable pairs given in the Introduction, but it yields no other amicable pairs for $k \leq 191600$ [10, $k \leq 20000$], [42, $k \leq 191600$]. Euler generalized this rule to one which finds *all* the amicable pairs of the form $(2^k pq, 2^k r)$:

Euler's Rule *2^kpq and 2^kr form an amicable pair if $p = 2^{k-l}f-1$, $q = 2^kf-1$ and $r = 2^{2k-l}f^2 - 1$ are all primes, with $f = 2^l + 1$ and $k > l \geq 1$.*

For $l = 1$, this is Thabit's Rule. For $l > 1$, two more solutions are known, namely, for $l = 7, k = 8$ (Legendre, Chebyshev [45]), and for $l = 11, k = 40$ (te Riele [50]).

Proof of Euler's Rule By (1.1), k, p, q, and r must satisfy the two equations

$$(p+1)(q+1) = (r+1) \text{ and } (2^{k+1}-1)(p+1)(q+1) = 2^k(pq+r).$$

It follows that

$$r = pq + p + q$$

and

$$\left[p - (2^k - 1)\right]\left[q - (2^k - 1)\right] = 2^{2k}. \tag{4.1}$$

By writing the right-hand-side of (4.1) as AB, where $A = 2^{k-l}$ and $B = 2^{k+l}$ for some integer $l \in [1, k-1]$, all the possible solutions of (4.1) can be written as

$$p = 2^k - 1 + 2^{k-l},\ q = 2^k - 1 + 2^{k+l},$$

and we obtain an amicable pair if the three integers $p = 2^{k-l}(2^l+1) - 1$, $q = 2^k(2^l+1) - 1$, and $r = pq + p + q = 2^{2k-l}(2^l+1)^2 - 1$ are all prime. □

Euler's Rule requires that *three* numbers are prime simultaneously. Walter Borho [5] has studied rules to construct amicable pairs which require *two* numbers to be prime simultaneously. Borho's study was motivated by the question whether the set $\mathfrak{M}(b_1, b_2, p)$ of amicable pairs of the form

$$(m_1, m_2) = (b_1p^kq_1,\ b_2p^kq_2),$$

where b_1 and b_2 are positive integers and p is a prime not dividing b_1b_2, can be *infinite* in the sense that there are infinitely many positive integers k and primes $q_1 = q_1(k)$, $q_2 = q_2(k)$ for which (m_1, m_2) is an amicable pair. Borho found that a *necessary* condition for $\mathfrak{M}(b_1, b_2, p)$ to be infinite is that

$$\frac{p}{p-1} = \frac{b_1}{\sigma(b_1)} + \frac{b_2}{\sigma(b_2)}. \tag{4.2}$$

This led him to the following

Borho's Rule *Let $p, b_1, b_2 \in \mathbb{N}$ be given, where p is a prime not dividing b_1b_2, satisfying (4.2). If for some $k \in \mathbb{N}$ and for $i = 1, 2$,*

$$q_i = \frac{p^k(p-1)(b_1+b_2)}{\sigma(b_i)} - 1, \tag{4.3}$$

is a prime not dividing b_ip, then $(b_1p^kq_1,\ b_2p^kq_2)$ is an amicable pair.

This is an example of what Borho calls a *Thabit-rule* [5]: a statement, for $k = 1, 2, \ldots$, on amicable pairs involving powers p^k of a prime p. Of crucial importance in Borho's Rule is that the numbers q_1 and q_2 in (4.3) are integral.

Example 4.1 The triple $b_1 = 2^2 \cdot 5 \cdot 11$, $b_2 = 2^2$, $p = 127$ indeed satisfies (4.2) and, moreover, the numbers q_1 and q_2 are integral, giving the Thabit-rule:

$(2^2 \cdot 127^k \cdot 5 \cdot 11 \cdot q_1, 2^2 \cdot 127^k \cdot q_2)$ is an amicable pair for each $k \in \mathbb{N}$ for which both $q_1 = 56 \cdot 127^k - 1$ and $q_2 = 56 \cdot 72 \cdot 127^k - 1$ are prime.

For $k = 2$ indeed both q_1 and q_2 are prime, so that this yields the AP

$$2^2 \cdot 127^2 \begin{cases} 5 \cdot 11 \cdot 903223 \\ 65032127 \end{cases} .$$

□

Notice that one of the members of this pair is divisible by 220, the smaller member of Pythagoras's AP. This is no accident: Borho discovered that if we start with an amicable pair of the form (au, as), with $\gcd(a, us) = 1$ and s a prime, then in Borho's Rule we may choose $b_1 = au$ and $b_2 = a$ and we obtain

$$\frac{b_1}{\sigma(b_1)} + \frac{b_2}{\sigma(b_2)} = \frac{u+s+1}{u+s}.$$

Now if $u + s + 1 =: p$ is a prime, then the triple b_1, b_2, p satisfies (4.2), the numbers q_1 and q_2 turn out to be integral, and we have obtained

Borho's Rule, special case *Let (au, as) be an amicable pair with $\gcd(a, us) = 1$ and s a prime, and let $p = u + s + 1$ be a prime not dividing a. If for some $k \in \mathbb{N}$ both $q_1 = p^k(u + 1) - 1$ and $q_2 = p^k(u + 1)(s + 1) - 1$ are primes not dividing a, then $(aup^k q_1, ap^k q_2)$ is an amicable pair.*

Example 4.2 Take the amicable pair $(3^4 \cdot 5 \cdot 11 \cdot 29 \cdot 89, 3^4 \cdot 5 \cdot 11 \cdot 2699)$, so $a = 3^4 \cdot 5 \cdot 11$, $u = 29 \cdot 89$, and $s = 2699$. Now $p = u + s + 1 = 5281$ is a prime not in a, giving the Thabit-rule:

$(3^4 \cdot 5 \cdot 11 \cdot 5281^k \cdot 29 \cdot 89 \cdot q_1,\ 3^4 \cdot 5 \cdot 11 \cdot 5281^k \cdot q_2)$ *is an amicable pair for each $k \in \mathbb{N}$ for which both $q_1 = 2582 \cdot 5281^k - 1$ and $q_2 = 2582 \cdot 2700 \cdot 5281^k - 1$ are prime.*

For $k = 1$ indeed both q_1 and q_2 are prime, so that this yields the amicable pair, found by Lee [5]:

$$3^4 \cdot 5 \cdot 11 \cdot 5281 \begin{cases} 29 \cdot 89 \cdot 13635541 \\ 36815963399 \end{cases} .$$

Te Riele [50] found that $k = 19$ is the next value of k for which this rule gives an AP (being 152D). Borho [10] showed that there are no other values of $k \leq 267$ for which this rule yields APs. □

Currently, there are more than 2000 amicable pairs of the form required by the special case of Borho's Rule. The numbers q_1 and q_2 in this rule grow very quickly with k so that very often at least one of them is composite. Only a few amicable pairs have actually been found in this way [5, 50, 15, 8, 10, 11].

The requirement in Borho's Rule that $\gcd(a, us) = 1$ with s a prime implies that $\gcd(a, u) = 1$. We notice that this requirement is not necessary. For example, the amicable pair $(3^3 \cdot 5 \cdot 7 \cdot 13, 3 \cdot 5 \cdot 7 \cdot 139)$ is of the form (au, as) with $a = 3 \cdot 5 \cdot 7$, $u = 3^2 \cdot 13$ and $s = 139$ prime, but $\gcd(a, us) = 3 \neq 1$. Nevertheless, since $u + s + 1 = 117 + 139 + 1 = 257$ is prime, we have the Thabit-rule:

$(3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 257^k \cdot q_1,\ 3 \cdot 5 \cdot 7 \cdot 257^k \cdot q_2)$ *is an amicable pair for each $k \in \mathbb{N}$ for which both $q_1 = 118 \cdot 257^k - 1$ and $q_2 = 118 \cdot 140 \cdot 257^k - 1$ are prime.*

We conclude that the requirement in Borho's Rule, special case, that $\gcd(a, us) = 1$ (with s a prime) can be relaxed to: *s is a prime not dividing a.* In [10] Borho noticed that in the requirement that (au, as) is an amicable pair, that is, that $\sigma(au) = \sigma(a)(s+1) = a(u+s)$, *$s$ need not be a prime.* This situation is related to *Borho's Rule with breeders*, explained in Section 5.

Wiethaus [57] considered Borho's Rule with $b_1 = aS$, $b_2 = aq$, where $a, S, q \in \mathbb{N}$, S is squarefree, q a prime, and $\gcd(a, S) = \gcd(a, q) = \gcd(S, q) = 1$. The requirements (4.2) and (4.3) with q_i integral led him to the following

Wiethaus's Rule *Let $a, S \in \mathbb{N}$ with S squarefree,* $\gcd(a, S) = 1$, *and*

$$\frac{a}{\sigma(a)} = \frac{\sigma(S)}{S + \sigma(S) - 1}. \tag{4.4}$$

Write $\sigma(S)(S + \sigma(S) - 1) =: D_1 D_2$ with $D_1, D_2 \in \mathbb{N}$. If $p := D_1 + S + \sigma(S)$ and $q := D_2 + \sigma(S) - 1$ are distinct prime numbers with $\gcd(p, aS) = \gcd(q, a) = 1$, *then the following Thabit-rule holds:*

If for some $k \in \mathbb{N}$ the two numbers

$$q_1 := (p+q)p^k - 1 \text{ and } q_2 := (p - S)p^k - 1$$

are prime with $\gcd(q_1, aS) = \gcd(q_2, aq) = 1$, *then $(aSp^k q_1, aqp^k q_2)$ is an amicable pair.*

With help of this rule, Wiethaus [57] was able to generate more than 100,000 Thabit-rules and 10,000 new amicable pairs, including the first AP whose members have more than 1000 decimal digits. About ten years later, Zweers [60] and García used Wiethaus's Rule to establish new AP records. In Table 1 we list the nine consecutive APs of record size, starting with the pair of 1041 decimal digits, found by Wiethaus in 1988, and ending with the pair of 5577 decimal digits, found by García in 1997. We do not list the decimal representations of these large APs, but we give the values of a, S, D_1, and k to be chosen in Wiethaus's Rule by which the decimal representations of these APs can be reconstructed. All these APs are regular of type $(5, 2)$, except the third one (1923D) which is regular of type $(7, 2)$.

The condition in Wiethaus's Rule that S is squarefree is not necessary. Pedersen used this rule to find, in 2001, the largest known irregular amicable pair (651D), namely for

$$\begin{aligned} a &= 3^2 \cdot 5^2 \cdot 31^2 \cdot 331, \ S = 7^3 \cdot 743 \cdot 256651 \cdot 36276899, \\ D_1 &= 3101990448933961728, \ k = 16. \end{aligned}$$

5 Searches of amicable pairs of a special form

Euler [21] was the first to study the subject of amicable numbers in a systematic way. In fact, most amicable pairs known today have been found by methods which have their roots in Euler's study. Here, we only give a short outline of Euler's methods. For a detailed description of Euler's methods, we refer to [45, pp. 79–83].

Euler looked for amicable pairs of the form (aM, aN), where a is a given common factor and M and N are unknowns with $\gcd(a, MN) = 1$. By choosing $a = 2^k, k \in \mathbb{N}, M = pq, N = r$, where p, q, r are distinct primes, we obtain the rules of Thabit and of Euler, described in Section 4. Substitution of $(m, n) = (aM, aN)$ into the defining equations (1.1) yields the equations

$$\sigma(a)\sigma(M) = \sigma(a)\sigma(N) = a(M + N), \tag{5.1}$$

Table 1 Amicable pairs of record size (found with Wiethaus's Rule)

size	discoverer	a	S, D_1	k
1041D	Wiethaus [57]	2^9	$S = 569 \cdot 5039 \cdot 1479911 \cdot 30636732851$ $D_1 = 540109710022026120768000\textbf{0}$	20
1478D	Zweers [58]	2^{10}	$S = 1087 \cdot 17509 \cdot 2580653 \cdot 1220266291199$ $D_1 = 426458207232$	27
1923D	García [27]	105	$S = 11 \cdot 13 \cdot 37 \cdot 3779 \cdot 19994749 \cdot 6553914555541$ $D_1 = 1615208240046043904322043773115200$	28
2725D	Zweers [59]	2^{10}	$S = 1087 \cdot 17509 \cdot 2580653 \cdot 1220266291199$ $D_1 = 18329101258457088$	51
3193D	García [27]	2^9	$S = 569 \cdot 5039 \cdot 1479911 \cdot 30636732851$ $D_1 = 569031058361920000$	67
3383D	García [28]	2^9	$S = 569 \cdot 5023 \cdot 22866511 \cdot 287905188653$ $D_1 = 1164968493698251104480$	65
3766D	Zweers [59]	2^{10}	$S = 1087 \cdot 17509 \cdot 2580653 \cdot 1220266291199$ $D_1 = 335279558994820701878222848 00$	65
4829D	García [29]	2^{11}	$S = 2131 \cdot 51971 \cdot 168605317 \cdot 15378049151$ $D_1 = 1211082626633348448$	89
5577D	García [29]	2^{11}	$S = 2131 \cdot 51971 \cdot 168605317 \cdot 15378049151$ $D_1 = 18501732599907428352$	103

from which

$$\sigma(M) = \sigma(N).$$

Euler considered various combinations of variables in M and N.

Example 5.1 By choosing $a = 3^2 \cdot 7 \cdot 13$, $M = pq$, and $N = r$, Euler found the first amicable pair whose members are *odd*:

$$3^2 \cdot 7 \cdot 13 \begin{cases} 5 \cdot 17 = 69615 \\ 107 = 87633 \end{cases}.$$

□

Euler also considered a different approach, namely by assuming that M and N are *given* numbers, satisfying $\sigma(M) = \sigma(N)$, while a is to be found, satisfying $\sigma(a)/a = (M+N)/\sigma(M)$. If $\gcd(a, M) = \gcd(a, N) = 1$, then (aM, aN) is an amicable pair because

$$\sigma(aM) = \sigma(a)\sigma(M) = a(M+N) = aM + aN$$

and

$$\sigma(aN) = \sigma(a)\sigma(N) = \sigma(a)\sigma(M) = \sigma(aM).$$

Solving the equation $\sigma(x)/x = B/A$ with $\gcd(A, B) = 1$ may be done recursively as follows. If $p^n || A$ for some prime p and positive integer n, then $p^m | x$ for some $m \geq n$. Now fix some m and substitute $p^m y$ for x, and try to solve the resulting equation $\sigma(y)/y = (Bp^m)/(A\sigma(p^m))$ where the fraction of the right hand side has been reduced to its lowest terms. García found 153 new amicable pairs with help of this "unknown common factor method" [23, 45].

Lee [43] considered amicable pairs of the form $(m, n) = (Apq, Br)$ where p, q, r are primes with $\gcd(A, pq) = \gcd(B, r) = 1$. Substitution into (1.1) yields a bilinear equation in the unknowns p and q of the form

$$(c_1p - c_2)(c_1q - c_2) = c_3, \tag{5.2}$$

where

$$c_1 = A\sigma(B) - c_2,$$
$$c_2 = \sigma(A)(\sigma(B) - B),$$
$$c_3 = \sigma(B)(Bc_1 + Ac_2),$$

and

$$r = \frac{\sigma(A)(p+1)(q+1)}{\sigma(B)} - 1. \tag{5.3}$$

By writing the right hand side of (5.2) in all possible ways as a product of two positive integers, and equating with the left hand side, all possible solutions will be found. A very favourable situation arises when $c_1 = 1$ or a small positive integer. Many APs have been found in this case. For example, if $A = B$, then (5.2) reduces to

$$[(2A - \sigma(A))p - (\sigma(A) - A)]\,[(2A - \sigma(A))q - (\sigma(A) - A)] = A^2, \tag{5.4}$$

and if $A = B = 2^k, k \in \mathbb{N}$, then (5.2) reduces to (4.1).

In [52] te Riele noticed that if (b_1r_1, b_2r_2) is a *known* amicable pair where r_1, r_2 are primes with $\gcd(b_1, r_1) = \gcd(b_2, r_2) = 1$, then by choosing

$$A = b_1, \ B = b_2$$

and by dividing out common factors of c_1, c_2, c_3 from (5.2), the coefficient of p may become 1 or a small integer > 1.

Example 5.2 Given the AP [53, # 37]

$$2^3 \begin{cases} 17 \cdot 19 \cdot 281 \\ 53 \cdot 1879 \end{cases},$$

we may take $A = 2^3 \cdot 53, B = 2^3 \cdot 17 \cdot 19$, and find that $c_1 = 2^6 \cdot 3^3 \cdot 5$, $c_2 = 2^9 \cdot 3^4 \cdot 5 \cdot 11$, $c_3 = 2^{12} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 409$, so that (5.2) becomes:

$$(p - 264)(q - 264) = 5^2 \cdot 7 \cdot 409.$$

Writing the right hand side as $175 \cdot 409$, we obtain $p = 439$, $q = 673$, and, from (5.3), $r = 44483$, all primes, yielding the amicable pair

$$2^3 \begin{cases} 17 \cdot 19 \cdot 44483 \\ 53 \cdot 439 \cdot 673 \end{cases}.$$

□

A special case of the method of [52] leads to the following "mother-daughter" rule[2] by which many amicable pairs have been found.

[2]The "rules" given in this section are different from the Thabit-rules given in Section 4, in the sense that a Thabit-rule is an infinite set of statements on amicable pairs, namely, for $k = 1, 2, \ldots$, whereas the rules in this section do not depend on such a parameter.

te Riele's Rule *Let (au, ap) be a given amicable pair where p is a prime not dividing a. If a pair of distinct prime numbers r, s exists, with* $\gcd(a, rs) = 1$, *satisfying the bilinear equation*

$$(r-p)(s-p) = (p+1)(p+u),$$

and if a third prime q exists, with $\gcd(au, q) = 1$, *such that*

$$q = r + s + u,$$

then (auq, ars) is an amicable pair.

Example 5.3 For AP # 106 [48]:

$$(3^2 \cdot 5^3 \cdot 13 \cdot 11 \cdot 59, 3^2 \cdot 5 \cdot 13 \cdot 18719)$$

we have $a = 3^2 \cdot 5 \cdot 13$, $u = 5^2 \cdot 11 \cdot 59 = 16225$, $p = 18719$ and application of te Riele's Rule yields $(p+1)(p+u) = 2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13^2$. Writing this as $2688 \cdot 243360$, we obtain the three primes $r = 18719 + 2688 = 21407$, $s = 18719 + 243360 = 262079$, and $q = 21407 + 262079 + 16225 = 299711$, and thus the AP

$$(3^2 \cdot 5^3 \cdot 13 \cdot 11 \cdot 59 \cdot 299711, 3^2 \cdot 5 \cdot 13 \cdot 21407 \cdot 262079).$$

A second AP is obtained by writing $(p+1)(p+u)$ as $3120 \cdot 209664$. □

This rule, with the restriction that $\gcd(a, u) = 1$, was given in [51]. By applying it to the 152D AP mentioned in Example 4.2, te Riele found 11 new APs of record size (at that time), the largest being 282D.

In the case $\gcd(a, u) = 1$, the right hand side of the bilinear equation in te Riele's Rule can be written as

$$(p+1)(p+u) = (\sigma(u))^2 \frac{\sigma(a)}{a},$$

and we may expect this number to have more prime factors, hence more divisors, as u has more prime factors. So APs of type $(i, 1)$ with large value of i (which denotes the number of *different* prime factors of u) may be expected to be particularly suitable as input to te Riele's Rule. Succesful attempts to find APs of type $(i, 1)$ are described in [15, 16, 30, 31, 33].[3]

The largest value of i for which APs of type $(i, 1)$ are known is $i = 7$: in 2001, García found the first such AP [33], and one year later, Pedersen found two other examples of such APs [48]. The number of daughter APs generated with te Riele's Rule from García's AP of type $(7, 1)$ is 1433.[4] The mother pair with the largest number of daughters generated with te Riele's Rule, namely: 80136, is the AP of type $(5, 1)$ (found by Pedersen in 1997):

$$3^3 \cdot 5 \cdot 17 \cdot 29 \cdot 37 \begin{cases} 9619 \cdot 175649 \cdot 2174822171 \cdot 3699104087781354907 \cdot 552654745834954629043 \\ 7512723994458805334811002008295593545353601623096999231999 \end{cases}.$$

[3]Recently, Kohmoto communicated to one of us (JMP) the following interesting rule to generate APs of type $(2, 1)$ from *other* APs of type $(2, 1)$: *if (apq, ar) is an AP of type* $(2, 1)$ *and if $u = 2q + 1$, $s = (p+1)(2q+1) - 1$, and $t = (p+1)(s+1) - 1$ are distinct primes, with* $\gcd(a, ust) = 1$ *and $u \neq p$, then $(aups, aut)$ is also an AP of type* $(2, 1)$. We know six pairs of APs of type $(2, 1)$ which are "related" to each other by this rule. For example, from $(2^2 \cdot 5 \cdot 11, 2^2 \cdot 71)$, this rule generates Euler's AP $(2^2 \cdot 23 \cdot 5 \cdot 137, 2^2 \cdot 23 \cdot 827)$. Another example: from Borho's AP (found in 1983), given by $a = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 461$, $p = 5531$, $q = 38723$, $r = 214221167$, we find the three primes $u = 77447$, $s = 428436803$, and $t = 2370112399727$ which represents an AP found by García in 1995.

[4]We are currently generating and counting the daughters of Pedersen's two APs of type $(7, 1)$.

By replacing the common factor $3^3 \cdot 5$ in all these APs by $3^2 \cdot 7 \cdot 13$, another 80136 APs, isotopic to the former set, are identified.

In [48] a separate list is given of all the (currently: 2008) known regular and irregular APs of type $(i, 1)$, suitable as input for this rule. We notice that not all *irregular* amicable pairs of type $(i, 1)$ are suitable as input for this rule, like Euler's pair $(2^3 \cdot 19 \cdot 41, 2^5 \cdot 199)$.

Borho and Hoffmann [11] realized that the condition in te Riele's Rule that (au, ap) is an amicable pair, can be relaxed as follows.

Definition 5.4 A pair of positive integers (a_1, a_2) is called a *breeder* if the equations

$$a_1 + a_2 x = \sigma(a_1) = \sigma(a_2)(x+1) \qquad (5.5)$$

have a positive integer solution x. □

By replacing the assumption in te Riele's Rule that (au, ap) *is an amicable pair* by the assumption that (au, a) *is a breeder*, we obtain the more general

Borho's Rule with breeders *Let (au, a) be a breeder, with integer solution x. If a pair of distinct prime numbers r, s exists, with* $\gcd(a, rs) = 1$, *satisfying the bilinear equation*

$$(r-x)(s-x) = (x+1)(x+u),$$

and if a third prime q exists, with $\gcd(au, q) = 1$, *such that*

$$q = r + s + u,$$

then (auq, ars) is an amicable pair.

Borho made the restriction that $\gcd(a, u) = 1$, but this is not necessary (and we have left it out from Borho's Rule above).

From the definition of a breeder, it is clear that any method by which we may find amicable pairs of type $(i, 1), i \geq 1$, may be used to find breeders, because, if in (5.5) x is a prime not dividing a, then the pair $(a_1, a_2 x)$ is an amicable pair. As an example, let us consider Lee's method, described after Example 5.1, for finding amicable pairs of the form (Apq, Br), where we choose $A = B$. This yields equation (5.4) for p and q, while equation (5.3) for r becomes: $r = pq + p + q$. So for any two primes p and q not dividing A, and satisfying (5.4), we have found a breeder (Apq, A), and if r happens to be a prime, then we have found an amicable pair (Apq, Ar). We have applied Lee's method to all values of $A \leq 10^8$, and we have found 305 breeders, of which 75 are (known) amicable pairs of type $(2, 1)$. The smallest breeder (i.e., with smallest A-value) is $(2^2 \cdot 5 \cdot 11, 2^2)$ with $r = 71$, a prime, so this gives an amicable pair, and next comes the breeder $(2^3 \cdot 11 \cdot 23, 2^3)$ with $r = 287 = 11 \cdot 17$. Application of Borho's Rule to this breeder yields three amicable pairs of the form (Auq, Avw) (with $A = 2^3, u = 11 \cdot 23$), namely, for $(v, w, q) = (383, 1907, 2543), (467, 1151, 1871), (647, 719, 1619)$. These three APs were already found by Euler [21]. Application of Borho's Rule to *all* the 305 breeders (Apq, A) with $A \leq 10^8$ gave 4779 daughter amicable pairs, an average of 15.7 daughters per breeder. The 75 amicable pairs among these 305 breeders gave 929 daughters, an average of 12.4 daughters per AP-breeder.

With Stefan Battiato, Borho has extended his breeder method [4], namely by using the *output breeders* of Lee's method as *input* of a next application of Lee's method. With their experiments, Battiato and Borho showed an *exponential growth* of the number of breeders in subsequent "generations". In their main search they produced almost one million breeders and 26684 amicable pairs.[5] With their searches, Borho and Battiato were able to extend the number of amicable pairs, known in 1987, from about 13760 to about 51560. Their "champion" breeder is the breeder (au, a), with

$$a = 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 79 \quad \text{and} \quad u = 1013 \cdot 6180283 \cdot 2091919367,$$

from which Borho's Rule generates a total of 3634 APs.

Based on the search ideas described in this section, García [33] succeeded to find more than one million new amicable pairs. Although he formulated it somewhat differently, García applied Borho's Rule with breeders. In one particular case, namely from the breeder (au, a), with

$$a = 3^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 757 \cdot 3329$$

and

$$u = 1511 \cdot 72350721629 \cdot 2077116867246979,$$

García found 35279 new amicable pairs. In addition, each of these amicable pairs has six *isotopic* APs, obtained by replacing $3^3 \cdot 5^2 \cdot 19 \cdot 31$ by the six values given in Example 2.4.

Pedersen has built up a database of known breeders (au, a) and he has applied Borho's Rule to most of them. Table 2 surveys the status of this database at the time of writing of this paper. Of all the breeders in this database, there are 363 for which $\gcd(\sigma(a), u) > 1$. Not a single daughter was generated from these breeders. We can prove that indeed this is not possible, at least for those breeders for which $\sigma(a)/a > 3/2$ holds.[6] Closely related to this result is the fact (easy to prove) that if (m, n) is a regular AP with $a = \gcd(m, n)$, then $\gcd(\sigma(a), m/a) = \gcd(\sigma(a), n/a) = 1$. In fact, more than 90% of the amicable pairs currently known have been found with the help of te Riele's Rule and Borho's breeder versions of it.

Table 2 Status of Pedersen's database of breeders of the form (au, a)

		number of		
primes dividing u	breeders (au, a)	amicable pairs among these breeders	breeders to which Borho's Rule has been applied	daughter APs generated
2	1130	498	1124	59633
3	3619	919	3342	951482
4	1970	406	1832	843647
5	1144	108	812	580886
6	207	14	96	108479
7	44	3	11	3412
8	2	0	1	2124
	8116	1948	7218	2549663

[5]This number of APs could have been much larger if their precision had not been restricted to 29 decimal digits.

[6]For breeders (au, a) for which a is *even*, it is easy to see that $\sigma(a)/a > 3/2$; for *odd* a we do not know any breeders for which $\sigma(a)/a \leq 3/2$.

6 Exhaustive searches

One not particularly clever way to find amicable pairs is to compute for *all* the numbers m in a given interval $[A, B]$ the value of $\sigma(m) - m =: n$ followed by the computation of the value of $\sigma(n) - n$. If the latter equals m, we have found an amicable pair.[7] This involves one or two complete factorizations, in case m is deficient or abundant, respectively. However, a closer look reveals that it is sometimes possible to find out whether a given number m is deficient (hence cannot be the smaller member of an amicable pair) *without the need to factor it completely.* Moreover, once $\sigma(m)$ and n have been computed, it may be possible to discover that $\sigma(m) \neq \sigma(n)$ without the need to factor n completely. Te Riele used these ideas in an exhaustive search of all the amicable pairs with smaller member $\leq 10^{10}$ [53]. In an exhaustive search up to 10^{11}, Moews and Moews [47] (also see [46]) used a *sieve* to calculate $\sigma(m)$ for all m in a given interval.

Exhaustive amicable pair searches have been carried out up to the bound 10^{13}. Table 3 surveys the milestones in these searches. The two numbers between parentheses in the $A(x)$-column give the number $A_i(x)$ of *irregular* amicable pairs (m, n) with $m \leq x$, and the ratio $A_i(x)/A(x)$, respectively. This ratio seems to have stabilized near 22% at the end of the table. The number between parentheses in the last column indicates which fraction of the total number of APs found were new (at that time). In 2000, Einstein searched the interval $[10^{12}, 10^{13}]$ and found 8650 new amicable pairs. Since this search was not completely exhaustive, Chernych searched the same interval in 2002, but now in a completely exhaustive way. Chernych found ten new amicable pairs which were missed by Einstein.

Table 3 Exhaustive amicable pair searches

name(s)	x	$A(x)(A_i(x), A_i/A)$	# new APs found
Rolf [56]	10^5	13(4, 0.3077)	1 (0.077)
Alanen, Ore and Stemple [1]	10^6	42(11, 0.2619)	8 (0.276)
Bratley, Lunnon and McKay [13]	10^7	108(28, 0.2593)	14 (0.212)
Cohen [14]	10^8	236(55, 0.2331)	56 (0.438)
te Riele [53]	10^{10}	1427(345, 0.2418)	816 (0.685)
Moews and Moews [47]	10^{11}	3340(763, 0.2284)	1262 (0.659)
Moews and Moews [48]	2×10^{11}	4310(955, 0.2216)	860 (0.887)
Moews and Moews [48]	3×10^{11}	4961(1114, 0.2246)	463 (0.711)
Einstein and Moews [48]	10^{12}	7642(1682, 0.2201)	1965 (0.733)
Einstein and Chernych [48]	10^{13}	17519(3833, 0.2188)	8660 (0.877)

As in [53], we have compared $A(x)$ with $\sqrt{x}/\ln^i(x)$, for $i = 1, 2, 3$, see Table 4, but here we have added $i = 4$. From these figures it seems that at least for the three largest values of x in Table 4, the growth of $A(x)$ is characterized best by the function $\sqrt{x}/\ln^4(x)$.

[7] Walter Borho once characterized this "method" as catching fish from a pond by pumping out all the water.

Table 4 Comparison of $A(x)$ with $\sqrt{x}/\ln^i(x)$ for $i = 1, 2, 3, 4$

x	$A(x)$	$A(x)\ln(x)/\sqrt{x}$	$A(x)\ln^2(x)/\sqrt{x}$	$A(x)\ln^3(x)/\sqrt{x}$	$A(x)\ln^4(x)/\sqrt{x}$
10^5	13	0.473	5.45	62.7	722
10^6	42	0.580	8.02	111	1530
10^7	108	0.550	8.87	143	2305
10^8	236	0.435	8.01	148	2717
10^9	586	0.384	7.96	165	3418
10^{10}	1427	0.329	7.57	174	4011
10^{11}	3340	0.268	6.78	172	4347
10^{12}	7642	0.211	5.83	161	4454
10^{13}	17519	0.166	4.96	149	4448

7 Searches by finding many solutions of $\sigma(x) = S$

Erdős suggested the following way to find amicable pairs, which is based on (1.1): for given $S \in \mathbb{N}$, if $x_1, x_2, \ldots$ are solutions of the equation

$$\sigma(x) = S,$$

then any pair (x_i, x_j) $(i \neq j)$ for which $x_i + x_j = S$, is an amicable pair.[8] Heuristically, values of S for which the equation $\sigma(x) = S$ has *many* solutions have an increased chance to yield amicable pairs. Te Riele has worked out this idea [54] by developing an algorithm for finding as many as possible solutions of the equation $\sigma(x) = S$. A critical choice is the value of the pair sum S. Inspection of the pair sums of *known* amicable pairs revealed that in many cases these sums only have *small* prime divisors. In particular, among the 1427 APs below 10^{10} there are 37 *pairs* of APs (but no such triples) having the same pair sum, and in these 37 pair sums the largest occurring prime is 37. Suggested by this, many possible pair sums $S \in [4 \times 10^{11}, 2 \times 10^{12}]$ were generated having a similar prime structure and to each of these numbers S the algorithm was applied to find as many as possible suitable solutions of the equation $\sigma(x) = S$, followed by a search of pairs of solutions (x_1, x_2) summing up to S. As a result 565 new APs were found in the interval $[2 \times 10^{11}, 10^{12}]$ ([54] and Report NM-R9512). Of the APs found, 20.6% are irregular. This suggests that the method used here finds regular and irregular amicable pairs in about the same ratio as the *exhaustive* searches of Section 6.

We notice that the method of this section can be extended with help of the unknown common factor method, mentioned after Example 5.1: for those M, N with $\sigma(M) = \sigma(N)$ but $\sigma(M) \neq M + N$ we may try to find a such that $\sigma(a)/a = (M + N)/\sigma(M)$.

In [48] Pedersen keeps a list of pair sums S for which there are *at least two APs* with this pair sum. The current champion is

$$S = 1060088020992000 = 2^{18} \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19,$$

for which there are *eight* APs having this pair sum, six being odd-odd, two being even-even. The smallest odd-odd AP with this pair sum is

$$3 \cdot 5 \cdot 7 \begin{cases} 23 \cdot 29 \cdot 37 \cdot 83 \cdot 103 \cdot 23099 \\ 31 \cdot 71 \cdot 109 \cdot 1481 \cdot 14699 \end{cases}$$

[8] Erdős's idea was communicated to te Riele by Carl Pomerance.

and the smallest even-even AP with this pair sum is

$$2^2 \cdot 19 \begin{cases} 31 \cdot 41 \cdot 43 \cdot 53 \cdot 569 \cdot 4159 \\ 17 \cdot 191 \cdot 607 \cdot 1091 \cdot 3299 \end{cases} .$$

The numbers of known k-tuples of amicable pairs having the same pair sum are 1831, 180, 30, 5, 1 for $k = 2, 3, 4, 5, 6$, respectively.

8 Questions

We close with a number of unsolved questions.

1. Are there an infinite number of amicable pairs?
2. Is there an amicable pair whose members have opposite parity?
3. Is there an amicable pair whose members are relatively prime?
4. Is there an amicable pair with pair sum equal to 1 mod 3?
5. For any given prime p, is there an amicable pair whose members have no prime factors $< p$? According to Section 3.3 this is known to be true for $p = 3, 5$ and 7.
6. Are there any amicable pairs whose members have different smallest prime factors?
7. Are there amicable pairs for all possible types?

We trust that this paper has convinced the reader that the answer to Q1 is yes.[9] We also believe that the answer to Q5 is yes. We do not have an opinion on the other questions.

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