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An analysis of a new family of eighth-order optimal methods

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ABSTRACT

A new family of eighth order optimal methods is developed and analyzed. Numerical experiments show that our family of methods perform well and in many cases some members are superior to other eighth order optimal methods. It is shown how to choose the parameters to widen the basin of attraction.

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1. Introduction

There are many multistep methods for the solution of nonlinear equations, see e.g. Traub [1], and the recent book by Petković et al. [2]. The idea of optimality in such methods was introduced by Kung and Traub [3] who also developed optimal multistep method of increasing order. For example, the fourth-order optimal method given in [3] is

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}. \end{cases} \tag{1}$$

Based on this method Chun and Neta [4] constructed and analyzed the sixth order method

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}, \\ x_{n+1} = s_n - \frac{f(s_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)} - \frac{f(s_n)}{f(x_n)}\right]^2}. \end{cases} \tag{2}$$

In this paper we will use the idea of weight function to develop a family of optimal eighth order methods and show how to choose the parameters to obtain the best basins of attraction.

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2. An optimal eighth-order method

We consider here a generalization of the Chun–Neta sixth order scheme (2). The new family is constructed using the idea of weight functions. The multistep method is given by

$$\begin{cases} w_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{[1-r_n]^2}, \\ x_{n+1} = s_n - \frac{f(s_n)}{f'(x_n)} \frac{1}{[1-H(r_n)J(t_n)P(q_n)]^2}, \end{cases} \tag{3}$$

where $r_n = \frac{f(w_n)}{f(x_n)},\ t_n = \frac{f(s_n)}{f(x_n)},\ q_n = \frac{f(s_n)}{f(w_n)}$ and $H(r),\ J(t),\ P(q)$ are real-valued weight functions to be determined later.

For the method defined by (3), we have the following analysis of convergence.

Theorem 2.1. *Let $\xi \in I$ be a simple zero in an open interval I of a sufficiently differentiable function $f : I \to \mathbb{R}$. Let $e_n = x_n - \xi$. Then the new family of methods defined by (3) is of optimal eighth-order when*

$$\begin{aligned} &H(0)J(0)P(0) = 2, \\ &H'(0)P(0)J(0) = -1, \\ &H''(0)P(0)J(0) = -1, \\ &H'''(0)P(0)J(0) = 3, \\ &|H^{(4)}(0)| < \infty, \\ &J'(0) = -3J(0)/8, \\ &|J''(0)| < \infty, \\ &P'(0) = -P(0)/4, \\ &|P''(0)| < \infty. \end{aligned}$$

The error at the $(n+1)$th step, e_{n+1}, satisfies the relation

$$\begin{aligned} e_{n+1} = c_2\Bigg[&\left(\frac{2P''(0)}{P'(0)} - \frac{1}{4}\right)c_3^3 - c_2c_3c_4 + \left(4 - \frac{3P''(0)}{P'(0)}\right)c_2^2c_3^2 + 2c_2^3c_4 + \left(\frac{1}{3}P'(0)J(0)H^{(4)}(0) + \frac{6P''(0)}{P'(0)} - \frac{29}{4}\right)c_2^4c_3 \\ &+ \left(\frac{1}{2} - \frac{2}{3}P'(0)J(0)H^{(4)}(0) - \frac{4P''(0)}{P'(0)}\right)c_2^6\Bigg]e_n^8 + O(e_n^9), \end{aligned} \tag{4}$$

where c_i are given by

$$c_i = \frac{f^{(i)}(\xi)}{i!f'(\xi)}, \quad i \geqslant 1. \tag{5}$$

Proof. Let $e_n = x_n - \xi,\ e_n^w = w_n - \xi$ and $e_n^s = s_n - \xi$. Using the Taylor expansion of $f(x)$ around $x = \xi$ and taking $f(\xi) = 0$ into account, we get

$$f(x_n) = f'(\xi)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)\right] \tag{6}$$

and

$$f'(x_n) = f'(\xi)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)\right]. \tag{7}$$

Dividing (6) by (7) gives

$$\begin{aligned} u_n &= \frac{f(x_n)}{f'(x_n)} \\ &= e_n - c_2e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 + (10c_2c_4 - 4c_5 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_n^5 + (17c_4c_3 \\ &\quad - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 33c_2c_3^2 + 52c_3c_2^3 - 16c_2^5)e_n^6 + (-92c_3c_2c_4 + 22c_3c_5 - 18c_3^3 + 126c_3^2c_2^2 - 128c_3c_2^4 \\ &\quad + 12c_4^2 + 72c_4c_2^3 - 36c_5c_2^2 - 6c_7 + 16c_2c_6 + 32c_2^6)e_n^7 + (-7c_8 - 118c_5c_2c_3 + 348c_4c_3c_2^2 + 19c_2c_7 - 64c_2c_4^2 \\ &\quad + 31c_4c_5 - 75c_4c_3^2 - 176c_4c_2^4 + 92c_5c_2^3 + 27c_6c_3 - 44c_6c_2^2 + 135c_2c_3^3 - 408c_3^2c_2^3 + 304c_3c_2^5 - 64c_2^7)e_n^8 + O(e_n^9). \end{aligned} \tag{8}$$

From (8), we have

$$e_n^w = c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + (-10c_2c_4 + 4c_5 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_n^5 + (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 + (92c_3c_2c_4 - 22c_3c_5 + 18c_3^3 - 126c_3^2c_2^2 + 128c_3c_2^4 - 12c_4^2 - 72c_4c_2^3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 32c_2^6)e_n^7 + (64c_2^7 + 7c_8 + 118c_5c_2c_3 - 348c_4c_3c_2^2 - 19c_2c_7 + 64c_2c_4^2 - 31c_4c_5 + 75c_4c_3^2 + 176c_4c_2^4 - 92c_5c_2^3 - 27c_6c_3 + 44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 - 304c_3c_2^5)e_n^8 + O(e_n^9). \quad (9)$$

Writing the Taylor's expansion for $f(w_n)$ and using (9), we obtain

$$\begin{aligned} f(w_n) &= f'(\xi)[e_n^w + c_2(e_n^w)^2 + c_3(e_n^w)^3 + c_4(e_n^w)^4 + O((e_n^w)^5)] \\ &= f'(\xi)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + (-10c_2c_4 + 4c_5 - 6c_3^2 + 24c_3c_2^2 - 12c_2^4)e_n^5 \\ &\quad + (-17c_4c_3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_n^6 + (18c_3^3 - 64c_2^6 + 6c_7 + 104c_3c_2c_4 \\ &\quad - 16c_2c_6 - 22c_3c_5 - 160c_3^2c_2^2 + 206c_3c_2^4 - 12c_4^2 - 104c_4c_2^3 + 44c_5c_2^2)e_n^7 + (144c_2^7 + 7c_8 + 134c_5c_2c_3 \\ &\quad - 455c_4c_3c_2^2 - 19c_2c_7 + 73c_2c_4^2 - 31c_4c_5 + 75c_4c_3^2 + 297c_4c_2^4 - 134c_5c_2^3 - 27c_6c_3 + 54c_6c_2^2 - 147c_2c_3^3 \\ &\quad + 582c_3^2c_2^3 - 552c_3c_2^5)e_n^8 + O(e_n^9)]. \end{aligned} \quad (10)$$

Dividing (10) by (7) gives

$$\frac{f(w_n)}{f'(x_n)} = c_2e_n^2 + (2c_3 - 4c_2^2)e_n^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_n^4 + (-20c_2c_4 + 4c_5 - 12c_3^2 + 64c_3c_2^2 - 38c_2^4)e_n^5 + (104c_2^5 + 5c_6 - 26c_2c_5 - 34c_4c_3 + 90c_4c_2^2 + 103c_2c_3^2 - 240c_3c_2^3)e_n^6 + O(e_n^7). \quad (11)$$

Dividing (10) by (6) gives

$$\begin{aligned} r_n &= \frac{f(w_n)}{f(x_n)} \\ &= c_2e_n + (2c_3 - 3c_2^2)e_n^2 + (3c_4 - 10c_2c_3 + 8c_2^3)e_n^3 + (-14c_2c_4 + 4c_5 - 8c_3^2 + 37c_3c_2^2 - 20c_2^4)e_n^4 \\ &\quad + (-22c_3c_4 + 51c_4c_2^2 - 18c_2c_5 + 55c_2c_3^2 - 118c_3c_2^3 + 5c_6 + 48c_2^5)e_n^5 + (6c_7 + 150c_4c_2c_3 - 22c_2c_6 - 15c_4^2 - 163c_4c_2^3 \\ &\quad - 28c_5c_3 + 65c_5c_2^2 - 252c_3^2c_2^2 + 344c_3c_2^4 + 26c_3^3 - 112c_2^6)e_n^6 + O(e_n^7). \end{aligned} \quad (12)$$

Using (8), (11) and (12), we find

$$\begin{aligned} e_n^s &= e_n^w - \frac{f(w_n)}{f'(x_n)} \frac{1}{[1-r]^2} \\ &= (-c_2c_3 + 2c_2^3)e_n^4 + (-2c_2c_4 + 14c_3c_2^2 - 2c_3^2 - 10c_2^4)e_n^5 + (31c_2^5 - 3c_2c_5 - 7c_4c_3 + 21c_4c_2^2 + 30c_2c_3^2 - 72c_3c_2^3)e_n^6 \\ &\quad + (20c_3^3 - 74c_2^6 + 88c_3c_2c_4 - 4c_2c_6 - 10c_3c_5 - 188c_3^2c_2^2 + 246c_3c_2^4 - 6c_4^2 - 100c_4c_2^3 + 28c_5c_2^2)e_n^7 + O(e_n^8), \end{aligned} \quad (13)$$

so that, after elementary calculation,

$$\begin{aligned} f(s_n) &= f'(\xi)[e_n^s + c_2(e_n^s)^2 + c_3(e_n^s)^3 + O((e_n^s)^4)] \\ &= f'(\xi)[(-c_2c_3 + 2c_2^3)e_n^4 + (-2c_2c_4 + 14c_3c_2^2 - 2c_3^2 - 10c_2^4)e_n^5 + (31c_2^5 - 3c_2c_5 - 7c_4c_3 + 21c_4c_2^2 + 30c_2c_3^2 - 72c_3c_2^3)e_n^6 \\ &\quad + (20c_3^3 - 74c_2^6 + 88c_3c_2c_4 - 4c_2c_6 - 10c_3c_5 - 188c_3^2c_2^2 + 246c_3c_2^4 - 6c_4^2 - 100c_4c_2^3 + 28c_5c_2^2)e_n^7 + O(e_n^8)]. \end{aligned} \quad (14)$$

An easy calculation then produces

$$\frac{f(s_n)}{f'(x_n)} = (-c_2c_3 + 2c_2^3)e_n^4 + (-2c_2c_4 + 16c_3c_2^2 - 2c_3^2 - 14c_2^4)e_n^5 + (59c_2^5 - 3c_2c_5 - 7c_4c_3 + 25c_4c_2^2 + 37c_2c_3^2 - 110c_3c_2^3)e_n^6 + (26c_3^3 - 192c_2^6 + 112c_3c_2c_4 - 4c_2c_6 - 10c_3c_5 - 310c_3^2c_2^2 + 508c_3c_2^4 - 6c_4^2 - 158c_4c_2^3 + 34c_5c_2^2)e_n^7 + O(e_n^8), \quad (15)$$

$$\begin{aligned} t_n &= \frac{f(s_n)}{f(x_n)} \\ &= (-c_2c_3 + 2c_2^3)e_n^3 + (-2c_2c_4 - 2c_3^2 + 15c_3c_2^2 - 12c_2^4)e_n^4 + (-3c_2c_5 - 7c_3c_4 + 33c_2c_3^2 - 89c_3c_2^3 + 23c_4c_2^2 + 43c_2^5)e_n^5 \\ &\quad + (98c_4c_2c_3 - 4c_2c_6 - 6c_4^2 - 125c_4c_2^3 - 10c_5c_3 + 31c_5c_2^2 - 236c_3^2 * c_2^2 + 347c_3c_2^4 + 22c_3^3 - 117c_2^6)e_n^6 \\ &\quad + (130c_5c_2c_3 - 651c_4c_3c_2^2 - 5c_2c_7 - 17c_5c_4 - 162c_5c_2^3 - 13c_6c_3 + 39c_6c_2^2 + 72c_2c_4^2 + 95c_4c_3^2 + 468c_4c_2^4 - 266c_2c_3^3 \\ &\quad + 1087c_3^2c_2^3 - 1042c_3c_2^5 + 266c_2^7)e_n^7 + O(e_n^8) \end{aligned} \quad (16)$$

and

$$\begin{aligned} q_n &= \frac{f(s_n)}{f(w_n)} \\ &= (-c_3+2c_2^2)e_n^2+(-2c_4+8c_2c_3-6c_2^3)e_n^3+(-3c_5+7c_3^2-25c_3c_2^2+11c_2c_4+9c_2^4)e_n^4+(18c_3c_4-4c_6 \\ &\quad -30c_4c_2^2+14c_2c_5-32c_2c_3^2+36c_3c_2^3-2c_2^5)e_n^5+(22c_5c_3-71c_4c_2c_3-5c_7-36c_5c_2^2+17c_2c_6+11c_4^2 \\ &\quad +37c_4c_2^3-13c_3^3+41c_3^2c_2^2+26c_3c_2^4-28c_2^6)e_n^6+O(e_n^7). \end{aligned} \tag{17}$$

We now expand $H(r_n),\ J(t_n),\ P(q_n)$ into Taylor series about 0 to obtain

$$\begin{aligned} H(r_n) &= H(0)+H'(0)r_n+\frac{H''(0)}{2}r_n^2+\frac{H'''(0)}{6}r_n^3+O(r_n^4), \\ J(t_n) &= J(0)+J'(0)t_n+\frac{J''(0)}{2}t_n^2+O(t_n^3), \\ P(q_n) &= P(0)+P'(0)q_n+\frac{P''(0)}{2}q_n^2+O(q_n^3). \end{aligned} \tag{18}$$

Upon using the values

$$\begin{aligned} &H(0)J(0)P(0)=2, \\ &H'(0)P(0)J(0)=-1, \\ &H''(0)P(0)J(0)=-1, \\ &H'''(0)P(0)J(0)=3, \\ &J'(0)=-3J(0)/8, \\ &P'(0)=-P(0)/4 \end{aligned}$$

and Eqs. (11), (16) and (17) we obtain

$$\begin{aligned} \frac{1}{[1-H(r_n)J(t_n)P(q_n)]^2} &= 1+2c_2e_n+3c_3e_n^2+4c_4e_n^3 \\ &\quad +\left[\left(2\frac{P''(0)}{P'(0)}+\frac{1}{3}P'(0)J(0)H^{(4)}(0)-\frac{9}{4}\right)c_2^4+5c_5-c_2c_4+\left(\frac{9}{2}-2\frac{P''(0)}{P'(0)}\right)c_3c_2^2+\frac{1}{2}\left(\frac{P''(0)}{P'(0)}-\frac{1}{2}\right)c_3^2\right]e_n^4 \\ &\quad +O(e_n^5). \end{aligned} \tag{19}$$

Therefore, from (13), (15) and (19), we obtain

$$\begin{aligned} e_{n+1} &= e_n^s-\frac{f(s_n)}{f'(x_n)}\frac{1}{[1-H(r)J(t)P(q)]^2} \\ &= c_2\left[\left(\frac{2P''(0)}{P'(0)}-\frac{1}{4}\right)c_3^3-c_2c_3c_4+\left(4-\frac{3P''(0)}{P'(0)}\right)c_2^2c_3^2+2c_2^3c_4+\left(\frac{1}{3}P'(0)J(0)H^{(4)}(0)+\frac{6P''(0)}{P'(0)}-\frac{29}{4}\right)c_2^4c_3\right. \\ &\quad \left.+\left(\frac{1}{2}-\frac{2}{3}P'(0)J(0)H^{(4)}(0)-\frac{4P''(0)}{P'(0)}\right)c_2^6\right]e_n^8+O(e_n^9), \end{aligned} \tag{20}$$

this completing the proof. □

Now we can choose

$$\begin{aligned} H(r) &= \frac{a+br+cr^2}{1+dr+gr^2}, \\ J(t) &= \frac{\alpha+\beta t}{1+\gamma t}, \\ P(q) &= \frac{A+Bq}{1+Cq}. \end{aligned}$$

It is easy to see that $B=A(C-1/4),\ a=\frac{2}{A\alpha},\ b=\frac{1-4g}{A\alpha},\ \beta=\alpha(\gamma-3/8),\ c=\frac{1}{2}\frac{8g-3}{A\alpha}$, and $d=1-2g$. Substituting these values in $H,\ J$ and P, we have

$$H(r)=\frac{1}{2}\frac{4+(2-8g)r+(8g-3)r^2}{A\alpha(1+(1-2g)r+gr^2)},$$

$$J(t)=\frac{1}{8}\frac{\alpha(8+(8\gamma-3)t)}{1+\gamma t},$$

$$P(q) = \frac{1}{4}\frac{A(4+(4C-1)q)}{1+Cq}.$$

Therefore we have 5 parameters α, γ, A, C, g. We can choose $C = -2/3$ and $g = 41/36$ by requiring that $3 - 4P'(0)J(0)H^{(4)}(0) - \frac{24P''(0)}{P'(0)} = 0$ and $4 - \frac{3P''(0)}{P'(0)} = 0$, which eliminate the terms of c_2^6 and $c_2^2c_3^2$ in the error Eq. (4). Thus we have

$$H(r) = \frac{2(36-64r+55r^2)}{A\alpha(36-46r+41r^2)},$$

$$J(t) = \frac{1}{8}\frac{\alpha(8+(8\gamma-3)t)}{1+\gamma t},$$

$$P(q) = \frac{1}{4}\frac{A(11q-12)}{2q-3}.$$

This family of methods is denoted by OM.

Since we only use the product $H(r)J(t)P(q)$, it is easy to see that $A\alpha$ cancels out.

$$H(r)J(t)P(q) = \frac{1}{2}\frac{4+(2-8g)r+(8g-3)r^2}{A\alpha(1+(1-2g)r+gr^2)}\frac{1}{8}\frac{\alpha(8+(8\gamma-3)t)}{1+\gamma t}\frac{1}{4}\frac{A(4+(4C-1)q)}{1+Cq}.$$

Thus we have the 3 free parameters g, γ and C. This family is denoted OMN. Clearly OM is a special case since it has a specific choice of g and C. In the next section we give a comparative numerical study of several members of our families OM and OMN as well as other well known optimal eighth order methods. We will also discuss the basins of attraction of our families.

3. Numerical examples

In this section we present some numerical experiments using our newly found methods and compare these results to other schemes. All computations were done using MAPLE using 128 digit floating point arithmetic (Digits := 128). Given an initial guess x_0 we decide that the method converges if the sequence $\{x_n\}$ generated by the iterative method has a residual $|f(x_n)|$ less than a tolerance $\epsilon = 10^{-25}$ in a maximum of 100 iterations, otherwise we consider the method to be divergent. We used the following test functions and display the approximate zero x_* found up to the 28th decimal places.

Index	Test function	Root
1	x^3+4x^2-10	1.365230013414096845760806829 0
2	$\sin^2(x)-x^2+1$	1.404491648215341226035086817 8
3	$(x-1)^3-1$	2.0
4	x^3-10	2.154434690031883721759293566 5
5	$xe^{x^2}-\sin^2(x)+3\cos(x)+5$	−1.207647827130918927009416758 4
6	$e^{x^2+7x-30}-1$	3.0
7	$\sin(x)-\frac{x}{2}$	1.895494267033980947144035738 1
8	$x^5+x-10000$	6.308777129972689094767571771 8
9	$\sqrt{x}-\frac{1}{x}-3$	9.633595562832695192406312709 2
10	e^x+x-20	2.842438953784447067816585940 2
11	$\ln(x)+\sqrt{x}-5$	8.309432694231571795346955682 7
12	x^2-e^x-3x+2	.2575302854398607604553673049
13	$e^x\sin(x)+\ln(1+x^2)$	0
14	$e^{-x^2+x+2}-1$	−1
15	$x^5+x^4+4x^2-15$	1.347428098968304981506715381

Compared to our methods OM and OMN with various parameters we have taken the following methods

- Kung–Traub eighth-order method (KT8) [3] defined by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ \tau_n &= y_n - \frac{f(x_n)}{f'(x_n)}\frac{f(y_n)f(x_n)}{[f(x_n)-f(y_n)]^2}, \\ x_{n+1} &= \tau_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n)f(y_n)f(\tau_n)}{[f(x_n)-f(y_n)]^2}\frac{f^2(x_n)+f(y_n)[f(y_n)-f(\tau_n)]}{[f(x_n)-f(\tau_n)]^2[f(y_n)-f(\tau_n)]}. \end{aligned} \tag{21}$$

- The method based on Kung–Traub optimal fourth-order method and Hermite interpolating polynomial (HKT8) [2]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ \tau_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{1}{[1 - f(y_n)/f(x_n)]^2}, \\ x_{n+1} &= \tau_n - \frac{f(\tau_n)}{H_3'(\tau_n)}, \end{aligned} \tag{22}$$

where

$$H_3'(\tau_n) = 2(f[x_n, \tau_n] - f[x_n, y_n]) + f[y_n, \tau_n] + \frac{y_n - \tau_n}{y_n - x_n}(f[x_n, y_n] - f'(x_n)). \tag{23}$$

- The method based on Kung–Traub optimal fourth-order method and Hermite interpolating polynomial replacing the function (HKN8) [5]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ \tau_n &= y_n - \frac{f(y_n)}{f'(x_n)} \frac{1}{[1 - f(y_n)/f(x_n)]^2}, \\ x_{n+1} &= \tau_n - \frac{H_3(\tau_n)}{f'(\tau_n)}, \end{aligned} \tag{24}$$

Table 1
Comparison of eighth-order iterative schemes.

f		KT8	HKT8	HKN8	N8	WM8	OM1	OM2
f_1	IT	3	3	3	3	3	3	3
$x_0 = 1.5$	$f(x_*)$	3.5e-69	4.8e-72	4.8e-72	1e-66	2.3e-60	−5.4e−86	−5.7e−83
f_2	IT	3	3	3	3	3	3	3
$x_0 = 1.37$	$f(x_*)$	−1.1e−90	−4.4e−94	−9.7e−59	−1.6e−86	−2.1e−80	1.1e−97	1.3e−97
f_3	IT	4	3	3	4	4	3	3
$x_0 = 2.5$	$f(x_*)$	0	9.5e−26	9.5e−26	0	0	−3.7e−29	−1.6e−31
f_4	IT	4	4	4	4	4	4	4
$x_0 = 4$	$f(x_*)$	1.1e−116	0	0	2.2e−104	6.4e−83	0	0
f_5	IT	4	4	4	4	4	3	3
$x_0 = -1.5$	$f(x_*)$	−1.2e−126	−1.2e−126	2.6e−101	−1.2e−126	−1.2e−126	3.5e−37	1.6e−28
f_6	IT	8	9	8	9	9	8	7
$x_0 = 4$	$f(x_*)$	3.7e−40	0	−4.8e−73	0	1.6e−33	−2e−126	−2.2e−42
f_7	IT	3	3	3	3	3	3	3
$x_0 = 2$	$f(x_*)$	−4,4e−71	−3e−76	-1.4e−49	−3.6e−68	−1.2e−63	2.4e−84	2.1e−86
f_8	IT	5	5	5	5	101	15	42
$x_0 = 4$	$f(x_*)$	1e−63	0	−4.9e−85	8.3e−47	9.8e17	0	−7e−124
f_9	IT	4	4	5	4	4	4	4
$x_0 = 1$	$f(x_*)$	−1e−127	2.9e−89	1.5e−70	0	−1.4e−71	0	1e−127
f_{10}	IT	4	4	4	4	7	4	4
$x_0 = 2.1$	$f(x_*)$	0	0	−1.4e−112	2.4e−117	3.8e−29	−2e−126	0
f_{11}	IT	3	4	4	3	4	4	4
$x_0 = 1$	$f(x_*)$	1.8e−34	−2.9e−106	2.5e−29	9.2e−30	−1.1e−96	−1e−127	0
f_{12}	IT	3	3	3	3	3	3	3
$x_0 = 0.5$	$f(x_*)$	1.2e−84	4.1e−87	3.9e−51	4.5e−86	5.3e−84	6e−84	6.1e−84
f_{13}	IT	4	4	4	4	4	4	4
$x_0 = 1$	$f(x_*)$	9.4e−67	5.8e−80	9.9e−55	5.4e−60	7.4e−49	−1.1e−93	−4.9e−107
f_{14}	IT	3	3	3	3	3	4	4
$x_0 = -0.85$	$f(x_*)$	2.3e−46	3.2e−49	6.4e−34	2.3e−43	3.2e−38	-1.9e−83	−5.8e−83
f_{15}	IT	3	3	3	3	3	3	3
$x_0 = 1.2$	$f(x_*)$	1.4e−43	2.9e−46	-6.9e−33	9.4e−38	5e−31	−1.1e−52	−4.7e−49

Table 2
Comparison of eighth-order iterative schemes.

f		OMN1	OMN2	OMN3	OMN6
f_1	IT	3	3	3	3
$x_0 = 1.5$	$f(x_*)$	−1.0e−62	−8.9e−67	−2.4e−75	−4.1−68
f_2	IT	3	3	3	3
$x_0 = 1.37$	$f(x_*)$	9.5e−82	1.6e−83	5.1e−92	9.6e−89
f_3	IT	4	3	3	4
$x_0 = 2.5$	$f(x_*)$	−8.9e−121	−6.4e−30	−1.7e−29	−3.0e−127
f_4	IT	4	4	4	4
$x_0 = 4$	$f(x_*)$	−2.1e−35	0	−1.4e−126	−7.8e94
f_5	IT	4	4	4	4
$x_0 = -1.5$	$f(x_*)$	5.5e−105	−1.1e−126	1.2e−126	−9.8e−75
f_6	IT	10	8	6	10
$x_0 = 4$	$f(x_*)$	−6.2e−46	−2.0e−126	−1.2e−38	0
f_7	IT	3	3	3	3
$x_0 = 2$	$f(x_*)$	2.8e−66	1.6e−71	1.0e−78	2.2e−68
f_8	IT	6	5	5	101
$x_0 = 4$	$f(x_*)$	0	−4.0e−72	−2.1e−96	−5.5e20
f_9	IT	5	4	4	4
$x_0 = 1$	$f(x_*)$	2.0e−111	−1.1e−52	−3.8e−82	1.8e−84
f_{10}	IT	4	4	4	4
$x_0 = 2.1$	$f(x_*)$	−1.9e−124	0	−3.3e−76	−2.1e−121
f_{11}	IT	4	4	4	4
$x_0 = 1$	$f(x_*)$	6.2e−90	2.2e−109	0	−9.3e−45
f_{12}	IT	3	3	3	3
$x_0 = 0.5$	$f(x_*)$	2.4e−78	1.1e−86	1.5e−87	1.5e−78
f_{13}	IT	4	4	4	4
$x_0 = 1$	$f(x_*)$	−4.3e−32	−1.3e−69	−8.7e−86	−1.1e−34
f_{14}	IT	4	4	4	4
$x_0 = -0.85$	$f(x_*)$	−2.5e−64	−8.6e−116	−4.6e−124	−1.7e−73
f_{15}	IT	3	3	3	3
$x_0 = 1.2$	$f(x_*)$	−1.9e−36	−6.1e−35	−7.6e−40	−2.7e−41

Table 3
The parameters for each member. The first 2 belong to OM and the last 6 are OMN members.

Case	g	γ	C
1	41/36	2	−2/3
2	41/36	−1	−2/3
3	−4	0	−4
4	−4	0	0
5	0	−4	0
6	0	0	−4
7	0	0	0
8	2.9	−4	−4

where

$$H_3(\tau_n) = f(x_n) + f'(x_n)\frac{(\tau_n - y_n)^2(\tau_n - x_n)}{(y_n - x_n)(x_n + 2y_n - 3\tau_n)} + f'(\tau_n)\frac{(\tau_n - y_n)(x_n - \tau_n)}{x_n + 2y_n - 3\tau_n} - f[x_n, y_n]\frac{(\tau_n - x_n)^3}{(y_n - x_n)(x_n + 2y_n - 3\tau_n)}. \tag{25}$$

- An eighth order (N8) optimal method proposed by Neta [6] and based on King's fourth order optimal method [7] with $\beta = 2$ given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ \tau_n &= y_n - \frac{f(y_n)}{f'(x_n)}\frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} + \gamma f^2(x_n) - \rho f^3(x_n), \end{aligned} \tag{26}$$

Table 4
Number of EFPs, minimum and maximum values of the absolute value of the real parts of EFPs.

Case	Number of EFPs	Min. value	Max. value
1	54	5.06e−2	0.507
2	54	8.6e−3	0.353
3	48	9.7e−5	3.038
4	48	1.e−8	0.838
5	48	2.85e−2	0.524
6	48	1.85e−2	3.040
7	48	1.96e−3	0.495
8	54	0.161	11.364

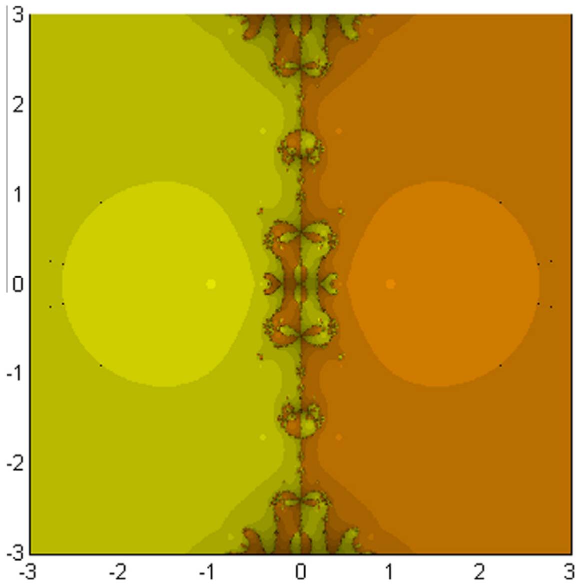

Fig. 1. Our method with $g = 41/36,\ \gamma = 2,\ C = -2/3,\ \alpha = 1$, and $A = -3$ for the roots of the polynomial $z^2 - 1$.

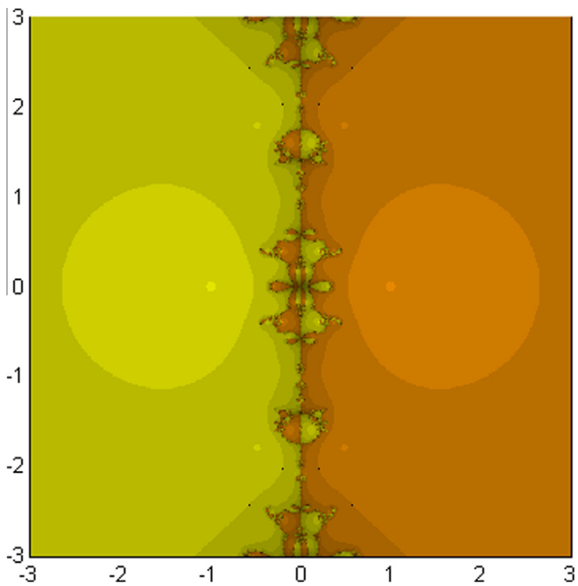

Fig. 2. Our method with $g = 41/36,\ \gamma = -1,\ C = -2/3,\ \alpha = 2$, and $A = 1$ for the roots of the polynomial $z^2 - 1$.

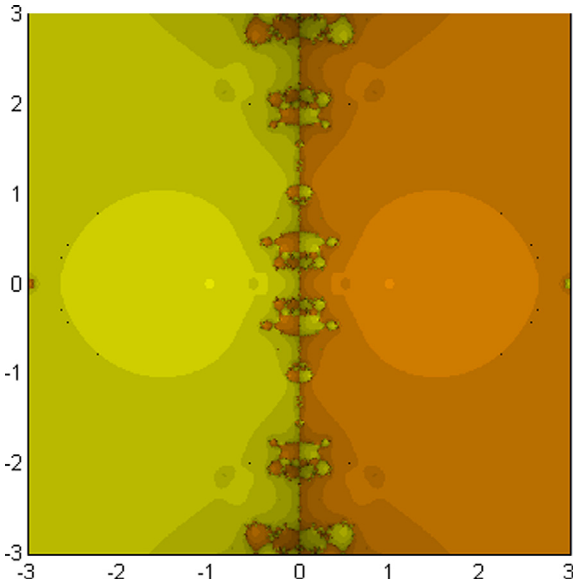

Fig. 3. Our method with $g = -4,\ \gamma = 0,\ C = -4$, and any α and A for the roots of the polynomial $z^2 - 1$.

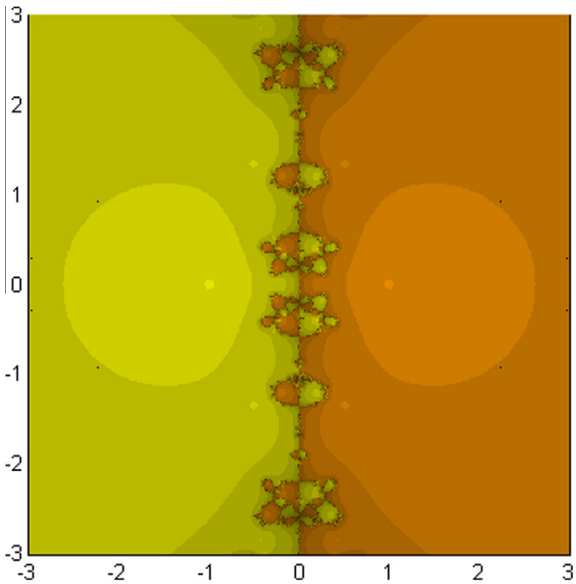

Fig. 4. Our method with $g = -4,\ \gamma = 0,\ C = 0$, and any α and A for the roots of the polynomial $z^2 - 1$.

where

$$\begin{aligned} &\rho = \frac{\phi_y - \phi_t}{F_y - F_t}, \quad \gamma = \phi_y - \rho F_y, \quad F_y = f(y_n) - f(x_n), \quad F_t = f(\tau_n) - f(x_n), \\ &\phi_y = \frac{y_n - x_n}{F_y^2} - \frac{1}{F_y f'(x_n)}, \quad \phi_t = \frac{\tau_n - x_n}{F_t^2} - \frac{1}{F_t f'(x_n)}. \end{aligned} \tag{27}$$

- A weight function based eighth order (WM8) optimal method [2] (using the fourth order Maheshwari's method [8]) given by

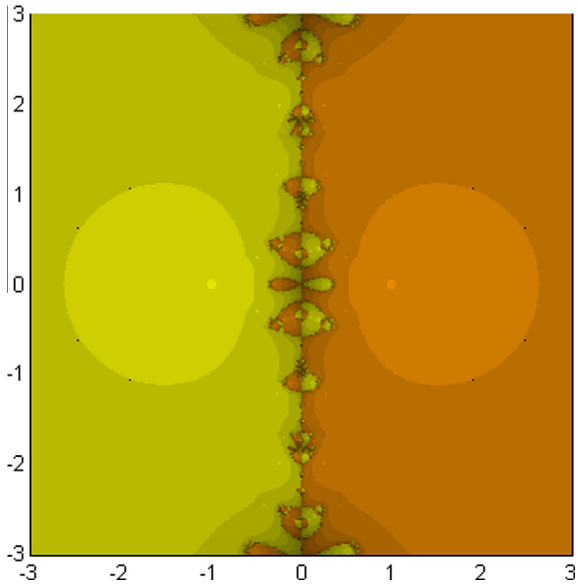

Fig. 5. Our method with $g = 0,\ \gamma = -4,\ C = 0$, and any α and A for the roots of the polynomial $z^2 - 1$.

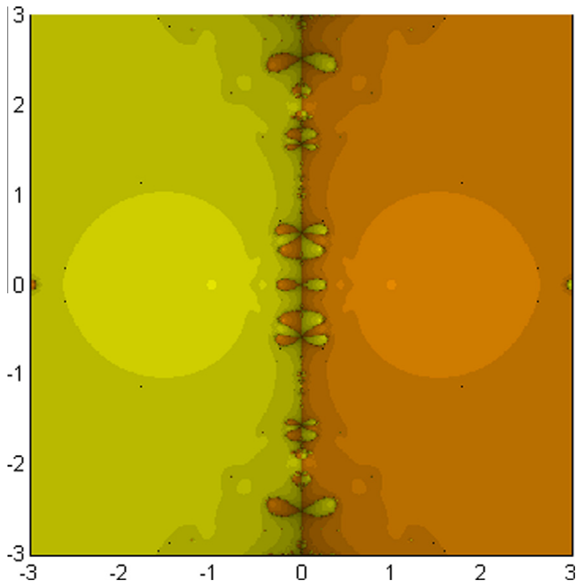

Fig. 6. Our method with $g = 2.9,\ \gamma = -4,\ C = -4$, and any α and A for the roots of the polynomial $z^2 - 1$.

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
\tau_n &= x_n - \left[\left(\frac{f(y_n)}{f(x_n)}\right)^2 - \frac{f(x_n)}{f(y_n) - f(x_n)}\right]\frac{f(x_n)}{f'(x_n)},\\
x_{n+1} &= \tau_n - \left[\phi\left(\frac{f(y_n)}{f(x_n)}\right) + \frac{f(\tau_n)}{f(y_n) - af(\tau_n)} + \frac{4f(\tau_n)}{f(x_n)}\right]\frac{f(\tau_n)}{f'(x_n)},
\end{aligned}
\tag{28}
$$

where ϕ is an arbitrary real function satisfying the conditions

$$\phi(0) = 1, \quad \phi'(0) = 2, \quad \phi''(0) = 4, \quad \phi'''(0) = -6.$$

We have taken $a = 1$ and $\phi(t) = 1 + 2t + 2t^2 - t^3$.

Table 5
Average number of iterations per point.

Case	Ex1	Ex2	EX3	Ex4	Ex5	Total
1	3.2708	4.7640	7.5009	6.0247	11.6231	33.1835
2	3.1197	4.3875	6.6839	5.8369	10.7186	30.7466
3	3.2751	4.3391	5.6630	5.6329	8.2313	27.1414
4	3.1885	4.1557	5.4185	5.8138	8.5328	27.1093
5	3.1163	4.3463	6.0809	5.5301	9.9465	29.0201
6	3.2751	4.3391	5.6630	5.6329	8.2313	27.1414
7	3.1885	4.1557	5.4185	5.8138	8.5328	27.1093
8	3.2452	5.0434	8.0473	5.9350	11.9265	34.1974

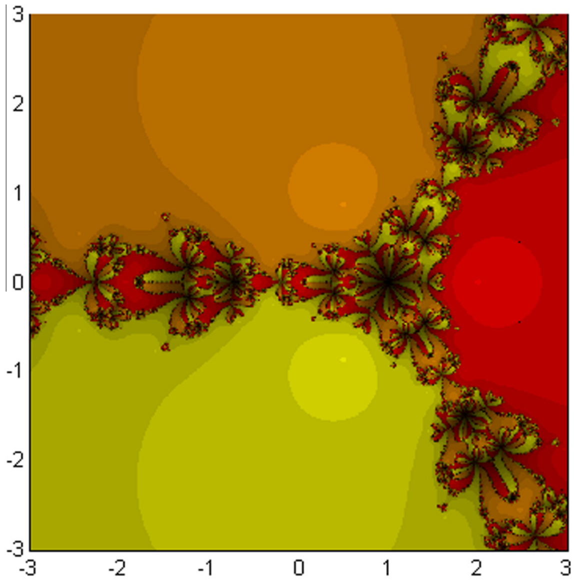

Fig. 7. Our method with $g = 41/36$, $\gamma = 2$, $C = -2/3$, $\alpha = 1$, and $A = -3$ for the roots of the polynomial $(z-1)^3 - 1$.

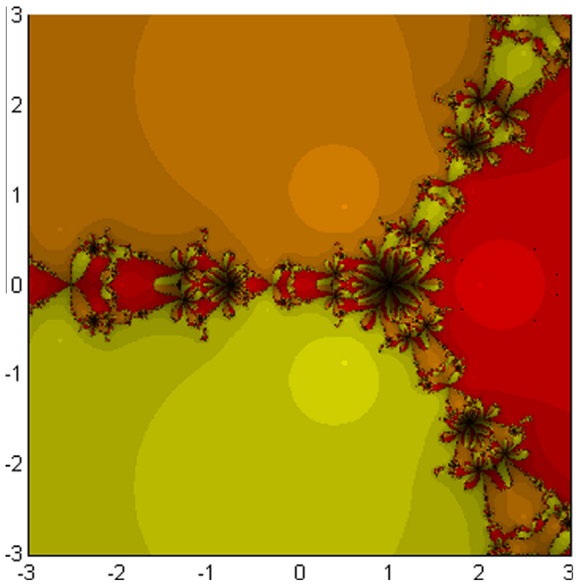

Fig. 8. Our method with $g = 41/36$, $\gamma = -1$, $C = -2/3$, $\alpha = 2$, and $A = 1$ for the roots of the polynomial $(z-1)^3 - 1$.

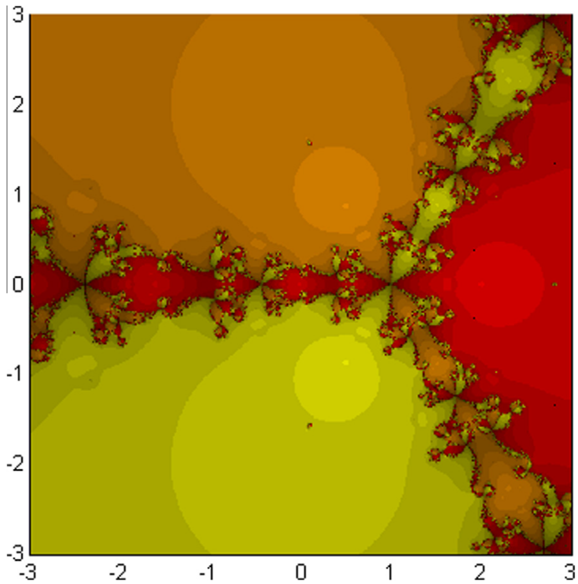

Fig. 9. Our method with $g = -4,\ \gamma = 0,\ C = -4$, and any α and A for the roots of the polynomial $(z-1)^3 - 1$.

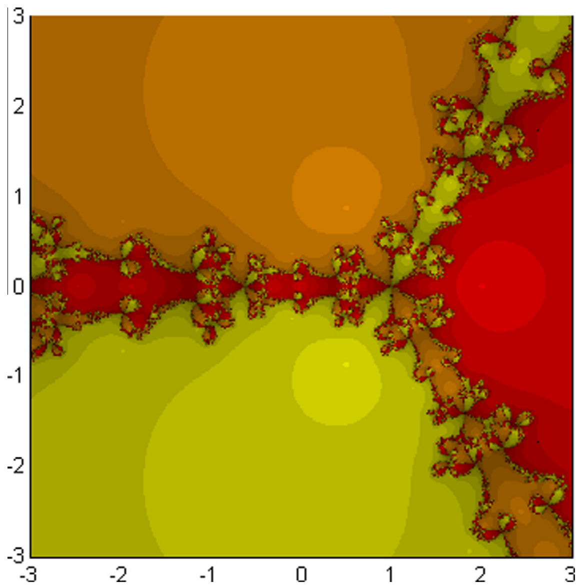

Fig. 10. Our method with $g = -4,\ \gamma = 0,\ C = 0$, and any α and A for the roots of the polynomial $(z-1)^3 - 1$.

In Table 1 we presented the results for KT8 (21), HKT8 (22), HKN8 (24), N8 (26) with $\beta = 2$, WM8 (28) and our new methods OM (OM1 is with $\alpha = 1, A = -3, \gamma = 2$ and OM2 is with $\alpha = 2,\ A = 1,\ \gamma = -1$). In Table 2 we also presented the results for our methods OMN1 (case 3, see Table 3), OMN2 (case 4), OMN3 (case 5) and OMN6 (case 8). The number of iterations *IT* required to converge is given along with the value of the function at the last iteration $f(x_*)$. It can be observed that for most of the considered test functions our methods show as good performance as the other methods in their convergence speed and also have reasonable smallness of the residuals. In fact, in one case (f_8) our methods converged even though WM8 diverged. We also found that in that example OMN6 diverged even though the other methods converged. We will show later that OMN6 is not a good choice. Therefore we can conclude that the new methods (OM1, OM2, OMN1–OMN3) are competitive with other eighth-order schemes being considered for solving nonlinear equations.

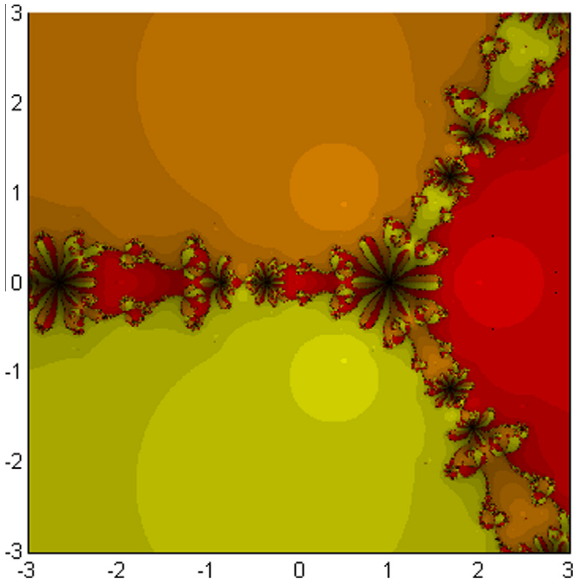

Fig. 11. Our method with $g = 0,\ \gamma = -4,\ C = 0$, and any α and A for the roots of the polynomial $(z-1)^3 - 1$.

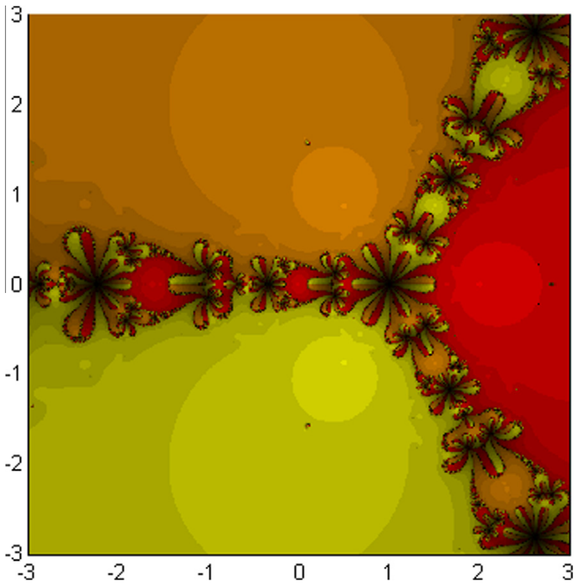

Fig. 12. Our method with $g = 2.9,\ \gamma = -4,\ C = -4$, and any α and A for the roots of the polynomial $(z-1)^3 - 1$.

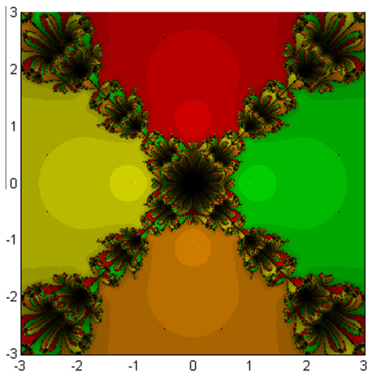

Fig. 13. Our method with $g = 41/36,\ \gamma = 2,\ C = -2/3,\ \alpha = 1$, and $A = -3$ for the roots of the polynomial $z^4 - 1$.

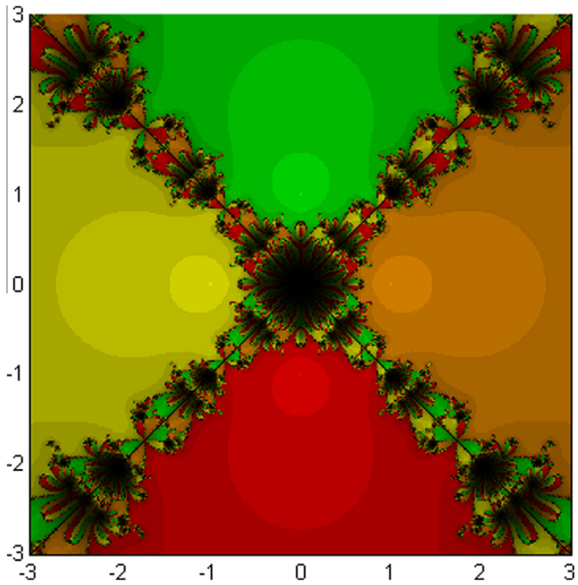

Fig. 14. Our method with $g = 41/36,\ \gamma = -1,\ C = -2/3,\ \alpha = 2$, and $A = 1$ for the roots of the polynomial $z^4 - 1$.

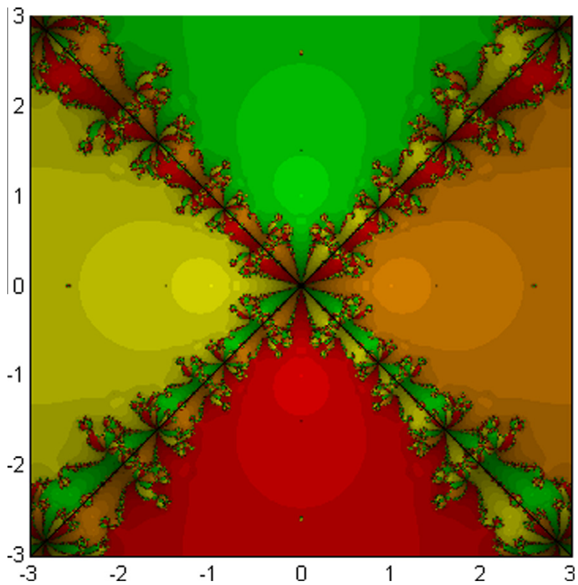

Fig. 15. Our method with $g = -4,\ \gamma = 0,\ C = -4$, and any α and A for the roots of the polynomial $z^4 - 1$.

In the next section, we analyze the basin of attraction of our eighth order family of methods to find out what is the best choice for the parameters. The idea of using basins of attraction was initiated by Stewart [9] and followed by the works of Amat et al. [10–13], Scott et al. [14], Chun et al. [15], Chicharro et al. [16], Cordero et al. [17], Neta et al. [18] and Chun et al. [19]. The only papers comparing basins of attraction for methods to obtain multiple roots is due to Neta et al. [20] and Neta and Chun [21–23].

4. Basins of attraction

In this section we give the basins of attraction of various members of the families OM and OMN. The 8 members are listed with their parameters in Table 3. The first 2 cases are of OM type and the last 6 cases are of OMN type.

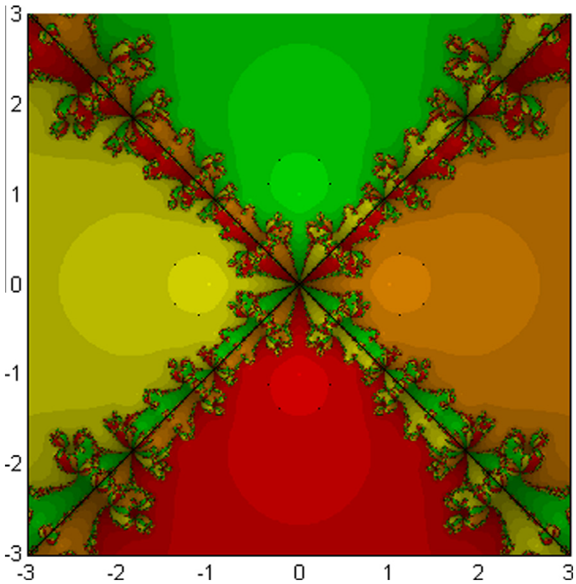

Fig. 16. Our method with $g = -4,\ \gamma = 0,\ C = 0$, and any α and A for the roots of the polynomial $z^4 - 1$.

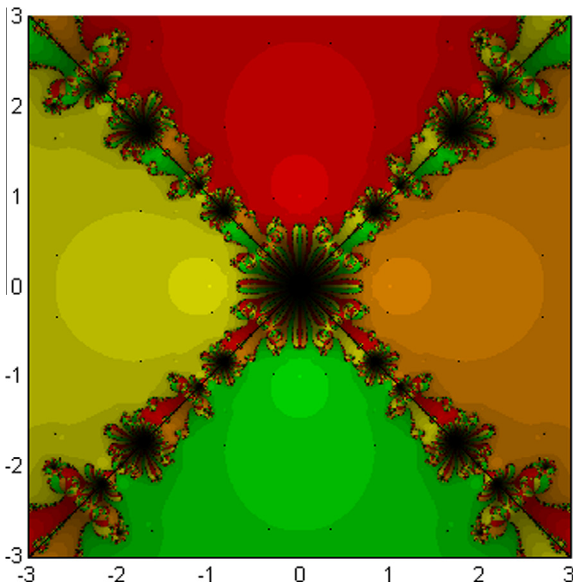

Fig. 17. Our method with $g = 0,\ \gamma = -4,\ C = 0$, and any α and A for the roots of the polynomial $z^4 - 1$.

The first 2 cases were chosen so that we annihilate two terms in the error constant. The first of those was arbitrarily picked. In order to understand the choice of the parameters in the other cases we discuss the extraneous fixed points. In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many multipoint iterative methods have fixed points that are not zeros of the function of interest. Thus, it is imperative to investigate the number of extraneous fixed points, their location and their properties. In the family of methods described in this paper, the parameters $g,\ \gamma$, and C can be chosen to position the extraneous fixed points on or close to the imaginary axis. This idea is due to Neta et al. [18] where they have shown an improvement in King's method by choosing the parameter that will position the extraneous fixed points on the imaginary axis. The second case was chosen so that we also have the extraneous fixed points as close as possible to the imaginary axis. Similarly, in the next 5 cases we have chosen the extraneous fixed

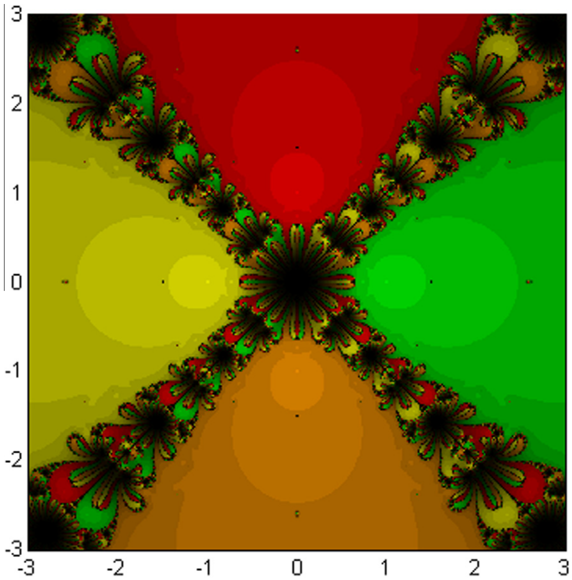

Fig. 18. Our method with $g = 2.9,\ \gamma = -4,\ C = -4$, and any α and A for the roots of the polynomial $z^4 - 1$.

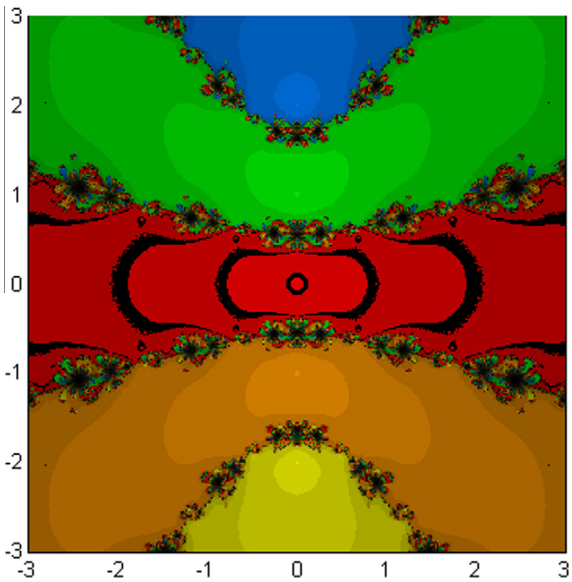

Fig. 19. Our method with $g = 41/36,\ \gamma = 2,\ C = -2/3,\ \alpha = 1$, and $A = -3$ for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.

points to be close to the imaginary axis. The last case violates that condition and one can see later that this violation increases the average number of iterations.

The eighth order family of methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n), \tag{29}$$

where

$$H_f(x_n) = 1 + \frac{r_n}{(1 - r_n)^2} + \frac{t_n}{(1 - H(r_n)J(t_n)P(q_n))^2}. \tag{30}$$

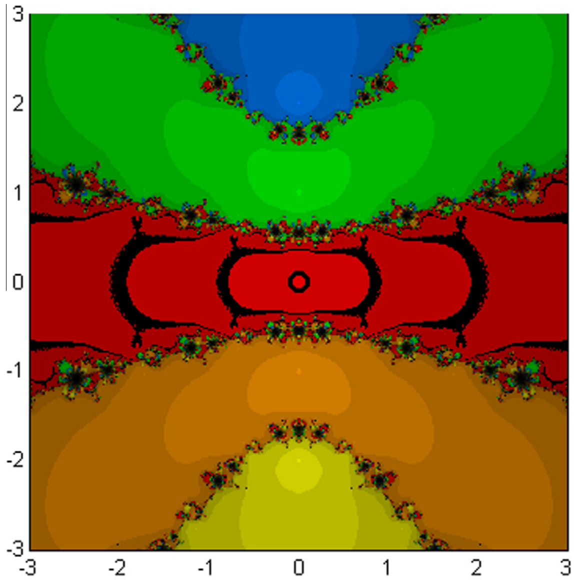

Fig. 20. Our method with $g = 41/36,\ \gamma = -1,\ C = -2/3,\ \alpha = 2$, and $A = 1$ for the roots of the polynomial $z(z^2+1)(z^2+4)$.

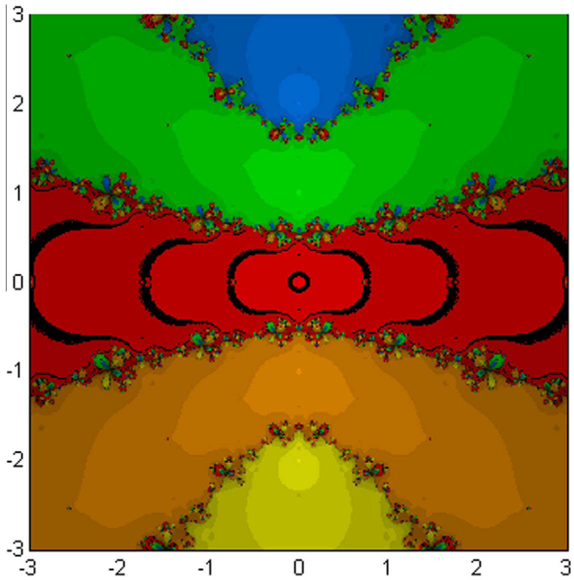

Fig. 21. Our method with $g = -4,\ \gamma = 0,\ C = -4$, and any α and A for the roots of the polynomial $z(z^2+1)(z^2+4)$.

Clearly the root ξ of $f(x)$ is a fixed point of the method. The points $\alpha \neq \xi$ at which $H_f(\alpha) = 0$ are also fixed points of the family. We have searched the parameter space and found that there are no point on the imaginary axis, but there are 5 cases with the smallest real part (these are denoted cases 3–7). To convince that this is a good choice we have taken a case where the real part of the extraneous roots is the largest (case 8). In Table 4 we have listed the number of extraneous fixed points (EFPs) and range of absolute value of the real parts of the EFPs for each case.

Example 1. In our first example we have used the polynomial $z^2 - 1$. The basins of attraction are given in Figs. 1–6. All the results are good. The best one is case 5 (Fig. 5) and the worst is case 3 (Fig. 3). The difference is barely noticeable. One can see it only when computing the average number of iterations per point (see Table 5.) All cases require between 3.1 and 3.3 iterations per point. Notice that cases 6 and 7 were not shown since we found that they yield identical results to cases 3 and 4, respectively.

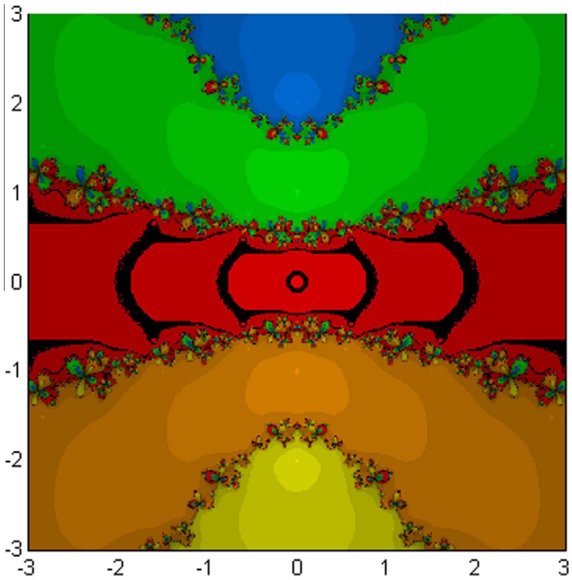

Fig. 22. Our method with $g = -4,\ \gamma = 0,\ C = 0$, and any α and A for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.

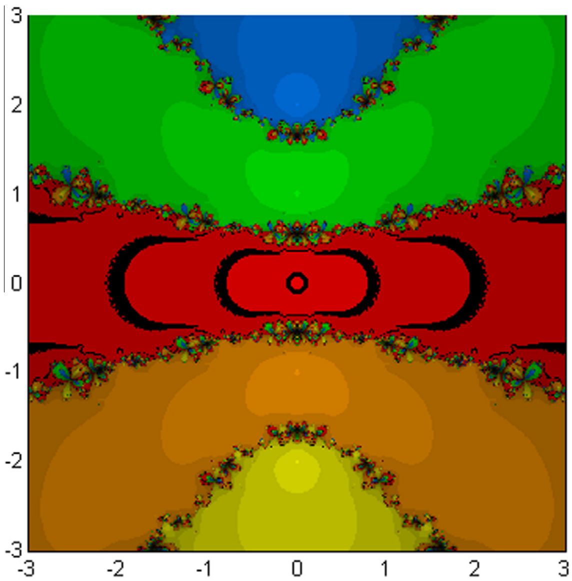

Fig. 23. Our method with $g = 0,\ \gamma = -4,\ C = 0$, and any α and A for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.

Example 2. In our next example we used the polynomial

$$p_2(z) = (z - 1)^3 - 1.$$

The basins are given in Figs. 7–12. In this case the best performer is case 4 (Fig. 10) and the worst is case 8 (Fig. 12).

Example 3. In our third example we have taken the polynomial

$$p_3(z) = z^4 - 1.$$

The basins are given in Figs. 13–18. Again the best performer is case 4 (Fig. 16) and the worst is case 8 (Fig. 18).

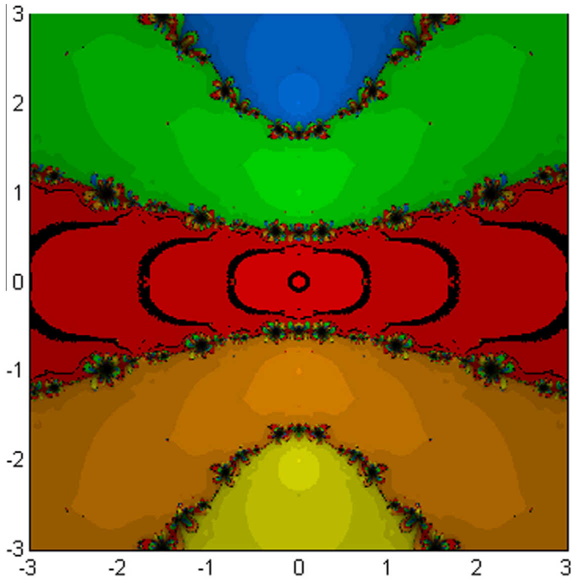

Fig. 24. Our method with $g = 2.9,\ \gamma = -4,\ C = -4$, and any α and A for the roots of the polynomial $z(z^2 + 1)(z^2 + 4)$.

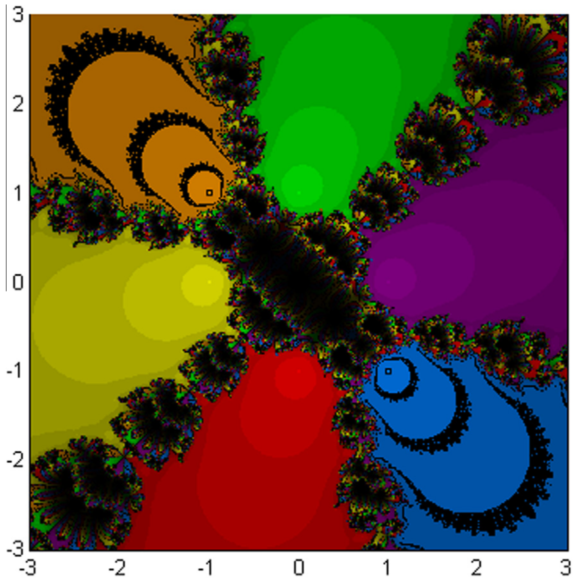

Fig. 25. Our method with $g = 41/36,\ \gamma = 2,\ C = -2/3,\ \alpha = 1$, and $A = -3$ for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

Example 4. In our fourth example we have taken the polynomial

$$p_4(z) = z(z^2 + 1)(z^2 + 4),$$

whose roots are $0,\ \pm i,\ \pm 2i$. The basins are given in Figs. 19–24. In this example the best is case 5 (Fig. 23) and the worst is case 1 (Fig. 19).

Example 5. In our last example we have taken the polynomial

$$p_5(z) = (z^2 - 1)(z^2 + 1)(z^2 + 2i)$$

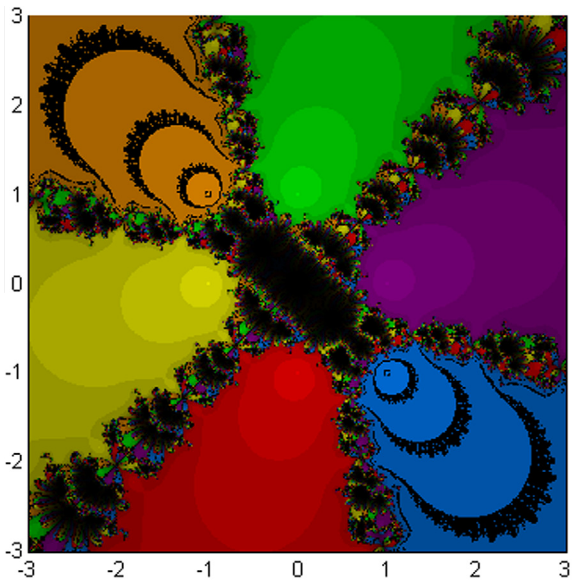

Fig. 26. Our method with $g = 41/36,\ \gamma = -1,\ C = -2/3,\ \alpha = 2$, and $A = 1$ for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

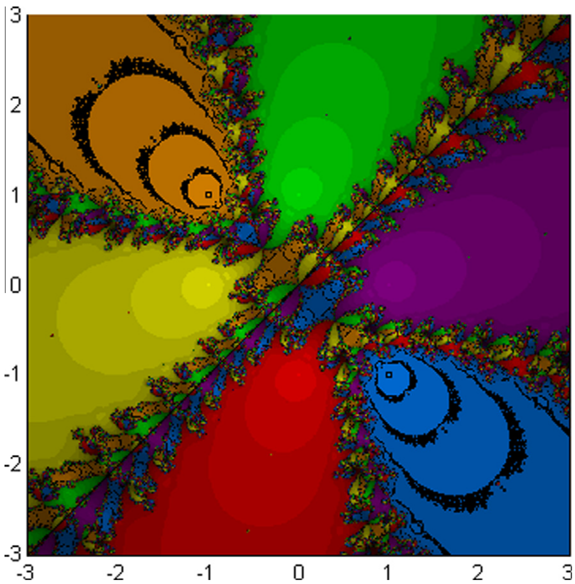

Fig. 27. Our method with $g = -4,\ \gamma = 0,\ C = -4$, and any α and A for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

whose roots are $\pm 1,\ \pm i,\ -1 + i$, and $1 - i$. The basins are given in Figs. 25–30. The best is case 3 (Fig. 28) and the worst is case 8 (Fig. 30).

Based on Table 5, the best case overall is case 4 for which $g = -4$ and $C = \gamma = 0$ and the worst is case 8 for which $g = 2.9$ and $C = \gamma = -4$. In general cases 3–5 (OMN) are better than cases 1–2 (OM). Case 8 is also OMN but there we picked parameters that lead to largest (in absolute value) real part of the extraneous fixed points. On the other hand, cases 3–5 have the smallest real part.

Remarks.

(1) Note that cases 3 and 6 yield identical results. Similarly cases 4 and 7 are identical. For this reason, we have not shown the results for cases 6 and 7.

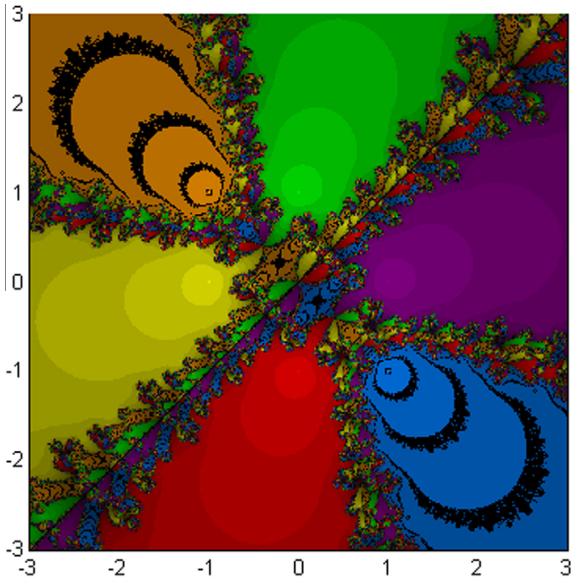

Fig. 28. Our method with $g = -4,\ \gamma = 0,\ C = 0$, and any α and A for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

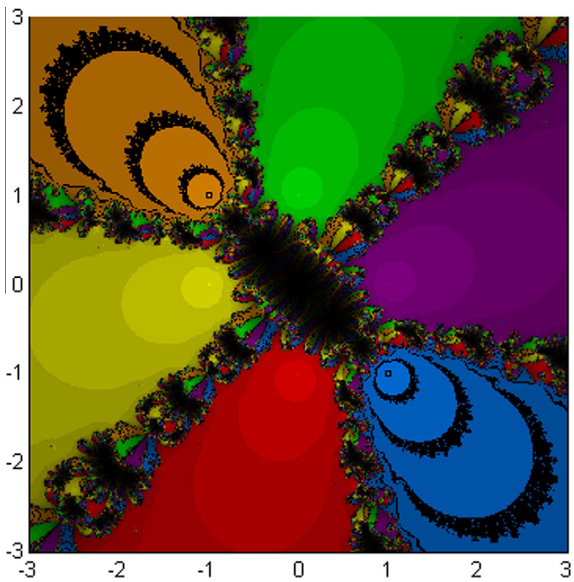

Fig. 29. Our method with $g = 0,\ \gamma = -4,\ C = 0$, and any α and A for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

(2) Out of the first 2 cases, were we annihilated two terms in the error constant, the best is case 2 were we also chose the extraneous fixed points to be closest to the imaginary axis.

5. Conclusions

In this paper we have developed a new family of optimal eighth order iterative method. The scheme is optimal in the sense that it satisfies the Kung–Traub conjecture. We have compared several members of our family to existing optimal eighth order schemes and found that some members of our family are competitive. We have shown how to choose the parameters of the family to find the best members by evaluating all the extraneous fixed points. We have shown that the

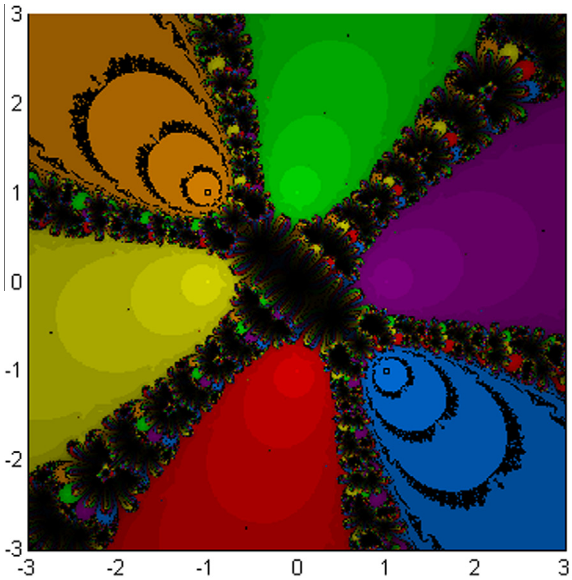

Fig. 30. Our method with $g = 2.9,\ \gamma = -4,\ C = -4$, and any α and A for the roots of the polynomial $(z^2 - 1)(z^2 + 1)(z^2 + 2i)$.

best members have extraneous fixed point close to the imaginary axis. One member (case 8) for which this is not true was the worst perform.

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