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Article

Experimental Assessment of a 2-D Entropy-Based Model for Velocity Distribution in Open Channel Flow

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Abstract: Velocity distribution in an open channel flow can be very useful to model many hydraulic phenomena. Among the others, several 1D models based on the concept of entropy are available in the literature, which allow estimating the velocity distribution by measuring velocities only in a few points. Nevertheless, since 1D models have often a limited practical use, a 2D entropy based model was recently developed. The model provides a reliable estimation of the velocity distribution for open channel flow with a rectangular cross section, if the maximum velocity and the average velocity are known. In this paper results from the proposed model were compared with measured velocities carried out from laboratory experiments. Calculated values were also compared with results inferred from a 2D model available in the literature, resulting in a greater ease of use and a more reliable estimate of the velocity profile.

Keywords: entropy; flow measurement; open-channel flow; Shannon entropy; streamflow; velocity distribution

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1. Introduction

The assessment of velocity distribution in open channel flow is required by many hydraulic applications, such as for the design of channel cross sections, the design of canal and river structures, the analysis of sediment and/or contaminant transport. The velocity field can be reliably assessed if enough velocity points are sampled into the flow area. Nevertheless, velocity measurements at high flows are very unusual, because both high discharges are fairly infrequent and measurements during floods are very difficult. Consequently, a simple approach for reliable estimation of velocity distribution could be very useful in practical applications.

It should also be considered that the velocity at a point is uncertain by its nature, due to natural and/or man-made causes, and should not be considered as a deterministic variable. Traditional approaches to the study of hydraulics follow the well known laws of mass and energy conservation. They lead to deterministic models, which do not account for such uncertainty. More modern approaches, mainly based on the concept of entropy, as in the fields of hydraulics and hydrology [1,2], consider velocity as a probabilistic variable, taking into account this uncertainty. [3] first obtained the equation for the velocity profile in a simplified 1D domain, based on the concept of Shannon entropy [4] and using the principle of maximum entropy [5]. This model was further extended and used both preserving the one-dimensionality of the flow [6–9] and considering more complex two-dimensional domains [10–12]. Nevertheless, the usefulness of the 1D models available in the literature is often limited, because of the many simplifications, whereas 2D models require several parameters to be calibrated, with no physical basis, making them difficult and unfriendly to use.

Moramarco *et al.* [13] developed a practical and simple method by assuming that the 2D model developed by Chiu, written for the vertical where the maximum velocity occurs, can also be applied to the other verticals. Such a model is simpler than the one proposed by Chiu, but still it requires a lot of information, including average velocity and position and magnitude of maximum velocity for each vertical.

It should be considered that once the model (any of these) is set, the function gives the value of the velocity and no information about the uncertainty of this value is given. Therefore it seems that the result is deterministic but by means of uncertainty analysis [14] some information about model reliability can be obtained.

In the light of this consideration, starting from the approach proposed by Chiu, Marini *et al.* [15] developed a full 2D model for a rectangular domain. The proposed model requires no calibration parameter and the velocity distribution can be calculated if the geometry of the cross section, the average velocity and position and value of the maximum velocity are known. Marini *et al.* [15] have validated the proposed model with a number of measurements available in the literature, obtaining good agreement. However, further experiments are required to validate the proposed model. To this aim, in the paper, the theoretical background and the model equations were summarized. The calculated and measured velocities from experiments on a laboratory flume for different flow discharges and water depth were given, showing a fairly good agreement.

2. 2D Velocity Distribution Model

The Principle of Maximum Entropy (POME) formulated by Jaynes [5] states that any system which is in an equilibrium state, subject to certain constraints, tends to maximize its entropy. For this principle, when a watercourse, taken as a reference system, reaches the stationary conditions, then it presents the maximum content of the entropy [16] for which also the velocity distribution will be determined by maximizing the entropy of the system subject to constraints.

2.1. Velocity Function Definition in 2D Domain

Let u be the temporally-averaged velocity and $f(u)$ its Probability Density Function (PDF). The Shannon entropy H can be written as:

$$H(u) = -\int f(u)\ln[f(u)]\mathrm{d}u \tag{1}$$

The entropy $H(u)$ defines the uncertainty of u. Maximizing the Equation (1) and using POME, it is possible derive $f(u)$ when the function constraints are known. One can write these constraints from mass or momentum or energy conservancy law. Chiu [3] and Barbé *et al.* [7] have observed that the mass conservation equation is sufficient to derive the following constraint equations:

$$\int_0^{u_{\max}} f(u)\mathrm{d}u = 1 \tag{2}$$

$$\int_0^{u_{\max}} u\, f(u)\mathrm{d}u = \bar{u} \tag{3}$$

in which $u_{\max}$ is the maximum velocity and $\bar{u}$ is the mean of the velocities. In particular, the Equation. (3) satisfies mass conservation [7]. The entropy $H(u)$, under constraints (2) and (3), can be maximized using Lagrange multipliers method, obtaining the following equation [3]:

$$f(u) = \exp(\lambda_1 + \lambda_2 u - 1) \tag{4}$$

where λ_1 and λ_2 are the Lagrange multipliers.

Marini *et al.* have recently extended one such approach to a 2D domain, by assuming the velocity distribution function $u = u(x, y)$, in which x is the transverse coordinate and y the vertical coordinate measured from the bed, positive upward [15]. Let $f[u(x, y)]$ be its PDF and $F[u(x, y)]$ the Cumulative Distribution Function (CDF). For the sake of brevity, only the expression of the velocity distribution for a generic 2D domain is shown here:

$$u(x, y) = \frac{u_{\max}}{G}\ln\left[1 + \left(e^{G} - 1\right)\cdot F\left(u(x, y)\right)\right] \tag{5}$$

where G can be determined using Equation (3) or by means of any similar equation. Equation (5) represents the velocity distribution in a two-dimensional domain in terms of $u_{\max}$, G and the CDF defined in the domain. It is noteworthy to note that Equation (5) formally coincides with the equation derived by Chiu for the 1D case.

2.2. Entropic Parameter G

The entropic parameter G can be derived by the constraints equation if maximum and average velocities are known. Equation (2) represents the velocity PDF in terms of $u_{\max}$. Equation (3) represents the mean of the probability distribution ($\bar{u}$), and it is generally different than the average velocity.

As pointed out by Marini *et al.* [15], G can be derived starting from the cross-sectional average velocity (u_{av}). That velocity has a clear physical meaning and can be calculated as:

$$u_{av} = \frac{1}{A}\int_A \frac{u_{\max}}{G}\ln\left[1 + \left(e^{G} - 1\right)\cdot F(u)\right]\mathrm{d}A \tag{6}$$

Equation (6) should be solved to obtain the value of G for each domain in which u_{av} and u_{max} are known, and then the velocity distribution can be derived when the CDF is defined. Usually, u_{av} can be calculated either by measuring the flow discharge and the cross section area, or from u_{max} if the ratio u_{av}/u_{max} is known, which some authors [8,13,17] have shown to be fairly constant.

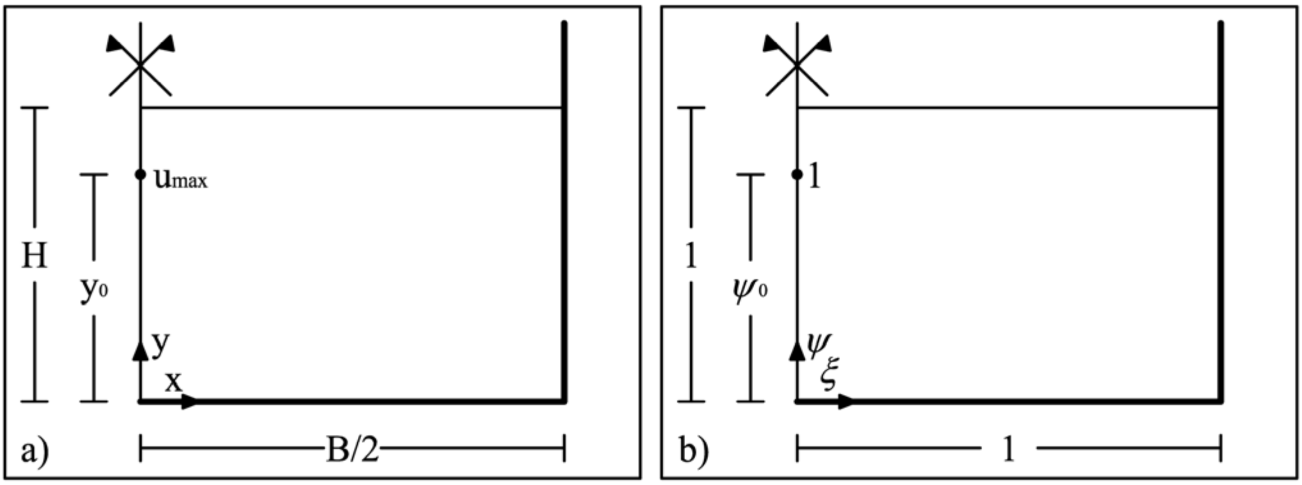
2.3. CDF for Symmetrical Rectangular Domain

The velocity CDF depends on the domain geometry. It has the following properties: (1) defined between 0 and 1, (2) continuous and differentiable, and (3) its value on the boundary must be null and have only one point in which it reaches unity.

Marini *et al.* proposed a CDF for a rectangular domain where $B/2$ is the half width and H is the flow depth (Figure 1a) [15]. A symmetrical velocity distribution with respect to the vertical axis was assumed, with the maximum velocity occurring at $y = y_0$. To obtain a more general CDF, a non-dimensional domain was considered, using the normalized variables $\xi = 2x/B$, $\psi = y/H$, and $\psi_0 = y_0/H$. Similarly, the ratio u/u_{max} instead of u was considered (Figure 1b). According to geometrical considerations, the authors obtained the following $F(u)$:

$$F(u) = \left(1-\xi^2\right)^{\frac{H}{B}} \cdot 4 \cdot \left[\left(\frac{\psi}{2}\right)^{\frac{\ln 2}{\ln 2-\ln(\psi_0)}} - \left(\frac{\psi}{2}\right)^{\frac{2\ln 2}{\ln 2-\ln(\psi_0)}}\right] \tag{7}$$

Figure 1. Rectangular symmetrical domain in (**a**) dimensional and (**b**) non-dimensional coordinates.



Equation (7) satisfies all the aforementioned properties: it is continuous and differentiable, varies between 0 and 1, and reaches unity when ξ is equal to 0 and ψ is equal to ψ_0. As example, a contour sketch of $F(u)$ is shown in Figure 2 for $H/B = 0.5$ and $\psi_0 = 0.8$.

3. Model Validation by Experimental Measurements and Comparison with Chiu's 2D Distribution

The model was validated with data collected from laboratory experiments. The experiments were carried out at the Hydraulic Laboratory of the Università degli Studi di Cassino. A 9 m long Perspex flume was used (Figure 3), with a rectangular cross section 40 cm wide and 70 cm tall. The slope of the flume can be changed by means of a pneumatic system and two floodgates at the upstream and downstream of the channel can be arranged to regulate flow discharge and cross section area. Two configurations having different roughness were analyzed: in the first configuration (hereafter $c = 0$) no additional roughness was added to the flume, whereas in the second one ($c = 1$) 10 mm diameter gravel was attached to the flume bottom. Velocities were measured using a two components Acoustic-Doppler Velocimeter (ADV). The ADV has a time resolution up to 100 Hz, and a spatial resolution given by a cylinder with diameter of 6 mm and height settable to 3, 6 or 9 mm. A time resolution of 10 Hz and a height 6 mm were set in the experiments.

Figure 2. Contour sketch of CDF $F(u)$ ($H/B = 0.5$ and $\psi_0 = 0.8$).

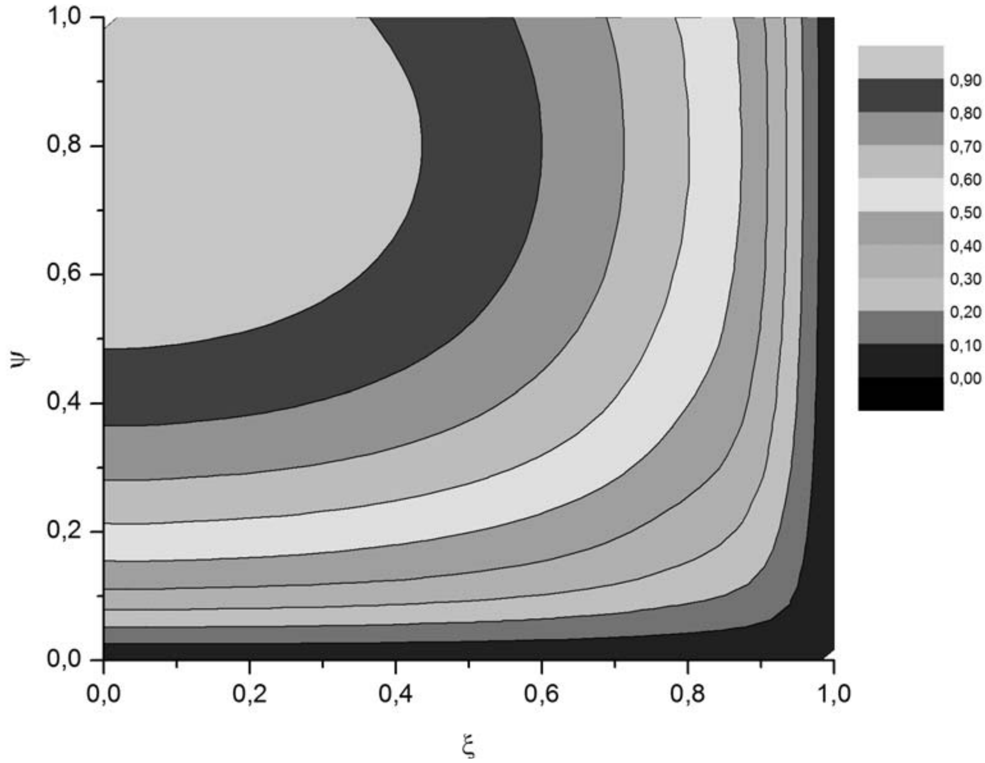


Figure 3. Flume used for experimental measurements.

In some tests, a Pitot tube was also used, showing acceptable differences between measures. An adequate time interval for sampling was assessed for velocity measurement, since ADV measures instantaneous velocities. A 30 s interval was used, which ensure a constant value also considering a time interval one order of magnitude greater. In the experiments the discharge and the flow depth were changed, and the velocities measured at several points. The measurements were carried out only over the left side of the flume because of the symmetry of the flow field. Although in the real case perfect symmetry is never verified, a simplified model was developed in the paper and no further generalization was considered. Obviously such an assumption restricts the proposed model, and a more general position of the maximum velocity point should be considered in the further development of the research.

For the sake of brevity, only the most significant data for each test were given in Table 1: the number of measurement points over the cross section, roughness configuration index *c*, water depth *H*, flow discharge *Q*, average and maximum velocity, entropic parameter *G*. The analyzed configurations provide variability in terms of flow discharge, water level, average and maximum velocity, showing also that the ratio u_{av}/u_{max} is fairly constant at varying the flow characteristics, as already pointed out by Chiu *et al.* [18] and Moramarco and Singh [19].

Table 1. Characteristics of velocity data, validation indexes and deviations for all experiments.

n°	Number of Data	*c*	H [cm]	Q [l/s]	u_{av} [m/s]	u_{max} [m/s]	G	c^2_{2d} [%]	c^2_{Chiu} [%]	E_{2D} [%]	E_{Chiu} [%]	ia_{2D} [%]	ia_{Chiu} [%]	d_{2D} [%]	d_{Chiu} [%]
1	54	0	13.2	30.6	0.638	0.579	4.75	82.9	49.4	70.9	−190.7	94.1	67.7	2.9	8.3
2	60	0	9.4	21.6	0.647	0.574	5.25	38.7	58.8	−60.7	−95.1	73.2	75.6	6.3	6.9
3	100	0	9.0	19.4	0.629	0.538	4.68	52.8	62.9	−56.4	1.0	77.0	84.0	6.7	5.0
4	70	0	9.2	17.0	0.524	0.463	3.22	56.0	57.0	18.5	−12.8	83.9	81.5	6.0	6.4
5	56	0	9.3	20.2	0.598	0.543	2.10	73.8	63.8	−95.0	−80.0	77.4	77.4	6.9	5.9
6	28	0	5.4	20.4	1.079	0.946	5.21	93.4	91.7	84.0	83.7	95.2	95.2	2.8	2.9
7	35	0	6.0	25.9	1.219	1.078	3.53	84.3	79.4	76.1	68.7	94.2	93.4	3.0	3.4
8	35	0	6.5	30.9	1.423	1.188	1.61	67.7	52.6	32.7	−29.2	85.8	73.2	6.7	8.9
9	35	0	5.5	26.1	1.383	1.186	2.21	61.5	56.5	20.7	1.1	83.4	79.0	6.0	6.8
10	28	0	5.1	26.5	1.497	1.299	2.89	65.0	61.5	23.7	16.3	83.3	81.7	5.6	5.8
11	49	1	8.2	20.2	0.715	0.615	3.75	81.3	86.6	76.8	80.2	91.5	92.8	5.1	4.3
12	49	1	8.0	20.7	0.773	0.647	2.23	90.3	89.4	89.5	75.4	97.2	92.8	3.1	4.7
13	49	1	10.2	24.9	0.687	0.610	3.58	90.0	76.7	88.8	63.0	97.1	92.3	2.3	4.1
14	49	1	9.6	28.6	0.878	0.745	3.01	90.4	85.0	87.2	74.0	97.1	94.0	2.9	4.0
15	56	1	9.2	26.1	0.768	0.710	3.89	84.4	35.5	79.6	−50.3	93.1	69.8	3.3	8.9
16	56	1	11.3	35.7	0.914	0.789	3.21	87.0	70.5	86.9	64.0	96.5	90.4	3.5	5.1
17	42	1	8.5	26.7	0.949	0.785	3.79	82.8	83.4	80.8	−19.9	95.2	82.8	4.6	10.6

The proposed model was also compared with the 2D distribution derived by Chiu [3,6,10]:

$$\frac{u}{u_{max}} = \frac{1}{M}\ln\left[1 + \left(e^{M} - 1\right)\cdot\frac{j - j_0}{j_{max} - j_0}\right] \tag{8}$$

in which j is a generic isovel with velocity u, j_0 and j_{max} are the isovels corresponding to a null and maximum velocity, respectively and M the entropic parameter.

Figure 4. Comparison between experimental and theoretical data calculated by 2D proposed model and Chiu's model (experiment n. 3).

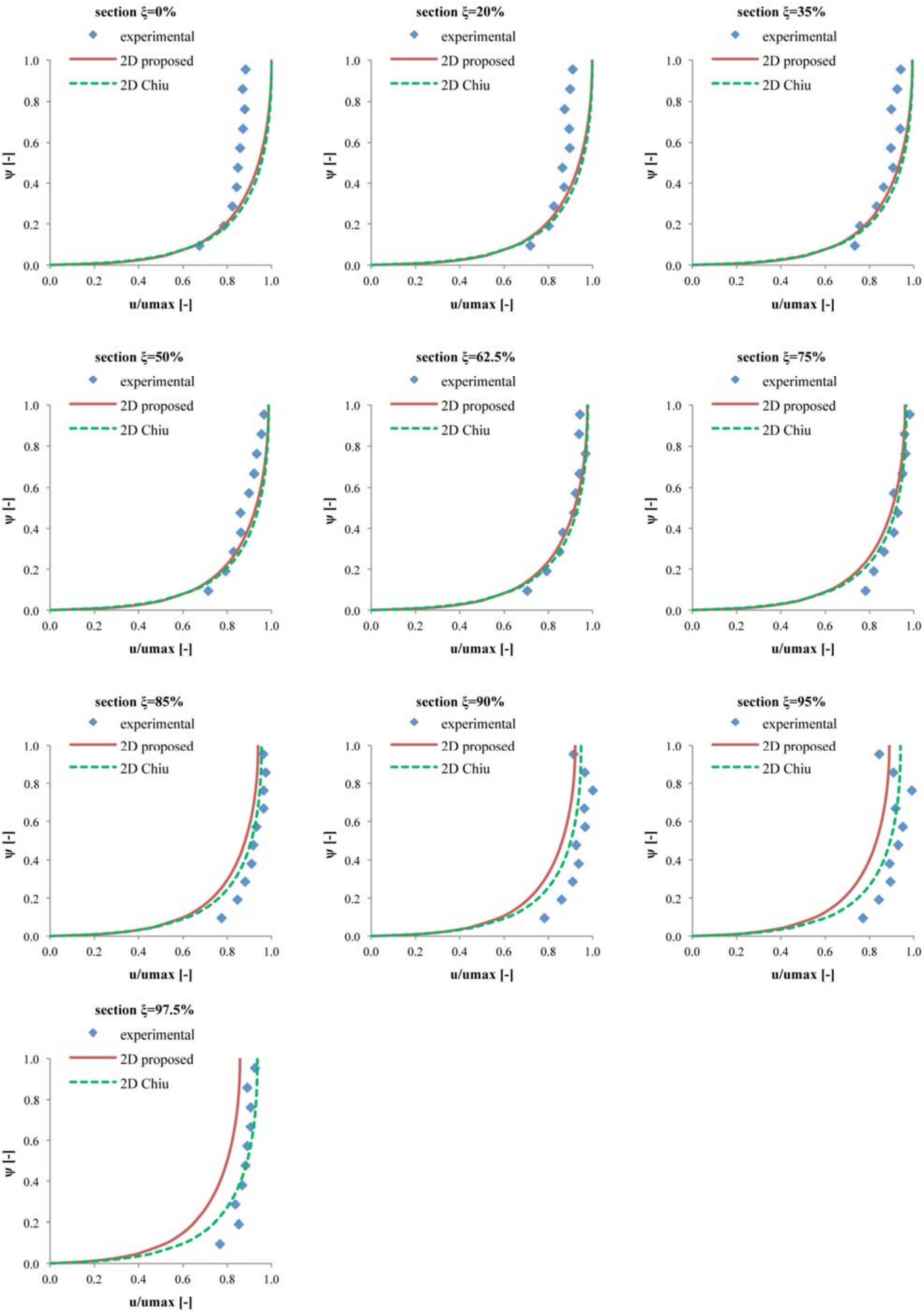
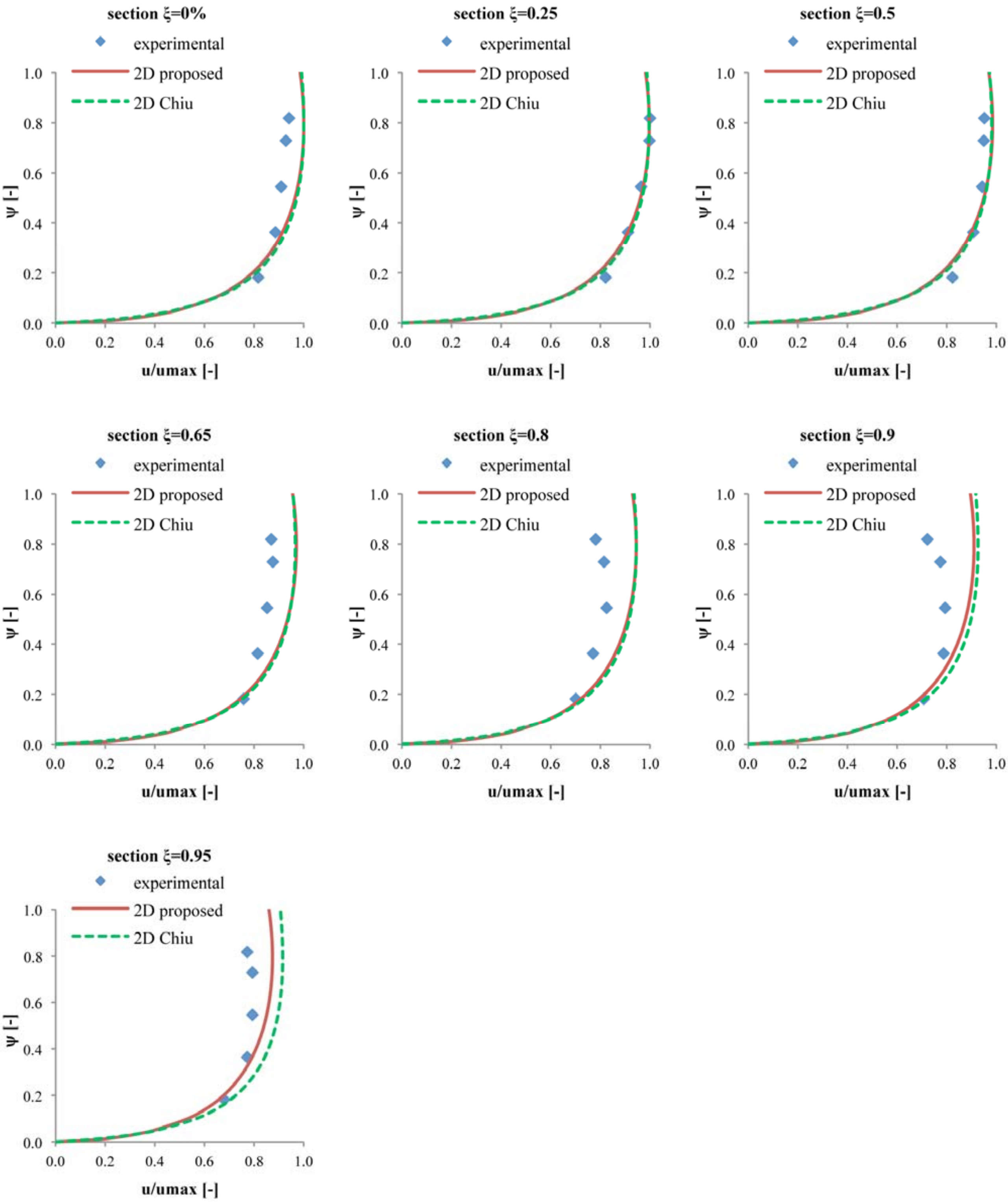


Figure 5. Comparison between experimental and theoretical data calculated by 2D proposed model and Chiu's model (experiment n. 17).

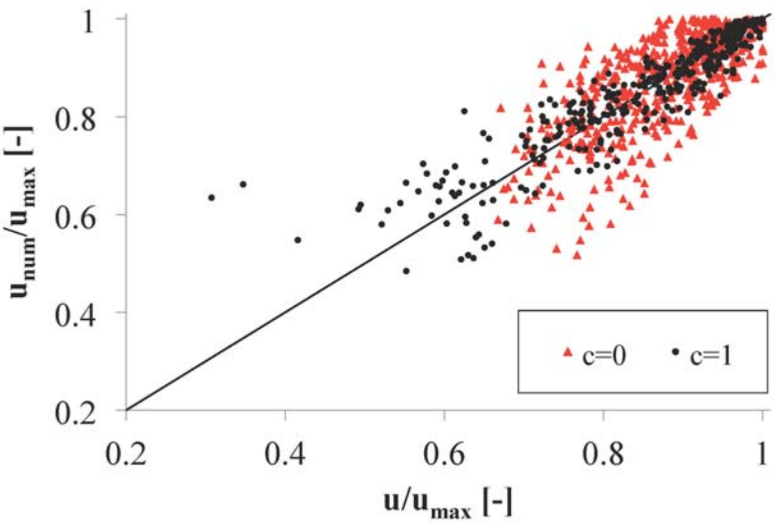


As example, in Figures 4 and 5 the measured velocities for the experiment n. 3 and 17 respectively were compared with data inferred from Chiu's model [3,6,10] and the proposed model. The plot shows that for experiment n. 3, related to $c = 0$, Chiu model gives better results, while for experiment n. 17, related to $c = 1$, the proposed model is better. Although the two models have approximately the same reliability, as highlighted in what follows, the proposed model has the advantage that no parameter calibration or experimentally-based equations are required for application.

A comprehensive comparison between measured and calculated velocities was given for the proposed model in Figure 6. For the sake of brevity, the same plot relating to the Chiu's model was omitted. For both models, data points are in fairly good agreement, and the greatest deviations occur near to the bottom and the wall of the channel, where lower values were measured. That probably depends (1) on the lower resolution of the ADV when approaching fixed boundary and (2) on the intrinsically probabilistic nature of any entropic based model. Banks and channel bottom significantly affect the velocity and the model is not able to explicitly account for the boundary effects. Consequently, such model are not able to provide reliable results at the cross section boundary.

The proposed distribution resulted in better agreement with measured data, as confirmed by the deviations with theoretical velocities. The average deviation and three efficiency indexes were calculated for each test. The average deviation (*d*) was calculated as $|u—u^*|/u_{max}$, where u^* is the theoretical value. The efficiency indexes used are the coefficient of correlation r^2, the Nash Sutcliffe Index *E*, the index of agreement *ia* [20]. The results for all the experiments were summarized in Table 1, showing the better agreement of the proposed model.

Figure 6. Comparison between experimental and theoretical data; triangle corresponds to $c = 0$, circle to $c = 1$.



4. Conclusions

A novel approach was recently presented for 2D velocity distribution in open channel flow. Based on the POME, the distribution was derived by assuming a rectangular 2D domain and a CDF for velocity was assumed. The reliability of the model was assessed by means of many experiments, carried out on a laboratory flume. Velocity profiles were measured in a rectangular cross section, by varying flow discharge, flow depth and channel slope in a fairly wide range of values. Different wall roughness was also considered. Measured data were compared with values inferred from the proposed model and Chiu's 2D model, showing a fairly good agreement. The proposed model shows smaller deviations than the Chiu's model, although both models exhibit the largest differences near to the cross section boundary.

The model allows a straightforward calculation of the velocity at any point of the domain and no parameter calibration or experimentally-based equations are required for application. When assuming wide channel geometry, the proposed model returns a velocity profile quite similar to the profile inferred from the 1D Chiu model.

Nevertheless, further validation is required for a definitive assessment of the model, by assuming a wider range for flow and cross section characteristics. Differently shaped cross sections will be also investigated, to assess the effectiveness of the model for deriving velocity distribution in natural river as well.

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