V. Conclusion

The problem of estimating the parameters in continuous-time stochastic signals, represented by CARMA processes, from discrete-time data has been studied. The suggested solution is to fit the covariance function, parameterized by the process parameters, to sample covariances. It has been shown that the estimation method gives consistent estimates, and an approximate covariance matrix for the estimated parameters has been derived. The validity of the derived expression was investigated in a numerical study, where it was seen that the variances are very close to the CRB for certain choices of the sampling interval and the number of covariance elements used in the criterion function. One important use of the derived covariance matrix is for choosing the user parameters of the method; by comparing the variances with the CRB for different choices of the user parameters, a suitable choice can be made.

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Extended Object Tracking Using Monte Carlo Methods

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***Abstract*—This correspondence addresses the problem of tracking extended objects, such as ships or a convoy of vehicles moving in urban environment. Two Monte Carlo techniques for extended object tracking are proposed: an interacting multiple model data augmentation (IMM-DA) algorithm and a modified version of the mixture Kalman filter (MKF) of Chen and Liu [1], called the mixture Kalman filter modified (MKFm). The data augmentation (DA) technique with finite mixtures estimates the object extent parameters, whereas an interacting multiple model (IMM) filter estimates the kinematic states (position and speed) of the manoeuvring object. Next, the system model is formulated in a partially conditional dynamic linear (PCDL) form. This affords us to propose two latent indicator variables characterizing, respectively, the motion mode and object size. Then, an MKFm is developed with the PCDL model. The IMM-DA and the MKFm performance is compared with a combined IMM-particle filter (IMM-PF) algorithm with respect to accuracy and computational complexity. The most accurate parameter estimates are obtained by the DA algorithm, followed by the MKFm and PF.**

***Index Terms*—Data augmentation, extended targets, mixture Kalman filtering, sequential Monte Carlo methods.**

I. Introduction

Most of the target tracking algorithms consider a single moving extended object as a point and estimate its center of mass based on the incoming sensor data, such as range and bearing. However, recent high-resolution sensor systems are able to resolve individual features or measurement sources on the extended object. Such an object can be modelled as a rigid or semi-rigid set of point sources, each of which may be the origin of a sensor measurement [2]. The possibility to additionally make use of the high-resolution measurements is referred to as *extended object tracking*. Knowledge of the object shape parameters is especially important for the object type classification.

The considered problem consists of both state and size parameters estimation of an extended target when tracking it. Estimation of *static* parameters in general nonlinear non-Gaussian state-space models is a long-standing problem [3]–[5]. Although in the literature there are results with Monte Carlo (particle filtering) methods, a well-known drawback of particle filtering for static parameters estimation is the degeneracy, the case when only one particle has a significant weight. Other solutions based on the expectation-maximization (EM) approach are also proposed. Some of the problems with the EM-type algorithm are due to the fact that it is of gradient type, and it can be trapped by local extremums.

The object extent parameters can be modeled in many different ways [2], [6]–[8]. The ellipsoidal object model proposed in [7] and [9] is adopted in our work. We are concerned with objects moving in a plane.

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The lengths of the major and minor axes of the ellipse have to be calculated, based on the measurements of the down-range extent. Shape parameters are included in [7] in the state vector together with kinematic parameters and are estimated by extended, unscented Kalman filters (EKFs, UKFs) and particle filtering. However, as pointed out in [7] and [9], the EKF is prone to divergence due to the presence of high nonlinearities, and the Monte Carlo (MC) approach can avoid this problem [10]. This motivates us to choose the MC framework and develop strategies that can cope with these problems.

The challenge of the problem under consideration is related with the complex target–observer geometry. Since the object maneuvers are giving rise to abrupt changes in the down-range extent measurements, a filter with augmented (state and static parameters) vector has longer transient periods and higher peak dynamic errors. It additionally decreases the accuracy of the target extent estimate. The influences of target maneuvers to size parameters evaluation can be reduced by implementing two different algorithms for state and parameters estimation.

The main contributions of this correspondence are in the estimation of parameters of nonpoint targets while tracking them. The added values and innovative aspects of this work as compared to previous investigations include the following:

i) proposition of two different algorithms for state and parameters estimation, accounting for specifics of this task;
ii) formulation of the problem for parameter estimation in PCDL form; this enables us to formulate the problem as parameter estimation of linear Markovian jump systems;
iii) estimation of the parameters with a data augmentation (DA) algorithm, with a mixture Kalman filter modified (MKFm) and a particle filter (PF).

The performance of the algorithms is assessed in terms of accuracy and computational complexity. We demonstrate that the DA algorithm outperforms the MKFm with respect to accuracy but is more computationally expensive. The developed MC techniques can be used in different tracking problems, where the extended target measurements are nonlinear related to the target parameters of interest, such as maritime and ground targets' surveillance.

The first developed technique combines the advantages of an *IMM filter* and of the *Markov Chain Monte Carlo* (MCMC) approach. The idea of combining the multiple model approach with MCMC for finite mixture estimation is present in a different application [11] dealing with joint estimation of system states and transition probabilities of linear jump Markov systems.

The object kinematic states (position and speed) are estimated in our correspondence by an IMM filter. The DA algorithm for estimation of finite mixture distributions [5] is proposed for shape parameter evaluation. Also, a PF for size parameters computation is designed and implemented with the IMM filtering scheme. In addition, we formulate the system model in a PCDL form through a measurement coordinate system conversion and discretisation of the continuous set of object parameters. As a result, we propose an alternative strategy, based on the MKF [1], [12] and [13, ch. 11]. The developed MKFm generates recursively samples of indicator variables and integrates out the linear and Gaussian state variables conditioned on these indicators. Due to the marginalization, the MKFm is more accurate than the conventional PF and performs the state and parameter estimation in a common sequential MC framework. This is achieved by two latent indicator variables characterizing, respectively, the motion regimes and size type. An additional statistical model validation scheme is incorporated into the MKFm to confirm or reject the critical model, based on the measured data.

The correspondence is organized as follows. Section II-C describes the system dynamics and measurement models. Section III formulates the problem. The IMM-DA algorithm is presented in Section IV, and the MKFm is given in Section V. A model validation test is described in Section VI. A comparative analysis of the developed algorithms is presented in Section VII. Section VIII summarizes the results.

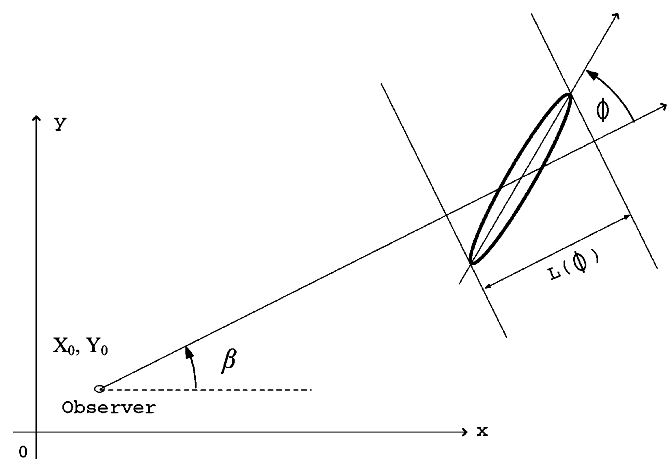

Fig. 1. Position of the ship versus the position of the observer.

II. System Dynamics and Measurement Models

A. System Model—General Form

Consider the following model:

$$\boldsymbol{x}_k = \boldsymbol{f}(m_k, \boldsymbol{x}_{k-1}, \boldsymbol{\theta}, \boldsymbol{w}_k) \tag{1}$$

$$z_k = \boldsymbol{h}(m_k, \boldsymbol{x}_k, \boldsymbol{\theta}, \boldsymbol{v}_k), \quad k = 1, 2, \ldots \tag{2}$$

of a discrete-time jump Markov system, describing the object dynamics and sensor measurements where $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$ is the *base (continuous) state* vector, with transition function $\boldsymbol{f}$, $z_k \in \mathbb{R}^{n_z}$ is the measurement vector with measurement function $\boldsymbol{h}$, and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is a vector, containing unknown static parameters. The noises $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are independent identically distributed (i.i.d.) Gaussian processes with characteristics $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$ and $\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R})$, respectively. The *modal (discrete) state* $m_k \in \mathbb{S} \triangleq \{1, 2, \ldots, s\}$ is a first-order Markov chain with transition probabilities $p_{ij} \triangleq \Pr\{m_k = j \,|\, m_{k-1} = i\}$, $(i, j \in \mathbb{S})$ and initial probability distribution $P_0(i) \triangleq \Pr\{m_0 = i\}, i \in \mathbb{S}$, such that $P_0(i) \geq 0$, and $\sum_{i=1}^{s} P_0(i) = 1; k = 1, 2, \ldots$ is a discrete time.

Consider a base state vector $\boldsymbol{x}_k = (x_k, \dot{x}_k, y_k, \dot{y}_k)'$, where x and y specify the position of an extended object, namely a ship, with respect to an observer's position, assumed known; $(\dot{x}, \dot{y})$ is the velocity in the Cartesian plane, centered at the observer's location (Fig. 1). All possible motion regimes s of the maneuvering ship are modeled by the modal state variable m. The static parameter vector $\boldsymbol{\theta} = (\ell, \gamma)'$ contains shape parameters: the major axis length ℓ of the ship ellipse and the aspect ratio between the minor and major axes γ. Based on *a prior* information about ship types, we assume that $\boldsymbol{\theta}$ takes values from a known discrete (size type) set $\boldsymbol{\theta} \in \mathbb{T} \triangleq \{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_t\}$ with known prior distribution: $P_{\boldsymbol{\theta}_0}(i) \triangleq \Pr\{\boldsymbol{\theta} = \boldsymbol{\theta}_i\}, i \in \{1, \ldots, t\}$.

B. Measurement Equation

Similarly to [7] and [9], we assume that a high-resolution radar provides measurements of range r, bearing β to the object centroid and the object down-range extent L along the observer–object line-of-sight (LOS) (Fig. 1). Here, (X_0, Y_0) is the location of the observer. The relationship between L and the angle ϕ between the major axis of the ellipse and the target–observer LOS is given by $L(\phi) = \ell\sqrt{\cos^2\phi + \gamma^2 \sin^2(\phi)}$. If the target ellipse is oriented so that its major axis is parallel to the velocity vector $(\dot{x}, \dot{y})$ then the along-range target extent $L(\phi)$ can be written as a function of the state vector $\boldsymbol{x}_k$ and $\boldsymbol{\theta}$ [7], [9] so that

$$L(\phi(\boldsymbol{x}_k)) = \boldsymbol{\theta}(1)\sqrt{\cos^2\phi(\boldsymbol{x}_k) + \boldsymbol{\theta}(2)^2\sin^2\phi(\boldsymbol{x}_k)} \tag{3}$$

where $\boldsymbol{\phi}(\boldsymbol{x}_k) = \arctan((x_k\dot{y}_k - \dot{x}_k y_k)/(x_k\dot{x}_k + y_k\dot{y}_k))$. For the measurement vector $z_k = (r_k, \beta_k, L_k)'$, the measurement function

$$\boldsymbol{h}(\boldsymbol{x}_k, \boldsymbol{\theta}) = \begin{pmatrix} \sqrt{(x_k - X_0)^2 + (y_k - Y_0)^2} \\ \arctan((y_k - Y_0)/(x_k - X_0)) \\ L(\phi(\boldsymbol{x}_k)) \end{pmatrix} \tag{4}$$

in (2) is highly nonlinear. The considered problem has its own particularities. Since the function $\boldsymbol{f}$ in (1) depends on the motion regimes only, the state evolution is *a priori* independent on $\boldsymbol{\theta}$. The kinematic states $\boldsymbol{x}_k$ and the modal states m_k can be estimated approximately through r and β, without using measurements of L. This is the rationale for proposing a separate algorithm for estimating $\boldsymbol{x}_k$ and m_k, like in the conventional tracking filters. The measurement function (4), and measurement vector, respectively, are split into two parts: $z_k = ((z_k^1)', z_k^2)'$, where $z_k^1 = (r_k, \beta_k)'$ is related to the kinematic states and $z_k^2 = L_k$ is related to the object shape. The shape parameters are estimated by the PF or DA based on the state estimate and z_k^2.

C. System Model—Partially Conditional Dynamic Linear Model

The general system model (1) can be presented in the form

$$\boldsymbol{x}_k = \boldsymbol{F}(m_k)\boldsymbol{x}_{k-1} + \boldsymbol{G}(m_k)\boldsymbol{w}_k(m_k) \tag{5}$$

$$z_k^1 = \boldsymbol{H}(m_k)\boldsymbol{x}_k + \boldsymbol{v}_k^1(m_k) \tag{6}$$

$$z_k^2 = L(\boldsymbol{\theta}_{\lambda_k}, \boldsymbol{x}_k) + v_k^2(\lambda_k), \quad k = 1, 2, \ldots \tag{7}$$

where the *modal (discrete) state* $m_k \in \mathbb{S}$ can be thought of as a *first indicator* variable. The *second indicator* variable λ_k takes values from the set $\mathbb{N}_t \triangleq \{1, 2, \ldots, t\}$ with probability $P(\lambda_k = i \mid \lambda_{k-1}) = P(\lambda_k = i) = 1/t$ and represents the *size type*. The noises have characteristics: $\boldsymbol{w}_k(m_k) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}(m_k)), \boldsymbol{v}_k^1(m_k) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}(m_k))$ and $v_k^2(\lambda_k) \sim \mathcal{N}(0, R_L(\lambda_k))$. The matrices $\boldsymbol{F}, \boldsymbol{G}$, and $\boldsymbol{H}$ are known, assuming that the indicator vector $\boldsymbol{\Lambda}_k = \{m_k, \lambda_k\}$ is known. Note that $\mathbb{S}$ is the set of the manoeuvring modes, whereas $\mathbb{N}_t$ is the set of index numbers of the discretized extent parameter. After converting the measurements from polar (r_k, β_k) to Cartesian (x_k, y_k) coordinates: $z_k^1 = (r_k\cos(\beta_k),\ r_k\sin(\beta_k))'$, the measurement (6) becomes linear with a simple measurement matrix $\boldsymbol{H}$ and with a covariance matrix $\boldsymbol{R}_c$ [14, p. 399].

Conditioned on the modal state m_k, (5) and (6) represent the DLM, and a KF can be applied for state and kinematic likelihood estimation. Conditioned on the indicator variable λ_k, the extent measurement likelihood can be calculated by (7), using the KF state estimate. A joint likelihood can be used to determine the most likely mode-size combination in the MKF framework. The above system (5)–(7) is referred to as a *partially* CDLM (PCDLM), since the nonlinear (7) takes part in the extent likelihood computation.

III. Problem Formulation

The *goal* is to estimate the *state* vector $\boldsymbol{x}_k$ and the *extent parameter* vector $\boldsymbol{\theta}$, based on measured data $\boldsymbol{Z}^k = \{z_1, z_2, \ldots, z_k\}$. If the *posterior joint state-size probability density function* (PDF)

$$p(\boldsymbol{x}_k, \boldsymbol{\theta}|\boldsymbol{Z}^k) = p(\boldsymbol{\theta}|\boldsymbol{x}_k, \boldsymbol{Z}^k)p(\boldsymbol{x}_k|\boldsymbol{Z}^k) \tag{8}$$

can be calculated, then the required estimate is given by

$$\begin{aligned} &E\{\boldsymbol{x}_k\boldsymbol{\theta} \mid \boldsymbol{Z}^k\} \\ &\quad = \int\int \boldsymbol{x}_k\boldsymbol{\theta}p(\boldsymbol{\theta}|\boldsymbol{x}_k, \boldsymbol{Z}^k)p(\boldsymbol{x}_k|\boldsymbol{Z}^k)d\boldsymbol{\theta}d\boldsymbol{x}_k \\ &\quad = \int\left\{\int \boldsymbol{\theta}p(\boldsymbol{\theta}|\boldsymbol{x}_k, \boldsymbol{Z}^k)d\boldsymbol{\theta}\right\}\boldsymbol{x}_k p(\boldsymbol{x}_k|\boldsymbol{Z}^k)d\boldsymbol{x}_k \\ &\quad = \int \boldsymbol{\zeta}_\theta(\boldsymbol{x}_k)\boldsymbol{x}_k p(\boldsymbol{x}_k|\boldsymbol{Z}^k)d\boldsymbol{x}_k \approx \boldsymbol{\zeta}_\theta(\hat{\boldsymbol{x}}_k)\hat{\boldsymbol{x}}_k \end{aligned} \tag{9}$$

where $\boldsymbol{\zeta}_\theta(\hat{\boldsymbol{x}}_k) = E(\boldsymbol{\theta}|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k)$ represents the expectation of the parameter vector $\boldsymbol{\theta}$, and $\hat{\boldsymbol{x}}_k \triangleq E\{\boldsymbol{x}_k|\boldsymbol{Z}^k\}$. Denote the lth mode history, realized by a Markovian jump system through time k as $m_k^l = \{m_0^l, m_1^l, \ldots, m_k^l\},\ l = 1, \ldots, s^k$. The state posterior PDF is obtained as a Gaussian mixture with an exponentially increasing number of terms [14]

$$p(\boldsymbol{x}_k|\boldsymbol{Z}^k) = \sum_{l=1}^{s^k} p\left(\boldsymbol{x}_k|m_k^l, \boldsymbol{Z}^k\right)P\left(m_k^l|\boldsymbol{Z}^k\right). \tag{10}$$

The exponential growth of computations can be avoided by different combinations of model histories. The generalized pseudo-Bayesian approaches (e.g., GPB1 and GPB2) [14] consider all possible models in the last one (two) sampling periods.

The IMM filter [14], [15] approximates the posterior state PDF

$$p(\boldsymbol{x}_k|\boldsymbol{Z}^k) \approx \sum_{j=1}^{s} p(\boldsymbol{x}_k|m_k = j, \boldsymbol{Z}^k)P(m_k = j|\boldsymbol{Z}^k) \tag{11}$$

by using s working in parallel KFs where each KF utilizes a different combination of the previous model-conditioned estimates. In the light of the considered problem, the IMM-DA and IMM-PF algorithms estimate the kinematic state based on part of the measurements $\tilde{\boldsymbol{Z}}^{(1,k)} = \{z_1^1, z_2^1, \ldots, z_k^1\}$. Hence, this kinematic state estimate is obtained as a sum of mode-conditioned state estimates $\boldsymbol{\mu}_k^j$,

$$\hat{\boldsymbol{x}}_k^{\text{imm}} \approx \sum_{j=1}^{s} \boldsymbol{\mu}_k^j P\left(m_k = j|\tilde{\boldsymbol{Z}}^{(1,k)}\right) \tag{12}$$

weighted by mode probabilities where

$$P\left(m_k = j \mid \tilde{\boldsymbol{Z}}^{(1,k)}\right) \propto p\left(z_k^1|m_k = j, \tilde{\boldsymbol{Z}}^{(1,k-1)}\right)P\left(m_k = j|\tilde{\boldsymbol{Z}}^{(1,k-1)}\right).$$

The state estimate $\hat{\boldsymbol{x}}_k^{\text{imm}}$ is applied in (9), instead of $\boldsymbol{x}_k$, for size parameters evaluation

$$\hat{\boldsymbol{\theta}}(\hat{\boldsymbol{x}}_k) \triangleq E\{\boldsymbol{\theta}|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k\} = \boldsymbol{\zeta}_\theta(\hat{\boldsymbol{x}}_k) = \int \boldsymbol{\theta}p(\boldsymbol{\theta}|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k)d\boldsymbol{\theta}. \tag{13}$$

The posterior PDF $p(\boldsymbol{\theta}|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k)$ in (13) can be approximated in different ways. Let us suppose that the shape parameter $\boldsymbol{\theta}$ is replaced by $\boldsymbol{\theta}_k$, which evolves according to the Markovian model $\boldsymbol{\theta}_k = \boldsymbol{\theta}_{k-1} + \boldsymbol{w}_k^\theta$, $\boldsymbol{w}_k^\theta \sim \mathcal{N}(\boldsymbol{w}_k^\theta; \boldsymbol{0}, \boldsymbol{Q}^\theta)$, where $\boldsymbol{w}_k^\theta$ is an artificial noise with covariance matrix $\boldsymbol{Q}^\theta$. Particle filtering provides a discrete weighted approximation to the true posterior PDF

$$p(\boldsymbol{\theta}_k|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k) \approx \sum_{j=1}^{N_p} w_k^{(j)}\delta\left(\boldsymbol{\theta}_k - \boldsymbol{\theta}_k^{(j)}\right);$$

$$\hat{\boldsymbol{\theta}}_k \approx \sum_{j=1}^{N_p} w_k^{(j)}\boldsymbol{\theta}_k^{(j)} \tag{14}$$

where $\boldsymbol{\theta}_k^{(j)}, j=1,\ldots,N_p$ is the set of supported points with the associated weights $w_k^{(j)} \propto w_{k-1}^{(j)} p(z_k^2|\boldsymbol{\theta}_k^{(j)}, \hat{\boldsymbol{x}}_k),\ j=1,\ldots,N_p$.

From the point of view of the DA, the posterior PDF $p(\boldsymbol{\theta}|\hat{\boldsymbol{x}}_k, \boldsymbol{Z}^k)$ is proportional to the PDF of the extent measurement $p(z_k^2|\boldsymbol{\theta}, \hat{\boldsymbol{x}}_k)$ which is considered as a t-component Gaussian mixture

$$p\left(z_k^2|\boldsymbol{\theta}, \hat{\boldsymbol{x}}_k\right) = \sum_{i=1}^{t} \pi_i \mathcal{N}\left(z_k^2; L(\boldsymbol{\theta}_i, \hat{\boldsymbol{x}}_k), R_L\right) \tag{15}$$

where $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_t)$ is a vector of mixture proportions (constrained to be non-negative and sum to unity) and $L(\boldsymbol{\theta}_i, \hat{\boldsymbol{x}}_k)$ is the measurement prediction, calculated according to (3). Thus, the task of size type determination is reduced to the well known *finite mixture estimation problem*: for the mixture model (15) with *known component* PDFs $\mathcal{N}(z_k^2; L(\boldsymbol{\theta}_i, \hat{\boldsymbol{x}}_k), R_L)$, one needs to estimate the *unknown weight* vector $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_t)$. At each time step k, the DA iteratively evaluates weights ($\boldsymbol{\pi}(k) \triangleq \boldsymbol{\pi}$), using the *joint* PDF of measurements over a sliding window $\{z_{k-ws+1}^2, z_{k-ws+2}^2 \ldots, z_k^2\}$, where ws is the window size. The mixture component with a *maximum weight* identifies the *most probable ship type*. The estimate of the extent parameters can be calculated as a sum of the parameter values $\boldsymbol{\theta}_i$ (from the discrete set), weighted by mixture proportions: $\hat{\boldsymbol{\theta}}_k = \sum_{i=1}^{t} \pi_i(k)\boldsymbol{\theta}_i$.

Unlike the PF and DA algorithms, which are implemented jointly with an IMM filter ($\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k^{\text{imm}}$), the MKFm provides estimates both of the system states and parameters. In contrast to (11), the MKFm approximates the posterior state PDF $p(\boldsymbol{x}_k|\boldsymbol{Z}^k)$ by a *random* mixture of N Gaussian distributions

$$p(\boldsymbol{x}_k|\boldsymbol{Z}^k) \approx \sum_{j=1}^{N} \tilde{w}_k^{(j)} \mathcal{N}\left(\boldsymbol{x}_k^{(j)}; \boldsymbol{\mu}_k^{(j)}, \boldsymbol{P}_k^{(j)}\right) \tag{16}$$

where the weights $\tilde{w}_k^{(j)}$ incorporate the properties of the posterior PDF of the indicator vector. The Gaussian mixture parameters, $(\boldsymbol{\mu}_k^{(j)}, \boldsymbol{P}_k^{(j)})$, respectively, the estimated mean $\boldsymbol{\mu}_k^{(j)}$ and state covariance matrix $\boldsymbol{P}_k^{(j)}$, are obtained by implementing a Kalman filter for a given mode history, as a part of the common mode-size type history.

Let $\widetilde{\boldsymbol{\Lambda}}_{k-1}^{(j)} = \left(\boldsymbol{\Lambda}_1^{(j)}, \boldsymbol{\Lambda}_2^{(j)}, \ldots, \boldsymbol{\Lambda}_{k-1}^{(j)}\right),\ j=1,\ldots,N$ be the set of indicator vectors, representing joint mode and size type history up to time $k-1$. Suppose that at time k, the indicator vector $\boldsymbol{\Lambda}_k^{(j)} = \{m_k^{(j)}, \lambda_k^{(j)}\}$ is generated based on $\tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}$, and the likelihoods of current kinematic and extent measurements. The parameter estimate $\hat{\boldsymbol{\theta}}_k$ is closely related to the posterior indicator probability $P(\lambda_k = i|\boldsymbol{Z}^k)$, which can be estimated as

$$P(\lambda_k = i|\boldsymbol{Z}^k) \approx \sum_{j=1}^{N} 1\left(\lambda_k^{(j)} = i\right) \tilde{w}_k^{(j)},\ i=1,\ldots t \tag{17}$$

where $1(\cdot)$ is an indicator function such that $1(\lambda_k = l) = 1$, if $\lambda_k = l$ and $1(\lambda_k = l) = 0$, otherwise. The state and extent parameter estimates are given by

$$\hat{\boldsymbol{x}}_k^{\text{mkf}} \approx \sum_{j=1}^{N} \tilde{w}_k^{(j)} \boldsymbol{\mu}_k^{(j)}, \quad \hat{\boldsymbol{\theta}}_k \approx \sum_{i=1}^{t} P(\lambda_k = i|\boldsymbol{Z}^k)\boldsymbol{\theta}_i. \tag{18}$$

While the PF looks for the solution in the *continuous interval* of shape parameters $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, the MKF and DA identify it among a *discrete set* of values $\boldsymbol{\theta} \in \mathbb{T}$, with a given prior distribution.

The proposed here technique comprises two steps. On receipt of a new measurement z_k, first the IMM algorithm (or MKF) is run with the previous state and mode estimates to update the current estimates, using the likelihood of kinematic measurements. Next, the current parameter estimate $\hat{\boldsymbol{\theta}}_k$ is found based on the previous, $\hat{\boldsymbol{\theta}}_{k-1}$, the current state and mode estimates and the extent measurement likelihood, respectively, by the PF, DA scheme, or the MKF.

IV. Extent Parameters Estimation by DA

The mixture model is given by the observation of n independent random variables $y_1,\ldots,y_n$ from a t-component mixture [5],

$$\digamma(y_k) = \sum_{i=1}^{t} \pi_i \digamma_i(y_k),\ k=1,\ldots,n \tag{19}$$

where the densities $\digamma_i,\ i=1,\ldots,t$ are known or are known up to a parameter. We consider the special case, where *only* the weights π_i have to be estimated. The DA algorithm approximates the mixture posterior distribution relying on the missing data structure of the mixture model. According to [5], the mixture model can always be expressed in terms of missing (or incomplete) data $\boldsymbol{\delta}(k)$. The vectors $\boldsymbol{\delta}(k) = (\delta_1(k), \delta_2(k), \ldots, \delta_t(k)), k=1,2,\ldots,n$ with components $\delta_i(k) \in \{0,1\},\ i=1,2,\ldots,t$ are defined to indicate that the measurement y_k has density $\digamma_i(y_k)$ [11]. The model is *hierarchical* with the true parameter vector $\boldsymbol{\pi}$ of the mixture, on the top level. Hence, the distribution $p(\boldsymbol{\delta}|\boldsymbol{\pi})$ of the missing data $\boldsymbol{\delta}$ depends on $\boldsymbol{\pi}$, i.e., $\boldsymbol{\delta} \sim p(\boldsymbol{\delta}|\boldsymbol{\pi})$. The observed data, $\boldsymbol{y} \sim p(\boldsymbol{y}|\boldsymbol{\pi}, \boldsymbol{\delta})$, are at the bottom level.

Starting with an initial value $\boldsymbol{\pi}^{(0)}$, the algorithm implements a *two-step iterative scheme*:

i) at the iteration $u, u=0,1,2,\ldots$, *generate* $\boldsymbol{\delta}^{(u)} \sim p(\boldsymbol{\delta}|\boldsymbol{y}, \boldsymbol{\pi}^{(u)})$ from a multinomial distribution with weights proportional to the observation likelihoods, $\delta_i^{(u)}(k) \propto \pi_i^{(u)} \digamma_i(y_k)$;

ii) then, *generate* $\boldsymbol{\pi}^{(u+1)} \sim p(\boldsymbol{\pi}|\boldsymbol{y}, \boldsymbol{\delta}^{(u)})$.

Since the conjugate priors on $\boldsymbol{\pi}$ are with Dirichlet distributions (DD), $\mathcal{D}(\alpha_1,\ldots,\alpha_t)$ [5], $\boldsymbol{\pi}^{(u+1)}$ is generated according to the DDs with parameters, depending on the missing data. Bayesian sampling produces an ergodic Markov chain $(\boldsymbol{\pi}^{(u)})$ with stationary distribution $p(\boldsymbol{\pi}|\boldsymbol{y})$. Thus, after u_0 initial (warming up) steps, a set of U samples $\boldsymbol{\pi}^{(u_0+1)},\ldots,\boldsymbol{\pi}^{(u_0+U)}$ are approximately distributed as $p(\pi|\boldsymbol{y})$. Due to ergodicity, averaging can be made with respect to time [5]. In the present implementation, the observation y_k coincides with the along-range extent measurement $z_k^2 \equiv L_k$ and $\digamma_i(z_k^2) \equiv \mathcal{N}(z_k^2; L(\boldsymbol{\theta}_i, \hat{\boldsymbol{x}}_k), R_L),\ k=1,2,\ldots,n,\ldots$. The joint IMM-DA scheme is given below.

Joint IMM—Data Augmentation Scheme

For $k=1,2,\ldots$

— Run the IMM algorithm with the previous state vector $\hat{\boldsymbol{x}}_{k-1}$, covariance matrix $\boldsymbol{P}_{k-1}$ and mode probabilities $P(m_k = j\,|\,Z^{k-1})$ to update the current estimate $\hat{\boldsymbol{x}}_k, \boldsymbol{P}_k$ and $P(m_k = j\,|\,\boldsymbol{Z}^k), j=1,\ldots,s$.

— Compute mixture components conditional PDFs

$$\tilde{G}_i(k) = \exp\left[-\left(z_k^2 - L\left(\boldsymbol{\theta}_i, \hat{\boldsymbol{x}}_k\right)\right)^2/(2R_L)\right], \quad i=1,\ldots,t.$$

— Implement data augmentation

— Initialization: $\boldsymbol{\pi}^{(0)} = \boldsymbol{\pi}(k-1)$;

— Iterations $(u=0,1,\ldots,u_0+U-1)$

- Missing data conditional probability mass functions

$$q_i^{(u)}(l) = \frac{\pi_i^{(u)}\tilde{G}_i(l)}{\sum_{i=1}^{t}\pi_i^{(u)}\tilde{G}_i(l)}, \quad \text{for } l=1,2,\ldots k, \quad i=1,2,\ldots t.$$

- Missing data generation (multinomial sampling)

$$\boldsymbol{\delta}^{(u)}(l) = (0,\ldots,0,1,0,\ldots,0) \sim \left\{q_i^{(u)}(l)\right\}_{i=1}^{t}, \quad l=1,2,\ldots k.$$

- Weights evaluation (Dirichlet distribution sampling)

$$\boldsymbol{\pi}^{(u+1)} \sim \mathcal{D}\left(\boldsymbol{\pi}; \alpha_1 + \sum_{l=1}^{k} \delta_1^{(u)}(l), \ldots, \alpha_t + \sum_{l=1}^{k} \delta_t^{(u)}(l)\right).$$

— Calculate the output estimates

$$\boldsymbol{\pi}(k) = \frac{1}{U}\sum_{\sigma=1}^{U} \boldsymbol{\pi}^{(u_0+\sigma)} \quad \text{and} \quad \hat{\boldsymbol{\theta}}_k = \sum_{i=1}^{t} \pi_i(k)\boldsymbol{\theta}_i.$$

V. Extended Object Tracking by MKFm

When substituting the indicator vectors $\boldsymbol{\Lambda}_k = \{m_k, \lambda_k\}, k = 1, 2, \ldots$, into the PCDLM (5)–(7), all vectors $\boldsymbol{x}_k, k = 1, 2, \ldots$ can be integrated out recursively by using a standard Kalman filter [1], [13]. If the MC sampling is performed in the space of indicator variables instead of in the space of the state variables, we obtain the MKF, which in principle gives more accurate results than the MC filters dealing with $\boldsymbol{x}_k$ directly.

Let a collection of N-Kalman filters, $\mathrm{KF}_{k-1}^{(1)}, \ldots, \mathrm{KF}_{k-1}^{(j)}, \ldots, \mathrm{KF}_{k-1}^{(N)}$ be run at time $k-1$. Each $\mathrm{KF}_{k-1}^{(j)}$ is characterized by the mean state vector $\boldsymbol{\mu}_{k-1}^{(j)}$ and its covariance matrix $\boldsymbol{P}_{k-1}^{(j)}$, i.e., with $(\boldsymbol{\mu}_{k-1}^{(j)}, \boldsymbol{P}_{k-1}^{(j)})$. Since the PCDLM (5), (6) is reduced to a DLM when conditioning on $\tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)} = (\boldsymbol{\Lambda}_1^{(j)}, \boldsymbol{\Lambda}_2^{(j)}, \ldots, \boldsymbol{\Lambda}_{k-1}^{(j)})$, the mean vector $\boldsymbol{\mu}_{k-1}^{(j)}$ and the covariance matrix $\boldsymbol{P}_{k-1}^{(j)}$, constitute a sufficient statistics at time $k-1$. Each filter is associated with a weight $w_{k-1}^{(j)}$ [1]. The update of the filter statistics $\mathrm{KF}_{k-1}^{(j)} \rightarrow \mathrm{KF}_k^{(j)}$ at time k is summarized below.

The first step begins with the computation of a trial sampling density for $m_k = i_1, \quad i_1 \in \mathbb{S}$:

$$\begin{aligned}
\mathcal{L}_{k,i_1}^{(j)} &\triangleq P\left(m_k = i_1 \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^k\right) \\
&\propto p\left(z_k^1 \,|\, m_k = i_1, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}\right) P\left(m_k = i_1 \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}\right), \\
&\quad P\left(m_k = i_1 \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}\right) \\
&= p\left(m_k = i_1 \,|\, m_{k-1}^{(j)}\right) = p_{m_{k-1}^{(j)}, i_1}.
\end{aligned}$$

The measurement $\boldsymbol{z}_k^1$ has a Gaussian density

$$\begin{aligned}
p\left(\boldsymbol{z}_k^1 \,|\, m_k = i_1, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}\right) &= p\left(\boldsymbol{z}_k^1 \,|\, m_k = i_1, \mathrm{KF}_{k-1}^{(j)}\right) \\
&\sim \mathcal{N}\left(\boldsymbol{z}_k^1; \boldsymbol{H}\boldsymbol{\mu}_{k\,|\,k-1}^{(j)}(m_k), \boldsymbol{S}_k^{(j)}(m_k)\right) \quad (20)
\end{aligned}$$

where $\boldsymbol{\mu}_{k\,|\,k-1}^{(j)}(m_k)$ is the predicted state vector and $\boldsymbol{S}_k^{(j)}(m_k)$ is the measurement prediction covariance matrix, calculated by a filter $\mathrm{KF}_{k-1}^{(j)}$, adjusted for $m_k = i_1$. Then, the indicator $m_k = i_1 \in \{1, \ldots, s\}$ is imputed with probability, proportional to $\mathcal{L}_{k,i_1}^{(j)}$. The mean vector $\boldsymbol{\mu}_k^{(j)}$ and covariance matrix $\boldsymbol{P}_k^{(j)}$ are updated only for the sampled index $i_1 = \ell_1$.

The second step comprises the computation of a trial sampling density for $\lambda_k = i_2, i_2 \in \mathbb{N}_t$:

$$\mathbb{L}_{k,i_2}^{(j)} \propto p\left(z_k^2 \,|\, \lambda_k = i_2, \mathrm{KF}_k^{(j)}\right) P\left(\lambda_k = i_2 \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}\right)$$

where $p(z_k^2 \,|\, \lambda_k = i_2, \mathrm{KF}_k^{(j)}) \sim \mathcal{N}(z_k^2; L(\theta_{i_2}, \boldsymbol{\mu}_k^{(j)}), R_L)$ and $P(\lambda_k = i_2 \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}, \boldsymbol{Z}^{k-1}) = P(\lambda_k = i_2) = 1/t$.

Finally, the weights for this updated filter estimate are calculated as

$$w_k^{(j)} = w_{k-1}^{(j)} \mathcal{L}_{k,\ell_1}^{(j)} \sum_{i_2=1}^{t} p\left(z_k^2 \,|\, \lambda_k = i_2, \mathrm{KF}_k^{(j)}\right) P(\lambda_k = i_2).$$

Based on the normalized weights $(\tilde{w}_k^{(j)})$, *estimates* $\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{\theta}}_k$, and *posterior indicator probabilities*, we calculate

$$\begin{aligned}
\hat{\boldsymbol{x}}_k^{mkf} &= \sum_{j=1}^{N} \boldsymbol{\mu}_k^{(j)} \tilde{w}_k^{(j)}, \\
\hat{\boldsymbol{\theta}}_k &= \sum_{i_2=1}^{t} P\left(\lambda_k = i_2 \,|\, \boldsymbol{Z}^k\right) \boldsymbol{\theta}_{i_2}, \\
P\left(m_k = i_1 \,|\, \boldsymbol{Z}^k\right) &= \sum_{j=1}^{N} 1\left(m_k^{(j)} = i_1\right) \tilde{w}_k^{(j)}, \qquad i_1 \in \mathbb{S} \\
P\left(\lambda_k = i_2 \,|\, \boldsymbol{Z}^k\right) &= \sum_{j=1}^{N} 1\left(\lambda_k^{(j)} = i_2\right) \tilde{w}_k^{(j)}, \quad i_2 \in \mathbb{N}_t.
\end{aligned}$$

Using (9), we modified the first step of the MKF, and the dimension of the indicator space $\boldsymbol{\Lambda}_k \in \mathbb{S} + \mathbb{N}_t$ is reduced compared with the MKF [1], [13, ch. 11] . We refer to this algorithm as a MKF *modified* (MKFm) and it is given below. The proposed MKFm differs from the MKF of Chen and Liu [1] in the way of calculating the trial sampling density $P(m_k, \lambda_k \,|\, \tilde{\boldsymbol{\Lambda}}_{k-1}, \boldsymbol{Z}^k)$ for the indicator vector $\boldsymbol{\Lambda}_k$. The MKF of Chen and Liu, applied to the extended object tracking, requires quite high-dimensional indicator sampling space $\boldsymbol{\Lambda}_k \in \mathbb{S} \times \mathbb{N}_t$, which increases the computational time [1, ch. 11].

MKFm for State and Size Parameters Estimation

1) Initialization, $k = 0$; For $j = 1, \ldots, N$: sample $m_0^{(j)} \sim \{P_0(i_1)\}_{i_1=1}^{s}$ and $\lambda_0^{(j)} \sim \{P_{\boldsymbol{\theta}_0}(i_2)\}_{i_2=1}^{t}$. Form $\tilde{\boldsymbol{\Lambda}}_0^{(j)} = \{m_0^{(j)}, \lambda_0^{(j)}\}$. Set $\mathrm{KF}_0^{(j)} = \{\boldsymbol{\mu}_0^{(j)}, \boldsymbol{P}_0^{(j)}\}$, where $\boldsymbol{\mu}_0^{(j)} = \hat{\boldsymbol{\mu}}_0$ and $\boldsymbol{P}_0^{(j)} = \boldsymbol{P}_0$ are the mean and covariance of the initial state $\boldsymbol{x}_0 \sim \mathcal{N}(\hat{\boldsymbol{\mu}}_0, \boldsymbol{P}_0)$. Set the initial weights $w_0^{(j)} = 1/N$. Set $k = 1$.
2) For $j = 1, \ldots, N$ complete:
   - For each $i_1 \in \mathbb{S}$ compute
     — one step prediction for each Kalman filter $\mathrm{KF}_{k-1}^{(j)}$: $(\boldsymbol{\mu}_{k\,|\,k-1}^{(j)})^{(i_1)}, \boldsymbol{P}_{k\,|\,k-1}^{(j)})^{(i_1)}, (\boldsymbol{S}_k^{(j)})^{(i_1)}$
     — on receipt of a measurement $\boldsymbol{z}_k^1$, calculate $\mathcal{L}_{k,i_1}^{(j)}$
     — sample $m_k^{(j)} \sim \{\mathcal{L}_{k,i_1}^{(j)}\}_{i_1=1}^{s}$; suppose that $m_k^{(j)} = \ell_1$
     — for ℓ_1 perform $\mathrm{KF}_k^{(j)}$ update: obtain $\boldsymbol{\mu}_k^{(j)}; \boldsymbol{P}_k^{(j)}$.
   - For each $i_2 \in \mathbb{N}_t$ and $z_k^2 = L_k$, calculate $\mathbb{L}_{k,i_2}^{(j)}$
     — sample $\lambda_k^{(j)} \sim \{\mathbb{L}_{k,i_2}^{(j)}\}_{i_2=1}^{t}$; suppose that $\lambda_k^{(j)} = \ell_2$. Append $\boldsymbol{\Lambda}_k^{(j)} = \{\ell_1, \ell_2\}$ to $\tilde{\boldsymbol{\Lambda}}_{k-1}^{(j)}$ and obtain $\tilde{\boldsymbol{\Lambda}}_k^{(j)}$.
   - Update the importance weights: $w_k^{(j)} = w_{k-1}^{(j)} \mathcal{L}_{k,\ell_1}^{(j)} \sum_{i_2=1}^{t} \mathbb{L}_{k,i_2}^{(j)}$;
   - Normalize the weights $\tilde{w}_k^{(j)} = w_k^{(j)} / \sum_{j=1}^{N} w_k^{(j)}$.
3) Compute the output estimates and posterior probabilities of indicator variables.
4) Resample with replacement to avoid possible degeneracy of the sequential importance sampling [15] when an estimate $N_{\mathrm{eff}} = 1/\sum_{j=1}^{N}(\tilde{w}_k^{(j)})^2$ of the effective sample size falls below a threshold N_{thresh}. If $N_{\mathrm{eff}} < N_{\mathrm{thres}}$, resample: $(\boldsymbol{\mu}_k^{(j)}, \boldsymbol{P}_k^{(j)}, \tilde{\boldsymbol{\Lambda}}_k^{(j)}), j = 1, \ldots, N$, according to the weights; set $w_k^{(j)} = 1/N$.
5) Set $k \longleftarrow k + 1$ and go to Step 2).

VI. Model Validation

The posterior indicator probabilities provide a *relative* measure for the most probable behavior mode and size type at each time step k. When the *detection of a particular object size type* is important, a model validation scheme can be incorporated into the MKF framework as an additional test to confirm or reject the existence of a certain size type.

Let Z_k^2 denote the random variable, associated with the scalar observation z_k^2. According to [16], under the null hypothesis that the model $M \triangleq \boldsymbol{\theta}_k = \boldsymbol{\theta}_i$ is the correct one, the sequence $\{u_k : k = 1, \ldots, n\}$ with $u_k \triangleq p(Z_k^2 \leq z_k^2 \,|\, \boldsymbol{Z}^{k-1}, M)$ is a realization of i.i.d. random variables in the interval $[0, 1]$. By letting $v_k = \Phi^{-1}(u_k)$, where Φ is a standard normal cumulative distribution function, a sequence of independent $\mathcal{N}(0, 1)$ random variables is generated. This result holds for any time series model and can be used to provide a direct statistical test of the adequacy of the model $\boldsymbol{\theta}_k = \boldsymbol{\theta}_i$. To obtain u_k, it is necessary to integrate out $\boldsymbol{\theta}_k$ by evaluating

$$u_k = \int p\left(Z_k^2 \leq z_k^2 \,|\, \boldsymbol{Z}^{k-1}, \boldsymbol{\theta}_k\right) p\left(\boldsymbol{\theta}_k \,|\, \boldsymbol{Z}^{k-1}\right) d\boldsymbol{\theta}_k.$$

For complex models, this integration cannot, in general, be carried out analytically [16], but an estimate of u_k can be obtained using a MC test as follows: a) a sample of particles $\boldsymbol{\theta}_k^{(j)}, j = 1, \ldots, N$ is generated from $p(\boldsymbol{\theta}_k \,|\, \boldsymbol{Z}^{k-1})$ and b) since $p(z_k^2 \,|\, \boldsymbol{Z}^{k-1}, \boldsymbol{\theta}_k^{(j)}) = \mathcal{N}(z_k^2; L(\boldsymbol{\theta}_k^{(j)}, \hat{\boldsymbol{x}}_k), R_L)$, the estimate [16]

$$\hat{p}\left(Z_k^2 \leq z_k^2 \,|\, \boldsymbol{Z}^{k-1}, \boldsymbol{\theta}_k^{(j)}\right) = 1 - \frac{1}{2}\mathrm{erfc} \times \left(\left(z_k^2 - L\left(\boldsymbol{\theta}_k^{(j)}, \hat{\boldsymbol{x}}_k\right)\right) / \sqrt{2R_L}\right)$$

can be evaluated analytically, using the complementary error function $\mathrm{erfc}(\cdot)$. Then, an estimate of u_k is given by

$$\hat{u}_k = \frac{1}{N}\sum_{j=1}^{N} \hat{p}\left(Z_k^2 \leq z_k^2 \,|\, \boldsymbol{Z}^{k-1}, \boldsymbol{\theta}_k^{(j)}\right).$$

A Kolmogorov–Smirnov test is applied to validate the Gaussianity of the sequence $\{v_k : k = 1, \ldots, n\}$. The null hypothesis is that the data have a standard normal distribution and the alternative hypothesis is that data does not have that distribution. The null hypothesis is rejected if the test is significant at the 5% level.

VII. Simulation Results

The performance of the designed algorithms is evaluated over trajectories comprising uniform motions and abrupt maneuvers [a typical scenario is shown in Fig. 2(a)]. The observer is static, located at the origin of (x, y) plane. The initial target state is $\boldsymbol{x}_0 = (10\,000, -16, 85\,000, 5.8)'$. The object performs two turn maneuvers with normal accelerations of ± 3.0 [m/s^2]. Its length is $\ell = 50\,[m]$, and the aspect ratio (between the minor and major axes) is $\gamma = 0.2$. The sensor parameters are as follows [9]: sampling interval $T = 0.2$ [s]; the measurement error covariances along range, azimuth and along-range extent are respectively: $\boldsymbol{R} = \mathrm{diag}\{5^2\ [\mathrm{m}]^2, 0.2^2\ [\mathrm{deg}]^2\}$ and $R_L = 5^2\ [\mathrm{m}]^2$.

Root-mean squared errors (RMSEs) [14] are chosen as a measure of the algorithms' accuracy. Results from 100 Monte Carlo runs are presented below. The set $\mathbb{S}$ of modal states contains $s = 3$ elements, corresponding to motion models, the first of which is for nearly constant velocity motion. The next two models are matched to nearly coordinated turn maneuvers with turn rates $\omega = \pm 0.18$ [rad/s]. The transition matrices $\boldsymbol{F}(m)$ in (5) have the form given in [14, p. 467], for the case of known turn rate. We assume that $\boldsymbol{\theta}$ takes values from a set $\mathbb{T} = \{(30, 0.15), (50, 0.2), (70, 0.25), (100, 0.3)\} (t = 4)$, with equal initial probabilities. Note that $\boldsymbol{\theta}_2$ corresponds to the true $\boldsymbol{\theta}$.

The DA design parameters are chosen as follows [5], [10]: the sliding window size is ws $= 160$; the number of iterations is 150, and the

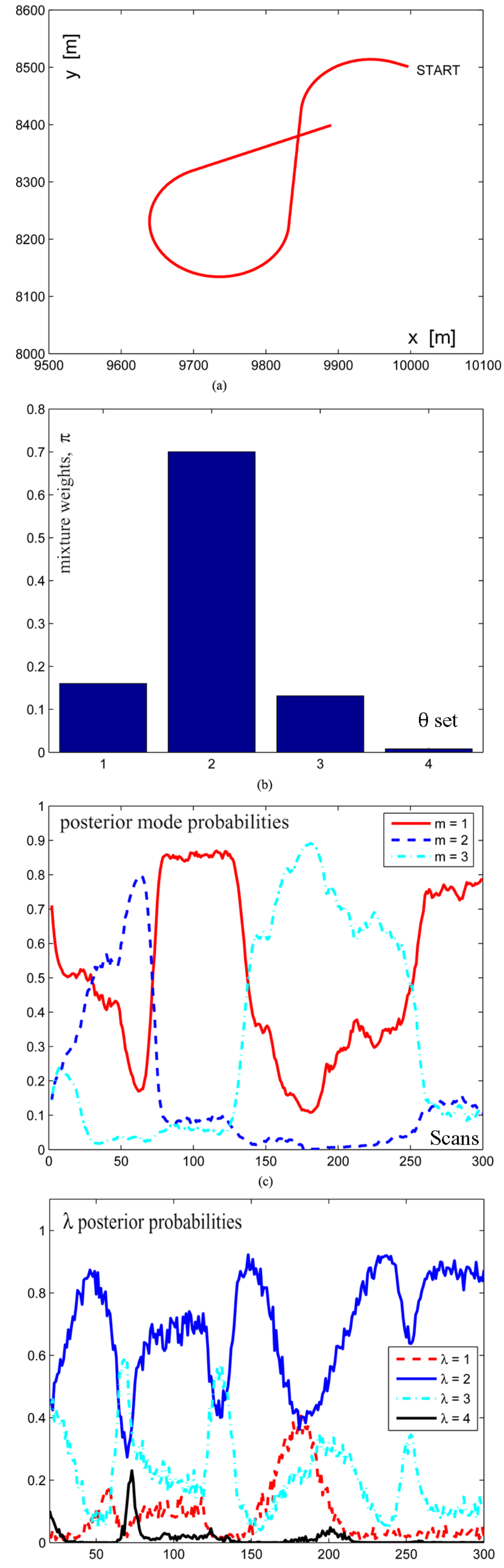

Fig. 2. (a) Testing scenario; (b) mixture proportions obtained by DA algorithms; (c) MKF posterior mode probabilities; and (d) MKF posterior size probabilities. In (c), $m = 1$ corresponds to turn rate $\omega = 0; \omega = \pm 0.18$ [rad/s] match to $m = 2$ and $m = 3$, respectively, and in (d) $\lambda = 2$ identifies the actual size $\boldsymbol{\theta}_2$.

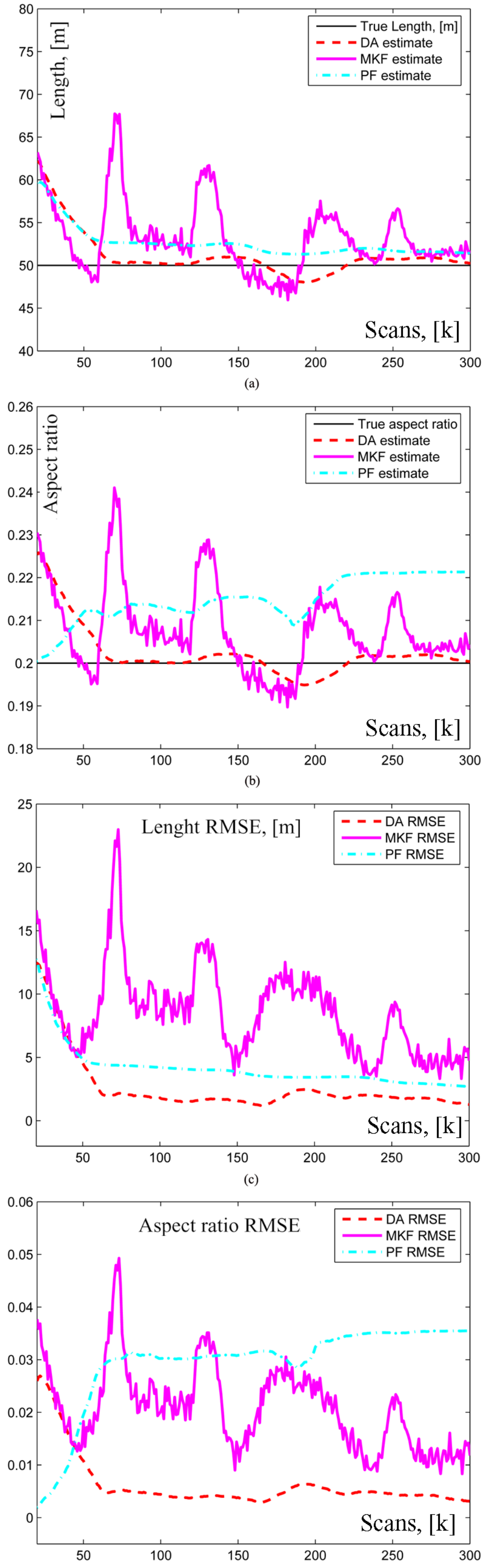

Fig. 3. Estimated extent parameters by DA, MKFm (with $N = 200$ particles), and PF (with $N_p = 300$ particles). (a) True and estimated length ℓ. (b) True and estimated γ. (c) RMSEs of the estimated ℓ (d) RMSEs of the estimated γ.

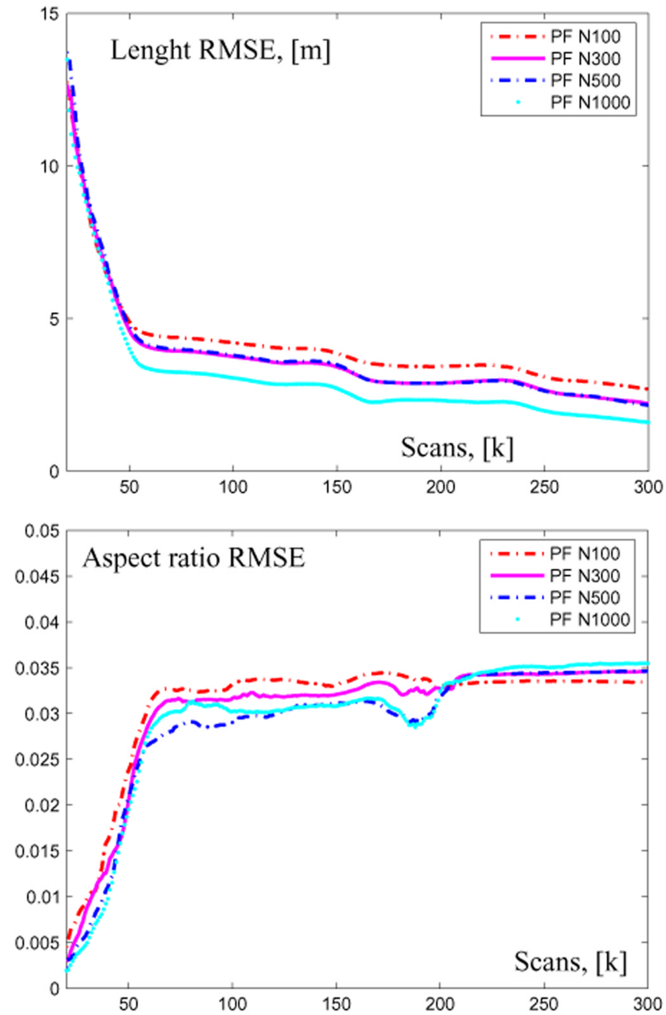

Fig. 4. PF results for $N_p = 100$, 300, 500 and 1000.

"warming up" initial interval is $u_0 = 70$. The mixture proportions $\pi_i, i = 1, \ldots, 4$, estimated over a single run, for $k = 100$ are given in Fig. 2(b). It can be seen that DA identifies the true $\boldsymbol{\theta}_2$ with a probability of $\pi_2 \approx 0.7$. This result confirms the reliability of the algorithm for classification tasks.

The MKFm is implemented with a sample size $N = 200$. The posterior mode probabilities $P(m_k = i_1 \,|\, \boldsymbol{Z}^k), i_1 = 1, 2, 3,\ k = 1, \ldots, 300$ are given in Fig. 2(c), and those of $P(\lambda_k = i_2 \,|\, \boldsymbol{Z}^k), i_2 = 1, 2, 3, 4,$ in Fig. 2(d). The switches between manoeuvring modes ($m = 2$ and $m = 3$) reproduce well the left and right turns performed by the extended object. The along-range object extent measurements depend on target–observer geometry and rapidly change during maneuvering phases. Fig. 2(d) shows that the posterior size type probabilities change due to the maneuvers, but the probability $P(\lambda_k = 2 \,|\, \boldsymbol{Z}^k)$, corresponding to the actual object size $\boldsymbol{\theta}_2$, remains maximum over the whole tracking interval.

A PF for extent parameters estimation is designed with $N_p = 300$ particles and $N_{\text{thresh}} = N_p/10$. Initially, N_p normally distributed particles $(\boldsymbol{\theta}^{(j)})_{j=1}^{N_p}$ are generated with mean, corresponding to the true $\boldsymbol{\theta}$. After that the particles are predicted according to the model, presented in Section III. Then the particle weights are evaluated using likelihoods of the received extent measurements and the $\hat{\boldsymbol{\theta}}$ estimate is calculated according to (14).

Comparative plots of the true and estimated ship parameters, ℓ and γ, obtained by the IMM-DA, MKFm, and IMM-PF, are presented in Fig. 3(a) and (b). The corresponding RMSEs are shown in Fig. 3(c) and (d). The maximum *speed* RMSEs are approximately 6.5 [m/s].

The proposed algorithms produce simultaneously stable tracking and size estimates converging to the true parameters. The DA procedure provides the most accurate results, since it processes the cumulative (in a window) measurement information which increases the computational time. The relative computational time IMM-DA:MKF:IMM-PF corresponds approximately to the proportions: 18:6:1. It should be noted that the PF involves an additional "artificial" noise, necessary for prediction. The proper choice of noise parameters can lead to a good result. However, the PF aspect ratio RMSE slowly increases over time [Fig. 3(d)]. This is observed over various scenarios and different sample sizes, as shown in Fig. 4. A similar tendency is indicated also in [9]. Taking this fact into consideration, we may conclude, that the MKFm provides a reasonable compromise between accuracy and computational time. The model validation scheme, incorporated within the MKFm, gives an additional size type information: if we are interested in the size type, which is not the true one, the Kolmogorov–Smirnov test certainly rejects this hypothesis. For example, if we want to check the hypothesis $\boldsymbol{\theta}_3 = \boldsymbol{\theta}_{\rm true}$, the estimated test statistic $ktest2 = 8$ definitely exceeds a 5% critical value of 1.36, since $\boldsymbol{\theta}_{\rm true} \equiv \boldsymbol{\theta}_2$.

VIII. Conclusion

A suboptimal solution to the problem of extended object tracking is proposed in this correspondence. MC algorithms (DA, MKFm and PF) are developed for the object extent parameter estimation, based on positional and along-range object extent measurements. The kinematic states are estimated with an IMM filter and with a MKFm, respectively. The approach of separation of states from parameters is implemented in the IMM-DA and IMM-PF. The overall state vector has a decreased dimension compared with the joint state-parameter estimation, the type of maneuver can be identified relatively quickly, and the kinematic states are estimated with small peak dynamic errors. The developed techniques offer a reasonable trade-off between accuracy and computational time and successfully deal with the complex target-observer geometry.

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On the Sensitivity of the Transmit MIMO Wiener Filter With Respect to Channel and Noise Second-Order Statistics Uncertainties

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***Abstract*—We consider the sensitivity of the transmit multiple-input multiple-output (MIMO) Wiener filter with respect to channel and noise second-order statistics (SOS) uncertainties. Using results from matrix perturbation theory, we derive second-order approximations to the excess mean-square error (EMSE) induced by using the channel or noise SOS estimates as if they were the true quantities. Assuming optimal training and sufficiently high signal-to-noise ratio (SNR), we develop simple and informative approximations to the EMSE, which indicate that the channel estimation errors are much more significant than the noise SOS estimation errors. Uncertainties due to channel time variations induce EMSE that increases with increasing SNR and asymptotically tends to a constant value.**

***Index Terms*—Multiple-input multiple-output (MIMO) systems, pre-equalization, Wiener filtering.**

I. Introduction

Joint optimization of transmit and receive filters for combatting frequency selectivity and/or interstream interference in multiple-input multiple-output (MIMO) or multiuser systems has been extensively studied (see, for example, [1] and the references therein). In order to keep the mobile units as simple as possible, we may consider separate transmit or receive processing. The transmit matched filter (TxMF), the transmit zero-forcing filter (TxZF) and the transmit Wiener filter (TxWF) are three linear pre-equalization (or precoding) structures that combat frequency selectivity and/or interstream interference and keep the receivers simple, because the only processing required at the receiver is a scalar scaling [1], [2].

The TxWF, which outperforms the two other structures in terms of mean-square error (MSE) and bit-error rate (BER) [1], can be computed if the channel and the input and noise second-order statistics (SOS) are perfectly known at the transmitter. This may happen, for

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