The set of autotopisms of partial Latin squares

R. M. Falcón

School of Building Engineering, University of Seville, Avda. Reina Mercedes 4 A, 41012 - Seville, Spain.

Abstract

Symmetries of a partial Latin square are primarily determined by its autotopism group. Analogously to the case of Latin squares, given an isotopism Θ, the cardinality of the set $\mathcal{PLS}_\Theta$ of partial Latin squares which are invariant under Θ only depends on the conjugacy class of the latter, or, equivalently, on its cycle structure. In the current paper, the cycle structures of the set of autotopisms of partial Latin squares are characterized and several related properties studied. It is also seen that the cycle structure of Θ determines the possible sizes of the elements of $\mathcal{PLS}_\Theta$ and the number of those partial Latin squares of this set with a given size. Finally, it is generalized the traditional notion of partial Latin square completable to a Latin square.

Keywords: Partial Latin square, autotopism group, cycle structure.
05B15, 20N05, 20D45

1. Introduction.

Every permutation π of the symmetric group S_n can be uniquely decomposed into product of disjoint cycles. Let n_π be the number of these cycles. The numbers λ_i^π of cycles of length i in this decomposition determine its *cycle structure* as the expression $z_\pi = n^{\lambda_n^\pi} \cdot \ldots \cdot 1^{\lambda_1^\pi}$, where any term of the form i^0 is omitted and any term of the form i^1 is replaced by i. The cardinality of the set $\mathcal{CS}_n$ of possible cycle structures of S_n is equal to the number $p(n)$ of partitions of n. Two permutations are conjugate if and only if they have the same cycle structure. Given $\pi \in S_n$, let λ_π and π_∞ be respectively its length and the union of its 1-cycles written in natural order. Hereafter, we suppose π to be represented by following its decomposition into a product $\pi_1 \ldots \pi_{n_\pi}$ of disjoint cycles in order of decreasing length, where each cycle π_i is written as $(p_{i,1} \ldots p_{i,\lambda_{\pi_i}})$, with $p_{i,1} = \min_j\{p_{i,j}\}$ and where $p_{i,1} < p_{j,1}$ whenever $i < j$

and $\lambda_{\pi_i} = \lambda_{\pi_j}$. Finally, given $a \in [n] = \{1, \ldots, n\}$, we write $a \in \pi_i$ if there exists $j \in [\lambda_{\pi_i}]$ such that $a = p_{i,j}$. Analogously, $a \in \pi_\infty$ means $\pi(a) = a$.

A *Latin square* of order n is an $n \times n$ array with elements chosen from a set of n distinct symbols such that each symbol occurs precisely once in each row and each column. Hereafter, $[n]$ is assumed to be this set of symbols and $\mathcal{LS}_n$ denotes the set of Latin squares of order n. Given $L = (l_{rc}) \in \mathcal{LS}_n$, its *orthogonal representation* $O(L)$ is the set of n^2 triples $\{(r, c, l_{rc})\colon\ r, c \in [n]\}$ defined by the rows r, columns c and symbols l_{rc} of L. This set satisfies the *Latin square condition*, that is, given two triples of $O(L)$ which coincide in two components, then the third component is also the same. Given $\pi \in S_3$, it is defined the Latin square L^π such that $O(L^\pi) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)})\colon\ (l_1, l_2, l_3) \in O(L)\}$, which is said to be *parastrophic* to L. Permutations of rows, columns and symbols also give rise to new Latin squares. Specifically, given three permutations α, β, γ of the symmetric group S_n, the triple $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n = S_n^3$ is called an *isotopism* and L^Θ is said to be *isotopic* to L, where $O(L^\Theta) = \{(\alpha(r), \beta(c), \gamma(s))\colon\ (r, c, s) \in O(L)\}$. To be isotopic is an equivalence relation, which will be denoted by $\sim$, and the set of Latin squares isotopic to L is its *isotopism class* $[L]$. The number of Latin squares and isotopism classes of $\mathcal{LS}_n$ are known for $n \leq 11$ [24, 18]. A list of representatives of isotopism classes for $n \leq 8$ is given in [25]. The *cycle structure* of Θ is the triple $z_\Theta = (z_\alpha, z_\beta, z_\gamma)$. Hereafter, given a subset $S \subseteq \mathfrak{I}_n$, $\mathcal{CS}_S$ denotes the set of cycle structures of the elements of S. Given $z = (z_1, z_2, z_3) \in \mathcal{CS}_S$, where $z_i = n^{z_{in}} \cdot \ldots \cdot 1^{z_{i1}}$, then n_{z_i} denotes the number of cycles of z_i, that is, $n_{z_i} = \sum_{j \in [n]} z_{ij}$. Finally, the *parastrophic class* of z is the set $[z] = \{z^\pi = (z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)})\colon\ \pi \in S_3\}$.

If $L^\Theta = L$, then Θ is said to be an *autotopism* of L. If $\alpha = \beta = \gamma$, then Θ is an *automorphism* of L and $\Theta = \alpha$ is written instead of (α, α, α). Let $\mathcal{LS}_\Theta, \Delta(\Theta)$, $\mathfrak{A}_n$ and $\mathcal{A}_n$ denote respectively the set of Latin squares which have Θ as an autotopism, its cardinality and the sets of autotopisms and automorphisms of at least one Latin square of order n. Necessary conditions for an isotopism to be an autotopism were given in [26, 24, 10, 29] and $\mathcal{A}_n$ was studied in [32, 1, 19, 20, 29]. If $\pi \in S_3$ and $L \in \mathcal{LS}_\Theta$, then $L^\pi \in \mathcal{LS}_{\Theta^\pi}$, so permutations on the components of Θ preserve $\Delta(\Theta)$. Moreover, this cardinality only depends on the conjugacy class of Θ [10] or, equivalently, on its cycle structure, so we also denote it by $\Delta(z_\Theta)$. A classification of $\mathcal{CS}_{\mathfrak{A}_n}$ is known for all $n \leq 17$ [10, 29]. Given $z \in \mathcal{CS}_{\mathfrak{A}_n}$, let $\mathfrak{I}_z = \{\Theta \in \mathfrak{I}_n\colon\ z_\Theta = z\}$ and $\mathcal{LS}_z = \bigcup_{\Theta \in \mathfrak{I}_z} \mathcal{LS}_\Theta$.

An *incidence structure* is a triple (P, B, I), where P and B respectively are finite sets of *points* and *blocks* and $I \subseteq P \times B$ is an *incidence relation*. It is r*-uniform* if every block contains exactly r points and it is s*-regular* if every point is in exactly s blocks. Two blocks are *equivalent* if they contain the same set of points and the *multiplicity* of a block is the size of its equivalence class. In the study of Latin squares, it can be defined a natural incidence relation I_n between $\mathcal{LS}_n$ and $\mathfrak{I}_n$, where, given $L \in \mathcal{LS}_n$ and $\Theta \in \mathfrak{I}_n$, then $(L, \Theta) \in I_n$ if and only if $L \in \mathcal{LS}_\Theta$. Hence, given $z \in \mathcal{CS}_{\mathfrak{A}_n}$, the triple $(\mathcal{LS}_z, \mathfrak{I}_z, I_n)$ is a $\Delta(z)$-uniform incidence structure such that every block have the same multiplicity [9]. Moreover, given $L \in \mathcal{LS}_n$, the triple $([L], \mathfrak{I}_z, I_n)$ is a uniform and regular incidence structure, where every block contains $\Delta_{[L]}(z)$ elements, whose exact value is known for order up to 6.

Although a general expression for the values of $\Delta(z)$ and $\Delta_{[L]}(z)$ remains unknown, some general and explicit formulas were given for the former [22, 23, 7, 27] and Gröbner bases were used to know its exact value for all autotopisms of Latin squares of order up to 7 [12]. For higher orders, Gröbner bases have problem with the exponential growth of data storage and the time of computation, for which the use of new combinatorial tools seems to be the key. Thus, for example, given $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$, Gröbner bases were used in [13] to obtain the value of $\Delta(z)$ for the majority of the cycle structures of autotopisms of $\mathfrak{A}_8$ and $\mathfrak{A}_9$, by solving the linear equation system formed after adding the constraints $x_{rcs} = x_{\alpha(r)\beta(c)\gamma(s)}$, for all $r, c, s \in [n]$, to those related to the *planar 3-index assignment problem* [6]:

$$\begin{array}{ll} \min \sum_{r,c,s\in[n]} w_{rcs} \cdot x_{rcs}, & \\ \quad \text{subject to} \quad & \sum_{r\in[n]} x_{rcs} = 1, \forall c, s \in [n], \\ & \sum_{c\in[n]} x_{rcs} = 1, \forall r, s \in [n], \\ & \sum_{s\in[n]} x_{rcs} = 1, \forall r, c \in [n], \\ & x_{rcs} \in \{0, 1\}, \forall r, c, s \in [n], \end{array} \qquad (3PAP_n)$$

where w_{rcs} are real weights for all $r, c, s \in [n]$ and whose set of feasible solutions are in $1-1$ correspondence with $\mathcal{LS}_n$ if we define the Latin square $L = (l_{rc})$ such that $l_{rc} = s$ if and only if $x_{rcs} = 1$.

All the previous concepts can be naturally extended to *partial Latin squares*, which are square arrays in which each cell is either empty or contains one element chosen from a set of n symbols, such that each symbol occurs at most once in each row and in each column. The *size* of a partial Latin square P is the number of its non-blank cells and is denoted by

$|P|$. Let $\mathcal{PLS}_n$ and $\mathcal{PLS}_{n,s}$ denote respectively the set of non-empty partial Latin squares of order n and its subset of arrays of size s. An upper bound of the elements of $\mathcal{PLS}_{n,s}$ is given in [14]. The *orthogonal representation* of $P \in \mathcal{PLS}_n$ is the set $O(P)$ of $|P|$ triples related to the non-blank cells of P. Parastrophic partial Latin squares have therefore the same size. Given $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n$, it is defined the partial Latin square P^Θ such that $O(P^\Theta) = \{(\alpha(r), \beta(c), \gamma(s))\colon (r,c,s) \in O(P)\}$, which is said to be *isotopic* to P and $[P]$ denotes its *isotopism class*. Thus, $|P^\Theta| = |P|$. Θ is said to be an *autotopism* of P if $P^\Theta = P$. Let $\mathfrak{A}_{\mathcal{P}n}$ and $\mathcal{PLS}_\Theta$ denote respectively the set of autotopisms of at least one partial Latin square of order n and that of non-empty partial Latin squares which have Θ as an autotopism. Besides, given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, $\mathcal{PLS}_z$ denotes the set $\bigcup_{\Theta \in \mathfrak{I}_z} \mathcal{PLS}_\Theta$.

A partial Latin square $P \in \mathcal{PLS}_n$ can be *completed* to a Latin square $L \in \mathcal{LS}_n$ if $O(P) \subseteq O(L)$. Given $\Theta \in \mathcal{CS}_{\mathfrak{A}n}$, the subset of $\mathcal{LS}_\Theta$ of Latin squares to which P can be completed is denoted by $\mathcal{LS}_{\Theta,P}$. The computation of $\Delta(z)$ can be then simplified [12] if a multiplicative factor $c_P \in \mathbb{N}$ is found such that $\Delta(z) = c_P \cdot |\mathcal{LS}_{\Theta,P}|$. Although this factor, which is called *P-coefficient of symmetry of* Θ, becomes crucial in the processing of high orders, no exhaustive study has been developed in this regard. Indeed, a comprehensive analysis of $\mathfrak{A}_{\mathcal{P}n}$ and $\mathcal{PLS}_\Theta$ has not been properly done until now.

The present paper deals with this last question. It is organized as follows: In Section 2, the set $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ is characterized and several related results exposed. In Section 3, given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, it is dealt with the possible sizes of a partial Latin square $P \in \mathcal{PLS}_z$. In Section 4, analogously to the case of Latin squares, it is proven that the number of partial Latin squares related to an autotopism only depends on the cycle structure of the latter, in such a way that the elements of $\mathcal{PLS}_n$ and $\mathfrak{I}_n$ can be respectively considered as points and blocks of incidence structures whose uniformity and regularity are studied. Moreover, new constraints are imposed to the $3PAP_n$ in order to obtain the set $\mathcal{PLS}_{z,s}$ of partial Latin squares of size $s \in [n]$ related to an autotopism of cycle structure $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$. Besides, by using Gröbner bases, its cardinality $\Delta_s(z)$ is obtained for $n \leq 4$. In Section 5, a theoretical ground for the coefficient of symmetry is exposed. Specifically, given $\Theta \in \mathfrak{I}_n$, it is studied the set of partial Latin squares of $\mathcal{PLS}_\Theta$ which can be completed to Latin squares of $\mathcal{LS}_\Theta$. Finally, since the current paper has a high dependence on notation, a glossary of symbols is shown in Appendix A.

2. The set $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$.

Autotopisms of partial Latin squares are uniquely determined by their cycle structures:

Lemma 2.1. $\Theta \in \mathfrak{A}_{\mathcal{P}n}$ *if and only if* $z_\Theta \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$.

Proof. The necessary condition holds by definition of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$. Now, if $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, then there must exist $\Theta_0 \in \mathfrak{I}_z$ and $P \in \mathcal{PLS}_{\Theta_0}$. Hence, given $\Theta \in \mathfrak{I}_z$, Θ and Θ_0 are conjugate and therefore there exists $\Theta' \in \mathfrak{I}_n$ such that $\Theta = \Theta'\Theta_0\Theta'^{-1}$. As a consequence, $P^{\Theta'} \in \mathcal{PLS}_\Theta$ and $\Theta \in \mathfrak{A}_{\mathcal{P}n}$. □

Let us define the set:

$$\mathrm{LCM}_n = \{(i,j,k) \in [n]^3 \colon \operatorname{lcm}(i,j) = \operatorname{lcm}(i,k) = \operatorname{lcm}(j,k) = \operatorname{lcm}(i,j,k)\}.$$

The next result characterizes the set $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ and can be considered as an immediate generalization for partial Latin squares of the necessary condition given by Stones, Vojtěchovský and Wanless in [29] for membership in $\mathfrak{A}_n$:

Lemma 2.2. *Given* $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{I}_n}$*, we have* $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ *if and only if there exists* $(i,j,k) \in \mathrm{LCM}_n$ *such that* $z_{1i} \cdot z_{2j} \cdot z_{3k} > 0$.

Proof. If $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, then there must exist $\Theta = (\alpha,\beta,\gamma) \in \mathfrak{I}_z$ and $P \in \mathcal{PLS}_\Theta$. Given $(r,c,s) \in O(P)$, let $(u,v,w) \in [n_\alpha] \times [n_\beta] \times [n_\gamma]$ be such that $r \in \alpha_u$, $c \in \beta_v$ and $s \in \gamma_w$. Since Θ is an autotopism of P, it must be $(\alpha_u^t(r), \beta_v^t(c), \gamma_w^t(s)) \in O(P)$, for all $t \in \mathbb{N}$. The necessary condition is then a consequence of the Latin square condition, by considering i,j,k to be, respectively, the lengths of α_u, β_v and γ_w.

To prove the converse, let $\Theta = (\alpha,\beta,\gamma) \in \mathfrak{I}_z$ and let α_u, β_v and γ_w be, respectively, i-, j- and k-cycles of α, β and γ. Let r,c,s be, respectively, elements of α_u, β_v and γ_w. The set of triples $\{(\alpha_u^t(r), \beta_v^t(c), \gamma_w^t(s)) \colon t \in [\operatorname{lcm}(i,j,k)]\}$ satisfies the Latin square condition because of being $(i,j,k) \in \mathrm{LCM}_n$ and therefore it is the orthogonal representation of a partial Latin square $P \in \mathcal{PLS}_\Theta$. □

Given $n > 1$, Lemma 2.2 implies $\mathfrak{A}_{\mathcal{P}n}$ to be a proper subset of $\mathfrak{I}_n$, because, for instance, $(1^n, 1^n, n^1) \notin \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$. Analogously, $\mathfrak{A}_n$ is a proper subset of $\mathfrak{A}_{\mathcal{P}n}$, because, for example, $(2,2,2) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_2}}$ and $(2 \cdot 1^{n-2}, 2 \cdot 1^{n-2}, 1^n) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, for $n > 2$, but neither of them are cycle structures of an autotopism of Latin square. Thus, the next claim is verified:

Proposition 2.3. $\mathfrak{A}_n \subset \mathfrak{A}_{\mathcal{P}n} \subset \mathfrak{I}_n$, $\forall n > 1$. $\square$

Moreover, $\mathfrak{A}_{\mathcal{P}n}$ and $\mathfrak{I}_n$ have asymptotically the same size. To see it, it is enough to assure that the cardinalities of the sets of their cycle structures coincide in the limit, which is proven in Theorem 2.6. Although $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ can be explicitly obtained for any order $n \in \mathbb{N}$ by implementing Lemma 2.2 in a computer procedure, a lower bound of its cardinality is determined by studying the following sets which partition $\mathcal{CS}_n$:

$$\mathcal{CS}_{n,m} = \{n^{z_n} \cdot \ldots \cdot 1^{z_1} \in \mathcal{CS}_n \colon\ z_m > 0 \text{ and } z_i = 0, \forall i \in [m-1]\},$$

where $m \in [n]$. The following results hold:

Lemma 2.4. $|\mathcal{CS}_{n,m}| = \begin{cases} 1, & \textit{if } m = n, \\ 0, & \textit{if } m \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n-1\}, \\ p(n-m) - \sum_{i=1}^{m-1} |\mathcal{CS}_{n-m,i}|, & \textit{otherwise.} \end{cases}$

Proof. The cases $m \geq \lfloor \frac{n}{2} \rfloor + 1$ are straightforward verified. Let $m \leq \lfloor \frac{n}{2} \rfloor$. Given $z \in \mathcal{CS}_{n,m}$, it is $z_m > 0$ and therefore it must be $z_{n-m+i} = 0$, for all $i \in [m]$. Hence, we can define $z' \in \mathcal{CS}_{n-m}$ such that $z'_i = z$, for all $i \in [n-m] - \{m\}$ and $z'_m = z_m - 1$. Specifically, since $z \in \mathcal{CS}_{n,m}$, it must be $z'_i = 0$, for all $i \in [m-1]$. Thus, $z' \in \bigcup_{i=m}^{n-m} \mathcal{CS}_{n-m,i} = \mathcal{CS}_{n-m} - \bigcup_{i=1}^{m-1} \mathcal{CS}_{n-m,i}$ and the claim is verified. $\square$

Proposition 2.5. $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}| \geq \sum_{(i,j,k) \in \mathrm{LCM}_n} |\mathcal{CS}_{n,i}| \cdot |\mathcal{CS}_{n,j}| \cdot |\mathcal{CS}_{n,k}|$.

Proof. Since the sets $\mathcal{CS}_{n,m}$ constitute a partition of $\mathcal{CS}_n$, the result is consequence of Lemma 2.2. $\square$

Theorem 2.6. $\lim_{n\to\infty} \frac{|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}|}{|\mathcal{CS}_{\mathfrak{I}_n}|} = 1$.

Proof. Since $(1,1,1) \in \mathrm{LCM}_n$, Proposition 2.5 implies that $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}| \geq |\mathcal{CS}_{n,1}|^3$ and thus, from Lemma 2.4, $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}| \geq p(n-1)^3$. Now, since $p(n)$ is equivalent to $\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$ when n tends to infinity [17] and $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}| \leq |\mathcal{CS}_{\mathfrak{I}_n}| = p(n)^3$, it is verified that:

n	$\lvert\mathcal{CS}_{\mathfrak{A}n}\rvert$	$\lvert\mathcal{CS}_{n,m}\rvert$ (m)								$\lvert\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}\rvert$	$\lvert[\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}]\rvert$
		1	2	3	4	5	6	7	8		
1	1									1	1
2	4	1								5	3
3	6	2								15	7
4	19	3	1							65	22
5	8	5	1							223	60
6	45	7	2	1						869	197
7	12	11	2	1						2535	526
8	87	15	4	1	1					7663	1492
9	43	22	4	2	1					21156	3937
10	89	30	7	2	1	1				60264	10850
11	21	42	8	3	1	1				150953	26628
12	407	56	12	4	2	1	1			385538	66984
13	27	77	14	5	2	1	1			915452	157398
14	141	101	21	6	3	1	1	1		2193225	374127
15	150	135	24	9	3	2	1	1		4928696	836154
16	503	176	34	10	5	2	1	1	1	11209311	1893607
17	40	231	41	13	5	3	1	1	1	24406191	4110132

Table 1: Cardinality of the sets of cycle structures, for $n \leq 17$ and $m \leq \lfloor \frac{n}{2} \rfloor$.

$$1 \geq \lim_{n\to\infty} \frac{|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}|}{|\mathcal{CS}_{\mathfrak{I}n}|} \geq \lim_{n\to\infty} \frac{p(n-1)^3}{p(n)^3} = 1.$$

$\square$

For $n \leq 17$, Table 1 shows the values $|\mathcal{CS}_{n,m}|$ and $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}|$, where $m \leq \lfloor \frac{n}{2} \rfloor$, in comparison with those of $|\mathcal{CS}_{\mathfrak{A}n}|$, which can be obtained by using the classification given in [10, 29]. The number $|[\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}]|$ of parastrophic classes of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}$ is also shown.

3. The size of a partial Latin square related to an autotopism.

Given $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ and $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$, any partial Latin square $P \in \mathcal{PLS}_\Theta$ can be decomposed into $n_{z_1} \cdot n_{z_2}$ blocks P_{ij} whose rows and columns are respectively determined by the elements of the cycle α_i of α and the cycle β_j of β, that is, $O(P_{ij}) = \{(r, c, s) \in O(P)\colon\ r \in \alpha_i \text{ and } c \in \beta_j\}$. It will be called the Θ-*decomposition of* P. Specifically, z determines not only the number of these blocks, but also their possible sizes and, consequently, a pair of bounds for the size of P. To see it, let us define the set:

$$\mathrm{LCM}_z = \{(i, j) \in [n]^2\colon\ \exists k \in [n] \text{ s.t. } (i, j, k) \in \mathrm{LCM}_n \text{ and } z_{1i} \cdot z_{2j} \cdot z_{3k} > 0\}.$$

The following results hold:

Lemma 3.1. *Given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, $\Theta \in \mathfrak{I}_z$, $P \in \mathcal{PLS}_\Theta$ and an $i \times j$-block B of the Θ-decomposition of P, there exists $\omega_B \in [\gcd(i, j)] \cup \{0\}$ such that $|B| = \omega_B \cdot \mathrm{lcm}(i, j)$. Furthermore, $\omega_B = 0$ if $(i, j) \notin \mathrm{LCM}_z$.*

Proof. Analogously to the proof of Lemma 2.2, the Latin square condition implies $|B| = 0$, whenever $(i, j) \notin \mathrm{LCM}_z$. Besides, given $(r, c, s) \in O(B)$, its orbit by the action of the group generated by $\Theta = (\alpha, \beta, \gamma)$ is the set of triples $(\alpha^t(r), \beta^t(c), \gamma^t(s)) \in O(B)$, for all $t \in [\mathrm{lcm}(i, j)]$. Thus, the Latin square condition implies $|B|$ to be a multiple of $\mathrm{lcm}(i, j)$. Finally, since there are $i \cdot j$ cells in B, the multiplicative factor must be at most $\gcd(i, j)$. $\square$

Proposition 3.2. *Given $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, let $z^{(23)} = (z_1, z_3, z_2)$ and $z^{(13)} = (z_3, z_2, z_1)$. Then, given $P \in \mathcal{PLS}_z$, it is $\mathfrak{l}_z \leq |P| \leq \mathfrak{u}_z$, where:*

$$\mathfrak{l}_z = \min_{(i,j) \in \mathrm{LCM}_z} \{\mathrm{lcm}(i, j)\},$$

$$\mathfrak{u}_z = \min\{ \sum_{(i,j) \in \mathrm{LCM}_z} z_{1i} \cdot z_{2j} \cdot i \cdot j, \sum_{(i,k) \in \mathrm{LCM}_{z^{(23)}}} z_{1i} \cdot z_{3k} \cdot i \cdot k, \sum_{(k,j) \in \mathrm{LCM}_{z^{(13)}}} z_{2j} \cdot z_{3k} \cdot j \cdot k\}.$$

Proof. Let $\Theta \in \mathfrak{I}_z$ be such that $P \in \mathcal{PLS}_\Theta$ and let B be a block of the Θ-decomposition of P such that $|B| > 0$. From Lemma 3.1, if B is an $i \times j$-block, where $(i, j) \in \mathrm{LCM}_z$, then $\mathrm{lcm}(i, j) \leq |B| \leq i \cdot j$ and thus $\mathfrak{l}_z \leq |P| \leq \sum_{(i,j) \in \mathrm{LCM}_z} z_{1i} \cdot z_{2j} \cdot i \cdot j$. Since the size of a partial Latin square is invariant by parastrophism and $P^\pi \in \mathcal{PLS}_{\Theta^\pi}$ for all $\pi \in S_3$, the number $\mathfrak{u}_z$ is an upper bound of $|P|$. $\square$

From the previous results, it is deduced that the possible sizes of the elements of $\mathcal{PLS}_z$ must be in the set:

$$\mathrm{Sizes}(z) = \left\{ \sum_{(i,j)\in \mathrm{LCM}_z} \omega_{ij} \cdot \mathrm{lcm}(i,j) \le \mathfrak{u}_z \colon \omega_{ij} \in [z_{1i} \cdot z_{2j} \cdot \gcd(i,j)] \right\}.$$

As an example, let us consider $z = (6, 3\cdot 2\cdot 1, 4\cdot 2) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}6}}$ and $\Theta = ((123456), (123)(45)(6), (1234)(56)) \in \mathcal{PLS}_z$. The Θ-decomposition of any partial Latin square $P \in \mathcal{PLS}_\Theta$ is then formed by three blocks, P_{11}, P_{12} and P_{13}, whose cells are respectively indicated by the symbols $\cdot$, $*$ and $\circ$ in the following diagram:

$$\begin{pmatrix} \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \\ \cdot & \cdot & \cdot & * & * & \circ \end{pmatrix}.$$

Besides, $\mathrm{LCM}_z = \{(6,3)\}$, $\mathrm{LCM}_{z^{(23)}} = \{(6,2)\}$ and $\mathrm{LCM}_{z^{(13)}} = \{(2,3)\}$. Hence, from Proposition 3.2, it must be $6 \le |P| \le \min\{18, 12, 6\} = 6$. Thus, $\mathrm{Sizes}(z) = \{6\}$ and $|P| = 6$. Specifically, there are six possibilities for P:

$$\begin{pmatrix} 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\begin{pmatrix} \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 5 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

4. The number of partial Latin squares related to an autotopism.

Given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$, $\Theta \in \mathfrak{I}_z$, $P \in \mathcal{PLS}_\Theta$ and $s \in [n]$, let us define the sets:

$$\mathcal{PLS}_{\Theta,[P]} = \mathcal{PLS}_\Theta \cap [P], \qquad \mathcal{PLS}_{\Theta,s} = \mathcal{PLS}_\Theta \cap \mathcal{PLS}_{n,s}.$$

In the current section, given $\Theta \in \mathfrak{I}_n$, the cardinality of the set $\mathcal{PLS}_\Theta$ is studied. The following result implies that it only depends on the cycle structure of Θ:

Lemma 4.1. *The number of isotopic partial Latin squares related to an autotopism only depends on the parastrophic class of the cycle structure of the latter.*

Proof. Let $\Theta_1, \Theta_2 \in \mathfrak{I}_n$ and $\pi \in S_3$ be such that $z_{\Theta_1} = z_{\Theta_2^\pi}$ and let $P \in \mathcal{PLS}_n$. Since Θ_1 and Θ_2^π are conjugate, there exists $\Theta \in \mathfrak{I}_n$ such that $\Theta_2^\pi = \Theta\Theta_1\Theta^{-1}$. Now, given $Q \in \mathcal{PLS}_{\Theta_1,[P]}$, it is $Q^\Theta \in \mathcal{PLS}_{\Theta_2^\pi,[P]}$ and therefore $\left(Q^\Theta\right)^{\pi^{-1}} \in \mathcal{PLS}_{\Theta_2,[P]}$. Thus, $|\mathcal{PLS}_{\Theta_1,[P]}| \leq |\mathcal{PLS}_{\Theta_2,[P]}|$. The opposite inequality is similarly proven. $\square$

The relation $\sim$ between isotopic partial Latin squares can be used in order to define equivalence classes in $\mathcal{PLS}_\Theta$. Since the size of a partial Latin square is preserved by isotopism, Lemma 4.1 implies the following cardinalities to be well-defined:

$$\Delta_{[P]}(z) = \left|\mathcal{PLS}_{\Theta,[P]}\right|,$$

$$\Delta_s(z) = |\mathcal{PLS}_{\Theta,s}| = \sum_{\substack{[Q] \,\in\, \mathcal{PLS}_\Theta/\sim \\ s.t.\ |Q|=s}} \Delta_{[Q]}(z),$$

$$\Delta_{\mathcal{P}}(z) = |\mathcal{PLS}_\Theta| = \sum_{[Q]\in\mathcal{PLS}_\Theta/\sim} \Delta_{[Q]}(z) = \sum_{s\in\mathrm{Sizes}(z)} \Delta_s(z),$$

It can be defined a natural incidence relation $I_{\mathcal{P}_n}$ between $\mathcal{PLS}_n$ and $\mathfrak{I}_n$, where, given $P \in \mathcal{PLS}_n$ and $\Theta \in \mathfrak{I}_n$, then $(P,\Theta) \in I_{\mathcal{P}_n}$ if and only if $P \in \mathcal{PLS}_\Theta$. Besides, let us denote by $\mathfrak{A}_P$ the set of autotopisms of P. The following results are then proven:

Proposition 4.2. *Let $P \in \mathcal{PLS}_n$ and $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$. The triples $([P], \mathfrak{I}_z, I_{\mathcal{P}_n})$, $(\mathcal{PLS}_{n,s}, \mathfrak{I}_z, I_{\mathcal{P}_n})$ and $(\mathcal{PLS}_n, \mathfrak{I}_z, I_{\mathcal{P}_n})$ are, respectively, $\Delta_{[P]}(z)$-, $\Delta_s(z)$- and $\Delta_{\mathcal{P}}(z)$-uniform incidence structures and all their blocks have the same multiplicity. Moreover, the former incidence structure is regular.*

Proof. From Lemma 4.1, it is enough to study the uniformity and multiplicity of $([P], \mathfrak{I}_z, I_{\mathcal{P}_n})$. Indeed, the uniformity is an immediate consequence of that lemma. Now, in order to see that all the blocks have the same multiplicity, let $\Theta_1, \Theta_1' \in \mathfrak{I}_z$ be such that $\mathcal{PLS}_{\Theta_1,[P]} = \mathcal{PLS}_{\Theta_1',[P]}$ and let us consider $\Theta_2 \in \mathfrak{I}_z$. Let $\Theta, \Theta' \in \mathfrak{I}_n$ be such that $\Theta_1 = \Theta\Theta_2\Theta^{-1}$ and $\Theta_1' = \Theta'\Theta_1\Theta'^{-1}$. Then, $\mathcal{PLS}_{\Theta_2,[P]} = \mathcal{PLS}_{\Theta^{-1}\Theta'\Theta,[P]}$, because $Q \in \mathcal{PLS}_{\Theta_2,[P]} \Leftrightarrow$

$Q^\Theta \in \mathcal{PLS}_{\Theta_1,[P]} \Leftrightarrow Q^{\Theta'\Theta} \in \mathcal{PLS}_{\Theta'_1,[P]} = \mathcal{PLS}_{\Theta_1,[P]} \Leftrightarrow Q^{\Theta^{-1}\Theta'\Theta} \in \mathcal{PLS}_{\Theta_2,[P]}$. Moreover, $\Theta^{-1}\Theta'\Theta = \Theta_2 \Leftrightarrow \Theta' = \Theta^{-1}\Theta_2\Theta = \Theta_1 \Leftrightarrow \Theta'_1 = \Theta_1$. Thus, the arbitrariness of Θ_1, Θ'_1 and Θ_2 implies the claim about the multiplicity.

Finally, in order to see that $([P], \mathfrak{I}_z, I_{\mathcal{P}_n})$ is a regular incidence structure, let us consider $Q_1, Q_2 \in [P]$ and let $\Theta \in \mathfrak{A}_{Q_1} \cap \mathfrak{I}_z$. Since Q_1 and Q_2 are isotopic, there must exist $\Theta' \in \mathfrak{I}_n$ such that $Q_1^{\Theta'} = Q_2$. Hence, $\Theta'\Theta\Theta'^{-1} \in \mathfrak{A}_{Q_2} \cap \mathfrak{I}_z$ and therefore $|\mathfrak{A}_{Q_1} \cap \mathfrak{I}_z| \leq |\mathfrak{A}_{Q_2} \cap \mathfrak{I}_z|$. The regularity holds because the opposite inequality is analogously proven. □

Theorem 4.3. *Let $P \in \mathcal{PLS}_n$. If $Q \in [P]$, then $|\mathfrak{A}_Q| = |\mathfrak{A}_P|$ and it coincides with the cardinality of the set $\mathfrak{I}_{P,Q}$ of isotopisms from P to Q.*

Proof. From Proposition 4.2, it is verified that $|\mathfrak{A}_Q| = \sum_{z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_n}}} |\mathfrak{A}_Q \cap \mathfrak{I}_z| = \sum_{z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_n}}} |\mathfrak{A}_P \cap \mathfrak{I}_z| = |\mathfrak{A}_P|$. Now, if $\mathfrak{I}_{P,Q}$ contains exactly m distinct isotopisms $\Theta_1, \ldots, \Theta_m \in \mathfrak{I}_n$, then the set $\{\Theta_1\Theta_1^{-1}, \ldots, \Theta_m\Theta_1^{-1}\}$ is formed by m distinct autotopisms of Q and therefore $|\mathfrak{I}_{P,Q}| \leq |\mathfrak{A}_Q|$. The opposite inequality is also verified, because, given $\Theta \in \mathfrak{I}_{P,Q}$, it is $\Theta'\Theta \in \mathfrak{I}_{P,Q}$, for all $\Theta' \in \mathfrak{A}_Q$. □

Hereafter, we focus our study on the values $\Delta_s(z)$. The values $\Delta_{[P]}(z)$ needs a comprehensive analysis of the isotopism classes of partial Latin squares and will be considered in a future study. Firstly, it raises the natural question of whether it is possible to obtain some general expression which determines these values for some specific size or cycle structure. Thus, for instance, it is immediate to see that $\Delta_s((1^n, 1^n, 1^n)) = |\mathcal{PLS}_{n,s}|$ and, since $\mathcal{PLS}_{n,n^2} = \mathcal{LS}_n$, it is also clear that $\Delta_{n^2}(z) = \Delta(z)$. In this regard, let us study some cases in which a general formula is given:

Proposition 4.4. *Let $s \in [n^2]$. It is verified that:*

$$\Delta_s((n, n, 1^n)) = \begin{cases} \frac{n!^2}{k! \cdot (n-k)!^2}, & \text{if } \exists k \in [n] \text{ s.t. } s = k \cdot n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\Theta = (\alpha, \beta, \mathrm{Id}) \in \mathfrak{I}_{(n,n,1^n)}$, where Id denotes the trivial permutation, and $P \in \mathcal{PLS}_{\Theta,s}$. Since the Θ-decomposition of P is only formed by P itself and $\mathrm{LCM}_z = \{(n, n)\}$, Lemma 3.1 implies $s = k \cdot n$, for some $k \in [n]$. Consequently, $O(P)$ is decomposed under the action of Θ into k orbits of length n. Specifically, there exist exactly k distinct columns $c_1, \ldots, c_k \in [n]$ and k distinct symbols $s_1, \ldots, s_k \in [n]$, such that $(1, c_i, s_i) \in O(P)$, for all $i \in [k]$. The k orbits of $O(P)$ under Θ are then the sets $\{(\alpha^t(1), \beta^t(c_i), s_i)\colon\ t \in [n]\}$, with $i \in [k]$.

Every element of $\mathcal{PLS}_{\Theta,k\cdot n}$ is therefore uniquely determined by the choice of the columns c_i and symbols s_i. Namely, there exist $\binom{n}{k}$ possible ways of choosing the k columns and, once they have been selected, there exist $\frac{n!}{(n-k)!}$ different ways of assigning k symbols to the cells $(1, c_1), \ldots, (1, c_n)$. Therefore, $\Delta_s(z) = \binom{n}{k} \cdot \frac{n!}{(n-k)!} = \frac{n!^2}{k! \cdot (n-k)!^2}$. □

Stones and Wanless defined in [30] a *partial orthomorphism of $\mathbb{Z}_n$ of size s* as an injective map $\nu : S \to \mathbb{Z}_n$ such that S is a subset of s elements of $\mathbb{Z}_n$ and $\nu(i) - i \not\equiv \nu(j) - j \pmod{n}$, for all distinct $i, j \in S$. Another particular case of interest in our study is then that of a cyclic automorphism $\Theta \in \mathfrak{I}_{(n,n,n)}$, for which, according to the results of Section 3, it must be $\mathrm{Sizes}(\Theta) = \{k \cdot n\colon\ k \in [n]\}$. In particular, the next result holds:

Proposition 4.5. *Given $k \in [n]$, the number $\Delta_{k\cdot n}((n, n, n))$ coincides with the number of partial orthomorphisms of $\mathbb{Z}_n$ of size k.*

Proof. Let $\Theta = (1 \ldots n) \in \mathcal{A}_n \cap \mathfrak{I}_{(n,n,n)}$ and $P \in \mathcal{PLS}_{\Theta,k\cdot n}$. Analogously to the case of Proposition 4.4, the set $O(P)$ is decomposed under the action of Θ into k orbits of length n in such a way that there exist k non-empty cells in each row of P. Thus, the set $S = \{i \in \mathbb{Z}_n\colon\ (1, (i \pmod{n} + 1), s) \in O(P)$ for some $s \in [n]\}$ has cardinality k. Given $i \in S$, let $s_i \in [n]$ be such that $(1, (i \pmod{n} + 1), s_i) \in O(P)$. Since P satisfies the Latin square condition, the map $\nu : S \to \mathbb{Z}_n$ defined by $\nu(i) = s_i \pmod{n}$ is injective. Moreover, since Θ is an autotopism of P, the Latin square condition also implies that $s_i \not\equiv s_j + i - j \pmod{n}$. Hence, ν is a partial orthomorphism of $\mathbb{Z}_n$ of size k. It establishes a 1-1 correspondence between partial Latin squares of $\mathcal{PLS}_{\Theta,k\cdot n}$ and partial orthomorphisms of $\mathbb{Z}_n$ of size k. Therefore, $\Delta_{k\cdot n}(\Theta)$ coincides with the number of such orthomorphisms and the result follows from Lemma 4.1. □

A general formula for the number of partial orthomorphisms of $\mathbb{Z}_n$ of a given size is shown by Stones in [28] (Equation (3.9) on page 81) and is also to appear in [31]. More specifically, Theorem 3.2.4 and Figure 3.5 of [28] establish by Proposition 4.5 explicit formulas of $\Delta_{k\cdot n}((n, n, n))$, for $k \leq 5$.

Let us now see a formula for the number of partial Latin squares of smallest size related to an autotopism:

Theorem 4.6. *Given $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$:*

$$\Delta_{\mathfrak{l}_z}(z) = \sum_{\substack{(i,j) \in \mathrm{LCM}_z \\ s.t.\ \mathrm{lcm}(i,j) = \mathfrak{l}_z}} z_{1i} \cdot z_{2j} \cdot \gcd(i,j) \cdot \sum_{\substack{k \in [n] \\ s.t.\ (i,j,k) \in \mathrm{LCM}_n}} k \cdot z_{3k}.$$

Proof. Given $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$, let $P \in \mathcal{PLS}_\Theta$ be such that $|P| = \mathfrak{l}_z$. From Lemma 3.1 and Proposition 3.2, there must exist only one non-empty block B in the Θ-decomposition of P. Specifically, B must be an $i \times j$-block of size $\mathrm{lcm}(i,j) = \mathfrak{l}_z$, where $(i,j) \in \mathrm{LCM}_z$. There exist $z_{1i} \cdot z_{2j}$ possible blocks in this way.

Moreover, $O(B)$ must be composed by all the triples of one of the $\gcd(i,j)$ orbits induced on B by the action of Θ. If $(r,c,s) \in [n]^3$ is one of these triples, then the symbol s must be one of the $k \cdot z_{3k}$ elements of a k-cycle of γ such that $(i,j,k) \in \mathrm{LCM}_n$. The result follows then by considering all the previous possibilities. $\square$

Corollary 4.7. *Let $P \in \mathcal{PLS}_{n,1}$. Given $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$:*

$$\Delta_{[P]}(z) = \Delta_1(z) = z_{11} \cdot z_{21} \cdot z_{31}.$$

Proof. Since there exists only one isotopism class of partial Latin squares of size 1, it is $\Delta_{[P]}(z) = \Delta_1(z)$. Now, if $\mathfrak{l}_z > 1$, then $(1,1) \notin \mathrm{LCM}_z$. Hence, $z_{11} \cdot z_{21} \cdot z_{31} = 0$ and the result holds. Finally, if $\mathfrak{l}_z = 1$, then it is enough to observe that it must be $(i,j,k) = (1,1,1)$ in the formula of Theorem 4.6. $\square$

The number $\Delta_s(z)$ can also obviously be obtained if the set $\mathcal{PLS}_{\Theta,n}$ is known for some $\Theta \in \mathfrak{I}_z$. In order to determine this set, let us observe that, analogously to the case of Latin squares [6], $\mathcal{PLS}_n$ can be identified [21] with the set $\mathcal{S}_{\mathcal{P}_n}$ of solutions of the equation system:

$$\begin{cases} \sum_{r\in[n]} x_{rcs} \leq 1, \forall c, s \in [n], \\ \sum_{c\in[n]} x_{rcs} \leq 1, \forall r, s \in [n], \\ \sum_{s\in[n]} x_{rcs} \leq 1, \forall r, c \in [n], \\ x_{rcs} \in \{0,1\}, \forall r, c, s \in [n]. \end{cases} \tag{1}$$

Specifically, it is enough to define the map $\varphi_n : \mathcal{PLS}_n \to \mathcal{S}_{\mathcal{P}_n}$, such that, given $P \in \mathcal{PLS}_n$, it is $\varphi_n(P) = (x^P_{111}, \dots, x^P_{11n}, x^P_{121}, \dots, x^P_{nnn})$, where, $x^P_{rcs} = 1$ if $(r, c, s) \in O(P)$ and 0, otherwise. The restriction of φ_n to $\mathcal{PLS}_\Theta$ and $\mathcal{PLS}_{\Theta,m}$ assures the truthfulness of the following result:

Proposition 4.8. *Given $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n$, there exists a bijection between $\mathcal{PLS}_\Theta$ and the set of solutions of the equation system which results after adding to* (1) *the constraints:*

$$x_{rcs} = x_{\alpha(r)\beta(c)\gamma(s)}, \forall r, c, s \in [n].$$

Moreover, given $m \in [n^2]$, if the equation:

$$\sum_{r,c,s\in[n]} x_{rcs} = m$$

is also added, then there exists a bijection between $\mathcal{PLS}_{\Theta,m}$ and the set of solutions of the resulting equation system. □

Proposition 4.8 implies that $\mathcal{PLS}_{\Theta,m}$ is determined by $2n^3 + 3n^2 + 1$ polynomial equations of degree 1 and 2 in n^3 variables:

Corollary 4.9. *Given $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n$ and $m \in [n^2]$, $\mathcal{PLS}_{\Theta,m}$ is the set of zeros of the ideal:*

$$I = \langle\, (\sum_{r\in[n]} x_{rcs}) \cdot (1 - \sum_{r\in[n]} x_{rcs}) \colon\ c, s \in [n] \,\rangle + \langle\, (\sum_{c\in[n]} x_{rcs}) \cdot (1 - \sum_{c\in[n]} x_{rcs}) \colon\ r, s \in [n] \,\rangle +$$

$$\langle\, (\sum_{s\in[n]} x_{rcs}) \cdot (1 - \sum_{s\in[n]} x_{rcs}) \colon\ r, c \in [n] \,\rangle + \langle\, x_{rcs} \cdot (1 - x_{rcs}) \,:\ r, c, s \in [n] \,\rangle +$$

$$\langle\, x_{rcs} - x_{\alpha(r)\beta(c)\gamma(s)} \colon\ r, c, s \in [n] \,\rangle + \langle\, m - \sum_{r,c,s\in[n]} x_{rcs} \,\rangle \subseteq \mathbb{Q}[\mathbf{x_n}] = \mathbb{Q}[x_{111}, \dots, x_{nnn}].$$

□

The ideal I of Corollary 4.9 is *zero-dimensional*, that is, there exists only a finite number of solutions of the corresponding system of polynomial equations. Moreover, $I \cap \mathbb{Q}[x_{rcs}] = \langle\, x_{rcs} \cdot (1 - x_{rcs}) \,\rangle \subseteq I$ for all $r, c, s \in [n]$ and, therefore, Proposition 2.7 of [4] implies I to be *radical*, that is, any

n	z	$\Delta_s(z)$									$\Delta_{\mathcal{P}}(z)$
		s									
		1	2	3	4	5	6	7	8	9	
1	$(1,1,1)$	1									1
2	(2,2,2)		4		0						4
	$(2,2,1^2)$		4		2						6
	$(1^2,1^2,1^2)$	8	16	8	2						34
3	(3,3,3)			9			9			3	21
	(3,3,2·1)			3							3
	$(3,3,1^3)$			9			18			6	33
	(2·1,2·1,2·1)	1	10	10	24	24	20	20	4	4	117
	$(2{\cdot}1,2{\cdot}1,1^3)$	3	6	18	6	18					51
	$(2{\cdot}1,1^3,1^3)$	9	18	6							33
	$(1^3,1^3,1^3)$	27	270	1278	3078	3834	2412	756	108	12	11775

Table 2: $\Delta_s(z)$ and $\Delta_{\mathcal{P}}(z)$ for each parastrophic class of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}$, where $n \leq 3$.

polynomial $p(\mathbf{x_n})$ belongs to I whenever there exists $t \in \mathbb{N}$ such that $p(\mathbf{x_n})^t \in I$. Since the affine variety defined by I is $V(I) = \mathcal{PLS}_{\Theta,m}$, Theorem 2.10 of [4] assures $\Delta_m(z_\Theta) = |V(I)| = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x_n}]/I)$, which can be computed from any Gröbner basis of I, with respect to any term ordering. In this regard, it has been implemented in Singular [5] a procedure called *PLST*, which has been included in the library *pls.lib* [11] and which has been used in order to obtain the values of $\Delta_s(z)$ and $\Delta_{\mathcal{P}}(z)$ for each parastrophic class of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}$, where $n \leq 4$. These values are shown in Tables 2 and 3, where the blank cells correspond to those $s \not\in \mathrm{Sizes}(z)$.

5. Θ-completable partial Latin squares.

In the Introduction, given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}n}$ and $\Theta \in \mathfrak{I}_z$, it has been indicated that a partial Latin square $P \in \mathcal{PLS}_\Theta$ can be used in the computation of $\Delta(z)$, if a multiplicative factor (P-coefficient of symmetry of Θ [12]) $c_P \in \mathbb{N}$ is found such that $\Delta(z) = c_P \cdot |\mathcal{LS}_{\Theta,P}|$. In this regard, let us finish the present study with a theoretical basis for this concept of coefficient of symmetry. To do it, it is necessary to generalize the traditional concept of completability. Specifically, P will be said to be Θ-*completable* if $\mathcal{LS}_{\Theta,P} \neq \emptyset$. Consequently, the traditional completability corresponds to the trivial isotopism $\Theta = (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$. Moreover, if a partial Latin square is Θ-completable, then it is also completable in the traditional way. Let us

z	$\Delta_s(z)$																$\Delta_{\mathcal{P}}(z)$
	s																
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
(4,4,4)				16				48				32				0	96
(4,4,3·1)				4													4
$(4,4,2^2)$				16				56				32				8	112
$(4,4,2\cdot1^2)$				16				64				64				8	152
$(4,4,1^4)$				16				72				96				24	208
(3·1,3·1,3·1)	1		18	18		90	90		165	165		99	99		9	9	763
$(3\cdot1,3\cdot1,2\cdot1^2)$	2		6	12		6	12										38
$(3\cdot1,3\cdot1,1^4)$	4		12	48		36	144		24	96							364
$(3\cdot1,2^2,2^2)$		8		8													16
$(3\cdot1,2^2,2\cdot1^2)$		4															4
$(3\cdot1,2\cdot1^2,2\cdot1^2)$	4	4	8	4													20
$(3\cdot1,2\cdot1^2,1^4)$	8	12															20
$(3\cdot1,1^4,1^4)$	16	72	96	24													208
$(2^2,2^2,2^2)$		32		352		1664		3552		3328		1408		256		32	10624
$(2^2,2^2,2\cdot1^2)$		32		360		1792		4152		4416		2048		384		32	13216
$(2^2,2^2,1^4)$		32		368		1920		4800		5760		3264		768		96	17008
$(2^2,2\cdot1^2,2\cdot1^2)$		24		192		640		880		416		32					2184
$(2^2,2\cdot1^2,1^4)$		16		72		96		24									208
$(2\cdot1^2,2\cdot1^2,2\cdot1^2)$	8	32	136	336	752	1440	1904	2856	2400	2608	1504	1056	448	224	64	16	15784
$(2\cdot1^2,2\cdot1^2,1^4)$	16	88	272	736	1344	1632	1728	1008									6824
$(2\cdot1^2,1^4,1^4)$	32	384	2208	6504	9792	7104	2112	216									28352
$(1^4,1^4,1^4)$	64	1728	25920	239760	1437696	5728896	15326208	27534816	32971008	25941504	13153536	4215744	847872	110592	9216	576	127545136

Table 3: $\Delta_s(z)$ and $\Delta_{\mathcal{P}}(z)$ for each parastrophic class of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}}4}$.

observe that Θ-completability is a particular case of the $\mathfrak{F}$-completability defined in [8], where $\mathfrak{F}$ is a set of autotopisms of partial Latin squares. A specific case of Θ-completability is the cyclically completability studied by Grüttmüller in [15, 16], where Θ would be a cyclic automorphism. Another case of Θ-completability is given in [3], where authors study the completion of Latin rectangles that admit an autotopism $(\mathrm{Id}, \alpha, \beta)$ to a Latin square related to such an autotopism.

If P is Θ-completable, it is not mandatory for Θ to be an autotopism of P. Thus, for instance, if $\Theta = ((12), (12), \mathrm{Id}) \in \mathfrak{A}_2$, the partial Latin square of orthogonal representation $\{(1, 1, 1)\}$ is Θ-completable but does not belong to $\mathcal{PLS}_\Theta$. Indeed, it can be easily checked that every partial Latin square of order $n \leq 2$ is Θ-completable whenever Θ is non-trivial.

An example of non-trivial isotopism for which there exists a related partial Latin square which is neither Θ-completable nor traditionally completable is $\Theta = ((12)(3), (12)(3), (12)(3)) \in \mathfrak{A}_3$. A partial Latin square in such conditions is:

$$\begin{pmatrix} 3 & \cdot & 2 \\ \cdot & 3 & 1 \\ 2 & 1 & \cdot \end{pmatrix}.$$

An example where it is possible to observe the difference between both concepts is given if $\Theta = ((12)(34), (12)(34), (12)(3)(4)) \in \mathfrak{A}_4$. In this case, the following partial Latin square is not Θ-completable, but it is completable in the traditional way:

$$\begin{pmatrix} 3 & 4 & \cdot & \cdot \\ 4 & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Finally, let us observe that, unlike traditional completability, Θ-completability is not invariant under isotopism. Thus, for instance, if $\Theta = (123)(4)(5)(6) \in \mathcal{A}_6$, then it can be checked that the first of the following two isotopic partial Latin squares of $\mathcal{PLS}_\Theta$ is Θ-completable, but the second is not:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Let C_Θ denote the set of Θ-completable partial Latin squares and let $C_{\Theta,s} = C_\Theta \cap \mathcal{PLS}_{\Theta,s}$. The cardinalities of these sets only depend on the parastrophic class of the cycle structure of Θ:

Lemma 5.1. *Let $\Theta_1, \Theta_2 \in \mathfrak{I}_n$ be such that $[z_{\Theta_1}] = [z_{\Theta_2}]$. Then, $|C_{\Theta_1,s}| = |C_{\Theta_2,s}|$, for all $s \in [n^2]$. As a consequence, $|C_{\Theta_1}| = |C_{\Theta_2}|$.*

Proof. Let $\pi \in S_3$ be such that $z_{\Theta_1} = z_{\Theta_2^\pi}$. Given $s \in [n^2]$ and $P \in C_{\Theta_1,s}$, there exists $L \in \mathcal{LS}_{\Theta_1}$ such that $O(P) \subseteq O(L)$. Besides, since Θ_1 and Θ_2^π are conjugate, then there exists $\Theta \in \mathfrak{I}_n$ such that $\Theta_2^\pi = \Theta\Theta_1\Theta^{-1}$. Thus, $(P^\Theta)^{\pi^{-1}} \in \mathcal{PLS}_{\Theta_2}$, $(L^\Theta)^{\pi^{-1}} \in \mathcal{LS}_{\Theta_2}$ and $O((P^\Theta)^{\pi^{-1}}) \subseteq O((L^\Theta)^{\pi^{-1}})$. Since $|(P^\Theta)^{\pi^{-1}}| = |P|$, it is verified that $|C_{\Theta_1,s}| \leq |C_{\Theta_2,s}|$. The opposite inequality is analogously proven and the consequence is immediate, because $|C_{\Theta_1}| = \sum_{s\in[n^2]} |C_{\Theta_1,s}| = \sum_{s\in[n^2]} |C_{\Theta_2,s}| = |C_{\Theta_2}|$. $\square$

From the previous result, it is natural to define the numbers $\mathfrak{c}_z$ and $\mathfrak{c}_{z,s}$ as the respective cardinalities of C_Θ and $C_{\Theta,s}$, for any $\Theta \in \mathfrak{I}_z$. Specifically, in the case of cyclic automorphisms, the results of Grüttmüller in [15] imply that $\mathfrak{c}_{(n,n,n)} = 0$ if n is even and $\mathfrak{c}_{(n,n,n),k\cdot n} = \Delta_{k\cdot n}((n,n,n))$, for all n odd and $k \leq 2$. Although it is not true in general if $k \geq 3$ [16], some partial results have been studied in this regard in [2, 30]. For a general cycle structure, the following result holds:

Theorem 5.2. *Given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ and $\Theta \in \mathfrak{I}_z$, let I be the ideal related to $\mathcal{PLS}_{\Theta,n^2}$ according to Corollary 4.9. Given $P \in \mathcal{PLS}_\Theta$, it is verified that:*

$$P \in C_\Theta \Leftrightarrow \dim_{\mathbb{Q}} \left(\mathbb{Q}[\mathbf{x_n}]/(I + \langle\, x_{rcs} - 1\colon\ (r,c,s) \in O(P)\,\rangle)\right) \neq 0.$$

Proof. From Corollary 4.9, the affine variety of the ideal $I + \langle\, x_{rcs} - 1\colon\ (r,c,s) \in O(P)\,\rangle$ coincides with $\mathcal{LS}_{\Theta,P}$. This ideal is radical by Proposition 2.7 of [4], because contains its intersection with $\mathbb{Q}[x_{rcs}]$, for all $r,c,s \in [n]$. Specifically, this intersection is either $\langle\, x_{rcs} - 1\,\rangle$ if $(r,c,s) \in O(P)$ or $\langle\, x_{rcs} \cdot (x_{rcs} - 1)\,\rangle$, otherwise. The result follows from Theorem 2.10 of [4]. $\square$

Gröbner bases have then been used to determine the numbers $\mathfrak{c}_{z,s}$ and $\mathfrak{c}_z$ (Table 4) for each non-trivial parastrophic class of $\mathcal{CS}_{\mathfrak{A}_n}$, where $n \leq 4$. Specifically, given $\Theta \in \mathfrak{I}_z$, the procedure *PLST* of [11] has been applied in order to obtain the partial Latin squares of $\mathcal{PLS}_{\Theta,s}$ and the dimension of each affine variety of Theorem 5.2.

n	z	$\mathfrak{c}_{z,s}$																$\mathfrak{c}_z$
		s																
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	$(1,1,1)$	1																1
2	$(2,2,1^2)$		4		2													6
3	$(3,3,3)$			9			9			3								21
	$(3,3,1^3)$			9			18			6								33
	$(2{\cdot}1,2{\cdot}1,2{\cdot}1)$	1	10	10	24	24	16	16	4	4								109
4	$(4,4,2^2)$				16				40				32				8	96
	$(4,4,2{\cdot}1^2)$				16				40				32				8	96
	$(4,4,1^4)$				16				72				96				24	208
	$(3{\cdot}1,3{\cdot}1,3{\cdot}1)$	1		18	18		90	90		90	90		45	45		9	9	505
	$(2^2,2^2,2^2)$		32		352		1408		2144		1792		896		256		32	6912
	$(2^2,2^2,2{\cdot}1^2)$		32		336		1344		2144		1792		896		256		32	6832
	$(2^2,2^2,1^4)$		32		368		1728		3792		4224		2496		768		96	13504
	$(2{\cdot}1^2,2{\cdot}1^2,2{\cdot}1^2)$	8	24	104	200	528	784	1328	1560	1760	1568	1248	800	448	192	64	16	10632

Table 4: $\mathfrak{c}_{z,s}$ and $\mathfrak{c}_z$ for each non-trivial parastrophic class of $\mathcal{CS}_{\mathfrak{A}_n}$, where $n \leq 4$.

Given $z \in \mathcal{CS}_{\mathfrak{A}_{\mathcal{P}_n}}$ and $\Theta \in \mathfrak{I}_z$, a set $\{P_1, \ldots, P_m\}$ of Θ-completable partial Latin squares will be said to be a *basis* of $\mathcal{LS}_\Theta$ if $\bigcup_{i\in[m]} \mathcal{LS}_{\Theta,P_i} = \mathcal{LS}_\Theta$ and $\mathcal{LS}_{\Theta,P_i} \cap \mathcal{LS}_{\Theta,P_j} = \emptyset$, whenever $i \neq j$. In this case, $\Delta(z) = \sum_{i\in[m]} |\mathcal{LS}_{\Theta,P_i}|$. Let us observe that, from a computational point of view, it is interesting to determine a basis of $\mathcal{LS}_\Theta$ such that the sizes of its elements are as great as it is possible, because then, for each P_i, it would be feasible to add to the constraints of Proposition 4.8, all those of the form $x_{rcs} = 1$, whenever $(r, c, s) \in O(P_i)$. The calculus of the corresponding Gröbner basis would be then more efficient and it would allow to obtain new values $\Delta(z)$. The next result holds:

Lemma 5.3. *Let $S \subseteq [n]^2$ and $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{A}_n$. The following sets are bases of $\mathcal{LS}_\Theta$:*

$$S_{RC} = \{P \in C_\Theta\colon\ (r, c, s) \in O(P) \Leftrightarrow (r, c) \in S\},$$

$$S_{RS} = \{P \in C_\Theta\colon\ (r, c, s) \in O(P) \Leftrightarrow (r, s) \in S\},$$

$$S_{CS} = \{P \in C_\Theta\colon\ (r, c, s) \in O(P) \Leftrightarrow (c, s) \in S\}.$$

Proof. Let us prove that S_{RC} is a basis of $\mathcal{LS}_\Theta$; the other cases are similar. Since $\Theta \in \mathfrak{A}_n$, it must be $\mathcal{LS}_\Theta \neq \emptyset$. Now, given $L \in \mathcal{LS}_\Theta$, let $P \in S_{RC}$ be such that $O(P) = \{(r, c, s) \in O(L)\colon\ (r, c) \in S\}$. Thus, $L \in \mathcal{LS}_{\Theta,P}$ and hence $\mathcal{LS}_\Theta = \bigcup_{Q\in S_{RC}} \mathcal{LS}_{\Theta,Q}$. Finally, given two distinct elements Q and Q' in S_{RC}, it must exist $(r, c) \in S$ and $s \in [n]$ such that $(r, c, s) \in$

$O(Q) - O(Q')$. It implies that $\mathcal{LS}_{\Theta,Q} \cap \mathcal{LS}_{\Theta,Q'} = \emptyset$ and therefore S_{RC} is a basis of $\mathcal{LS}_{\Theta}$. $\square$

A special case appears when $|\mathcal{LS}_{\Theta,P_i}| = |\mathcal{LS}_{\Theta,P_j}|$, for all $i, j \in [m]$. Such a basis will be called *homogeneous* and it follows that $\Delta(z) = m \cdot |\mathcal{LS}_{\Theta,P_i}|$, for all $i \in [m]$. The cardinality m of the homogeneous basis would be therefore the P_i-coefficient of symmetry of Θ, for all $i \in [m]$. Although a comprehensive study should be developed in this regard, let us finish the current paper with a result with gives a theoretical support to the majority of the coefficients of symmetry which were used in [12]:

Theorem 5.4. *Let $z = (z_1, z_2, z_3) \in \mathcal{CS}_{\mathfrak{A}_n}$ be such that $z_{11} \cdot z_{21} \cdot z_{31} \neq 0$. Let $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_z$ and $S = \{(i,j) \in [n]^2 \colon i \in \alpha_\infty, j \in \beta_\infty\}$. It is verified that S_{RC} is an homogeneous basis of $\mathcal{LS}_{\Theta}$ of cardinality $|\mathcal{LS}_{z_{11}}|$.*

Proof. From the hypothesis, it must be $z_{11} = z_{21} = z_{31}$ ([24], Theorem 1). Furthermore, given $P \in \mathcal{PLS}_{\Theta}$, the corresponding block $P_{\infty\infty}$ of the Θ-decomposition of P is a $z_{11} \times z_{11}$-array, such that each of its non-filled cells must contain one of the z_{11} fixed symbols of γ, that is, it is a Latin subsquare of P of order z_{11}. Thus, since $\Theta \in \mathfrak{A}_n$, Lemma 5.3 implies the set S_{RC} to be a basis of $\mathcal{LS}_{\Theta}$ of $|\mathcal{LS}_{z_{11}}|$ elements. Now, let us consider two distinct elements $Q, Q' \in S_{RC}$. Given $L \in \mathcal{LS}_{\Theta,Q}$, let us define the Latin square $L' \in \mathcal{LS}_n$ such that $O(L') = \{(r,c,s) \in [n]^3 \colon (r,c,s) \in O(Q')$ if $(r,c) \in S$, or $(r,c,s) \in O(L)$, otherwise$\}$, that is, the only difference of L' with respect to L is the block $L'_{\infty\infty}$, which is Q' instead of Q. Since $L \in \mathcal{LS}_{\Theta}$ and $Q' \in \mathcal{PLS}_{\Theta}$, it must be $L' \in \mathcal{LS}_{\Theta}$. Hence, $|\mathcal{LS}_{\Theta,Q}| \leq |\mathcal{LS}_{\Theta,Q'}|$. The opposite inequality is analogously proven and therefore S_{RC} is homogeneous. $\square$

6. Final remarks and further work.

In the current paper, it has been dealt with the set of autotopisms of partial Latin squares in order to develop further techniques which allow to improve some results about the set of autotopisms of Latin squares, such as those related with the obtaining of the values $\Delta(z)$. In Section 2, the cardinality of $\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}$ was studied and a lower bound was determined. Although it can be obtained by an exhaustive search once Lemma 2.2 is implemented in a computer procedure, it raises the question of whether it is possible to obtain a general formula for $|\mathcal{CS}_{\mathfrak{A}_{\mathcal{P}n}}|$. A similar question appears in Section 4 with the values $\Delta_{[P]}(z)$, for which a comprehensive study of isotopism classes of $\mathcal{PLS}_n$ would be necessary. It would also be useful in order to improve the computation and increase the order $n \leq 4$ which has been used in the examples of the present paper. Finally, once a theoretical basis has been exposed in Section 5 for the concept of coefficient of symmetry of an autotopism, it seems that an exhaustive study in this regard would be necessary to solve some of the problems of computation related to the calculus of the values $\Delta(z)$.

Acknowledgements.

The author is very grateful to the referees for their valuable comments and suggestions which have improved the original version of the current paper.

References

[1] D. Bryant, M. Buchanan and I. M. Wanless, The spectrum for quasigroups with cyclic automorphisms and additional symmetries, Discrete Math. 304 (2009), no. 4, 821-833.

[2] N. J. Cavenagh, C. Hamalainen and A. M. Nelson, On completing three cyclically generated transversals to a latin square, Finite Fields Appl. 15 (2009), no. 3, 294–303.

[3] N. J. Cavenagh and D. S. Stones, Near-automorphisms of Latin squares, J. Combin. Des. 19 (2011), no. 1, pp. 365–377.

[4] D. A. Cox, J. B. Little and D. O'Shea, Using Algebraic Geometry, Springer-Verlag, New York, 1998.

[5] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 3-1-3. A computer algebra system for polynomial computations, 2011. http://www.singular.uni-kl.de.

[6] R. Euler, R. E. Burkard and R. Grommes, On Latin squares and the facial structure of related polytopes, Discrete Math. 62 (1986), no. 2, 155-181.

[7] R. M. Falcón, Latin squares associated to principal autotopisms of long cycles. Application in Cryptography, Proceedings of Transgressive Computing 2006: a conference in honor of Jean Della Dora (2006), 213 - 230.

[8] R. M. Falcón, Study of Critical Sets in Latin Squares by using the Autotopism Group, Electronic Notes in Discrete Mathematics 29 (2007), 503–507.

[9] R. M. Falcón, Designs based on the cycle structure of a Latin square autotopism, Proceedings of 1st Hispano-Moroccan Days on Applied Mathematics and Statistics (2008), 479–484.

[10] R. M. Falcón, Cycle structures of autotopisms of the Latin squares of order up to 11, Ars Comb. (in press). Available from http://arxiv.org/abs/0709.2973.

[11] R. M. Falcón, SINGULAR library, `http://personal.us.es/raufalgan/LS/pls.lib`.

[12] R. M. Falcón and J. Martín-Morales, Gröbner bases and the number of Latin squares related to autotopisms of order ≤ 7, J. Symbolic Comput. 42 (2007), no. 11–12, 1142–1154.

[13] R. M. Falcón and J. Martín-Morales, The 3-dimensional planar assignment problem and the number of Latin squares related to an autotopism, Proceedings of XI Spanish Meeting on Computational Algebra and Applications (2008), 89–92.

[14] M. Ghandehari, H. Hatami and E. S. Mahmoodian, On the size of the minimum critical set of a Latin square, Discrete Math. 293 (2005), no. 1-3, 121–127.

[15] M. Grüttmüller, Completing partial latin squares with two cyclically generated prescribed diagonals, J. Combin. Theory Ser. A 103 (2003), no. 2, 349–362.

[16] M. Grüttmüller, Completing partial latin squares with prescribed diagonals, Discrete Appl. Math. 138 (2004), no. 1–2, 89–97.

[17] G. H. Hardy, S. and Ramanujan, Asymptotic Formulae in Combinatory Analysis, Proc. London Math. Soc. 17 (1918), 75–115.

[18] A. Hulpke, P. Kaski and P. R. J. Östergård, The number of Latin squares of order 11, Math. Comp. 80 (2011), no. 274, 1197–1219.

[19] B. Kerby and J. D. H. Smith, Quasigroup automorphisms and symmetric group characters, Comment. Math. Univ. Carol. 51 (2010), no. 2, 279–286.

[20] B. Kerby and J. D. H. Smith, Quasigroup automorphisms and the Norton-Stein complex, Proc. Amer. Math. Soc. 138 (2010), no. 9, 3079–3088.

[21] S. R. Kumar, A. Russell and R. Sundaram, Approximating Latin square extensions, Algorithmica 24 (1999), no. 2, 128–138.

[22] C. Laywine, An expression for the number of equivalence classes of Latin squares under row and column permutations, J. Combin. Theory Ser. A 30 (1981), no. 3, 317–321.

[23] C. Laywine and G. L. Mullen, Latin cubes and hypercubes of prime order, Fibonacci Quart. 23 (1985), no. 2, 139–145.

[24] B. D. McKay, A. Meynert and W. Myrvold. Small Latin Squares, Quasigroups and Loops, J. Combin. Des. 15 (2007), no. 2, 98– 119.

[25] B. D. McKay. http://cs.anu.edu.au/∼bdm/data/latin.html.

[26] A. A. Sade, Autotopies des quasigroupes et des systèmes associatives, Arch. Math. 4 (1968), no. 1, 1–23.

[27] D. S. Stones, The parity of the number of quasigroups, Discrete Math. 310 (2010), no. 21, 3033–3039.

[28] D.S. Stones, On the number of Latin rectangles, Ph.D. Thesis, Monash University, 2010. Available from http://arrow.monash.edu.au/hdl/1959.1/167114.

[29] D. S. Stones, P. Vojtěchovský and I. Wanless, Cycle structure of autotopisms of quasigroups and Latin squares. Preprint avaliable from http://www.du.edu/media/documents/nsm/mathematics/preprints/m1101.pdf

[30] D. S. Stones and I. M. Wanless, Compound orthomorphisms of the cyclic group, Finite Fields Appl. 16 (2010), no. 4, 277– 289.

[31] D. S. Stones and I. M. Wanless, A congruence connecting Latin rectangles and partial orthomorphisms, Ann. Comb., to appear.

[32] I. M. Wanless, Diagonally cyclic Latin squares, European J. Combin. 25 (2004), no. 3, 393–413.

Appendix A. Glossary of symbols.

$\mathcal{A}_n$	Set of automorphisms of at least one Latin square of order n.
$\mathfrak{A}_n$	Set of autotopisms of at least one Latin square of order n.
$\mathfrak{A}_{\mathcal{P}n}$	Set of autotopisms of at least one partial Latin square of order n.
$\mathcal{CS}_n$	Cycle structures of S_n.
$\mathcal{CS}_S$	Cycle structures of $S \subseteq \mathfrak{I}_n$.
$\mathcal{CS}_{n,m}$	$\{n^{z_n} \cdot \ldots \cdot 1^{z_1} \in \mathcal{CS}_n \colon\ z_m > 0$ and $z_i = 0, \forall i \in [m-1]\}$.
C_Θ	Θ-completable partial Latin squares.
$C_{\Theta,s}$	$C_\Theta \cap \mathcal{PLS}_{\Theta,s}$.
$\mathfrak{c}_z$	Cardinality of C_Θ, for all $\Theta \in \mathfrak{I}_z$.
$\mathfrak{c}_{z,s}$	Cardinality of $C_{\Theta,s}$, for all $\Theta \in \mathfrak{I}_z$.
$\mathfrak{I}_n$	Isotopisms of n elements.
$\mathfrak{I}_{P,Q}$	Isotopisms between two partial Latin squares P and Q.
$\mathfrak{I}_z$	Isotopisms with cycle structure z.
LCM_n	$\{(i,j,k) \in [n]^3 \colon\ \mathrm{lcm}(i,j) = \mathrm{lcm}(i,k) = \mathrm{lcm}(j,k) = \mathrm{lcm}(i,j,k)\}$.
LCM_z	$\{(i,j) \in [n]^2 \colon\ \exists k \in [n]$ s.t. $(i,j,k) \in \mathrm{LCM}_n$ and $z_{1i} \cdot z_{2j} \cdot z_{3k} > 0\}$.
$\mathcal{LS}_n$	Latin squares of order n.
$\mathcal{LS}_z$	Latin squares related to $\mathfrak{I}_z$.
$\mathcal{LS}_\Theta$	Latin squares related to $\Theta \in \mathfrak{A}_n$.
$\mathcal{LS}_{\Theta,P}$	Latin squares of $\mathcal{LS}_\Theta$ to which $P \in \mathcal{PLS}_n$ can be completed.
$\mathfrak{l}_z$	Lower bound of the size of any $P \in \mathcal{PLS}_z$ (Proposition 3.2).
$[n]$	$\{1,\ldots,\mathrm{n}\}$.
n_z	Number of cycles of $z \in \mathcal{CS}_n$.
n_π	Number of cycles of $\pi \in S_n$.
$O(P)$	Orthogonal representation of $P \in \mathcal{PLS}_n$.
$\|P\|$	Size of $P \in \mathcal{PLS}_n$.
$[P]$	Isotopism class of $P \in \mathcal{PLS}_n$.
$p(n)$	Number of partitions of n.
P^π	Parastrophic partial Latin square of $P \in \mathcal{PLS}_n$ w.r.t. $\pi \in S_3$.
P^Θ	Isotopic partial Latin square of $P \in \mathcal{PLS}_n$ w.r.t. $\Theta \in \mathfrak{I}_n$.

$\mathcal{PLS}_n$	Non-empty partial Latin squares of order n.
$\mathcal{PLS}_{n,s}$	Partial Latin squares of order n and size s.
$\mathcal{PLS}_z$	Partial Latin squares related to $\mathfrak{I}_z$.
$\mathcal{PLS}_\Theta$	Partial Latin squares related to $\Theta \in \mathfrak{A}_n$.
$\mathcal{PLS}_{\Theta,[P]}$	$\mathcal{PLS}_\Theta \cap [P]$.
$\mathcal{PLS}_{\Theta,s}$	$\mathcal{PLS}_\Theta \cap \mathcal{PLS}_{n,s}$, where $\Theta \in \mathfrak{I}_n$.
S_n	Symmetric group of n elements.
$\mathfrak{u}_z$	Upper bound of the size of any $P \in \mathcal{PLS}_z$ (Proposition 3.2).
z_Θ	Cycle structure of $\Theta \in \mathfrak{I}_n$.
$[z_\Theta]$	Parastrophic class of the cycle structure of $\Theta \in \mathfrak{I}_n$.
$\Delta(z)$	$\lvert\mathcal{LS}_\Theta\rvert$, for any $\Theta \in \mathfrak{I}_z$.
$\Delta_{[P]}(z)$	$\lvert\mathcal{PLS}_{\Theta,[P]}\rvert$, for any $\Theta \in \mathfrak{I}_z$.
$\Delta_s(z)$	$\lvert\mathcal{PLS}_{\Theta,s}\rvert$, for any $\Theta \in \mathfrak{I}_z$.
$\Delta_{\mathcal{P}}(z)$	$\lvert\mathcal{PLS}_\Theta\rvert$, for any $\Theta \in \mathfrak{I}_z$.
π_∞	Union of 1-cycles of $\pi \in S_n$ written in natural order.
$\sim$	Equivalence relation between isotopic partial Latin squares.