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Countable connected-homogeneous graphs

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ABSTRACT

A graph is connected-homogeneous if any isomorphism between finite connected induced subgraphs extends to an automorphism of the graph. In this paper we classify the countably infinite connected-homogeneous graphs. We prove that if Γ is connected countably infinite and connected-homogeneous then Γ is isomorphic to one of: Lachlan and Woodrow's ultrahomogeneous graphs; the generic bipartite graph; the bipartite 'complement of a complete matching'; the line graph of the complete bipartite graph $K_{\aleph_0,\aleph_0}$; or one of the 'treelike' distance-transitive graphs X_{κ_1,κ_2} where $\kappa_1, \kappa_2 \in \mathbb{N} \cup \{\aleph_0\}$. It then follows that an arbitrary countably infinite connected-homogeneous graph is a disjoint union of a finite or countable number of disjoint copies of one of these graphs, or to the disjoint union of countably many copies of a finite connected-homogeneous graph. The latter were classified by Gardiner (1976). We also classify the countably infinite connected-homogeneous posets.

1. Introduction

According to one usage, a mathematical structure M is *homogeneous* if any isomorphism between finite induced substructures of M extends to an automorphism of M. So in particular a *homogeneous graph* is a graph with the property that any isomorphism between finite induced subgraphs extends to an automorphism of the graph. This notion dates back to the fundamental work of Fraïssé; see [10], or originally, [9]. Homogeneous structures have interest for a variety of reasons. Their automorphism groups have rich and surprising group-theoretic properties; the sequences arising by counting orbits

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on k-sets often arise in combinatorial enumeration (see [1]); there are close connections, for example, to structural Ramsey theory (see [16]); and to extreme amenability of topological groups (see [18]).

For certain families of relational structure the study of homogeneity has led to classification results. The finite homogeneous graphs were determined by Gardiner in [11], and the countable homogeneous graphs were classified by Lachlan and Woodrow in [21]. The countable homogeneous posets were determined by Schmerl in [26], and the corresponding classification for tournaments was achieved by Lachlan [19]. Generalising the results for posets and tournaments, Cherlin [3] classified the homogeneous digraphs in a major piece of work.

There are various natural ways in which the condition of homogeneity can be relaxed. For graphs a vast array of symmetry conditions, weaker than homogeneity, have been considered; see [2] for a survey. For example, a graph is said to be k-homogeneous if we insist only that isomorphisms between subgraphs of size k extend to automorphisms. For any particular value of k, this gives a much larger class of graphs than the homogeneous ones. In fact, in [4] it was shown that for each k there are uncountably many countable graphs that are l-homogeneous for all $l \leqslant k$ but not $(k+1)$-homogeneous. An alternative direction is to weaken homogeneity to set-homogeneity, a condition saying that for any two finite isomorphic induced subgraphs, at least one isomorphism between them extends to an automorphism. The problem of classifying the countable set-homogeneous graphs is open, but in [5] the countable set-homogeneous graphs that are not 3-homogeneous are described (in fact, up to complementation there is only one such graph).

In this paper we consider a variant of homogeneity where we only require that isomorphisms between *connected* subgraphs extend to automorphisms. We say that a graph Γ is *connected-homogeneous* (or simply *C-homogeneous*) if any isomorphism between connected finite induced subgraphs extends to an automorphism. The (finite and infinite) locally-finite C-homogeneous graphs were classified in [13] and [6]; here a graph is locally-finite if every vertex has finite degree. Combined with those results, the main result of this paper completes the classification of the C-homogeneous graphs, in the countable case. It might be regarded as a little surprising that a classification result is possible here, especially when compared to other weakenings of homogeneity, like those mentioned above, where the resulting classes of graph are too unwieldy for full descriptions to be possible.

There is a connection between C-homogeneity and another well-studied property for graphs called distance-transitivity. A graph is *distance-transitive* if for any two pairs (u,v) and (u',v') with $d(u,v) = d(u',v')$, there is an automorphism taking u to u' and v to v'. It follows directly from the definition that any C-homogeneous graph is also distance-transitive. The infinite connected locally-finite distance-transitive graphs were classified by Macpherson [22] (a result strengthened to distance regular graphs by A. Ivanov [17], in independent work). They turn out to be precisely the graphs $X_{k,l}$ defined below (with $k, l \in \mathbb{N}$). However, in general the countably infinite distance-transitive graphs remain quite a mysterious family; see [2] for more on this. As we shall see below, the graphs $X_{k,l}$ also happen to be C-homogeneous (even when k or l is equal to $\aleph_0$) and hence they arise as part of our classification. Thus for infinite locally-finite graphs C-homogeneity and distance-transitivity are equivalent properties. Of course, this is not true for arbitrary countable graphs, and there are many examples of distance-transitive graphs that are not C-homogeneous; a concrete example being the graph whose vertex set is the collection of 2-element subsets of a countably infinite set, two vertices adjacent if intersecting in a singleton. In fact for countable graphs, there are uncountably many that are distance-transitive (see [2]) while, as a consequence of our main theorem below, only countably many of them are C-homogeneous. This demonstrates that the C-homogeneous graphs really are a very special subfamily of the distance-transitive graphs.

In this article we shall classify the countable C-homogeneous graphs. Before stating the main result we need a few definitions.

First note that in a C-homogeneous graph every connected component must be C-homogeneous, and since the graph is vertex transitive these components must be isomorphic to one another. Thus any countable C-homogeneous graph is a disjoint union of a countable (possibly finite) number of isomorphic copies of some fixed connected C-homogeneous graph. Therefore when investigating C-homogeneous graphs nothing is lost by restricting attention to those C-homogeneous graphs that are connected.

Given $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$, with $\kappa_1 \geqslant 2$, we construct a graph X_{κ_1,κ_2}. First consider the semiregular tree T_{κ_1,κ_2+1}, where all vertices in one bipartite block have valency κ_1 and all those in the other have valency $\kappa_2 + 1$. Now define X_{κ_1,κ_2} to be the graph with vertex set the bipartite block of T_{κ_1,κ_2+1} of vertices with valency κ_1 and two vertices adjacent in X_{κ_1,κ_2} if their distance in the tree T_{κ_1,κ_2+1} is 2. The graphs X_{κ_1,κ_2} are distance transitive, and all infinite locally-finite distance-transitive graphs are of this form, with κ_1, κ_2 finite (see [22]). In X_{κ_1,κ_2}, the neighbourhood of a vertex consists of κ_1 copies of the complete graph K_{κ_2}, with the complete graphs joined in a treelike way.

Let $\Gamma = X \cup Y$ be a countable bipartite graph with parts X, Y, and with the following property, where $\sim$ denotes adjacency:

($*$) For every distinct $a_0, \ldots, a_k$, $b_0, \ldots, b_l$ in X (respectively in Y) there exists a vertex u in Γ such that $u \sim a_i$ but $u \nsim b_j$ for all $i \leqslant k$, $j \leqslant l$.

By a routine back-and-forth argument (see for example [7, p. 98]) there is up to isomorphism a unique countable bipartite graph satisfying property ($*$). We call this graph the *countable generic bipartite graph*.

By 'the complement of a complete matching', we mean a bipartite graph with parts X, Y, such that, for some bijection $f : X \to Y$, each $x \in X$ is joined to $y \in Y$ if and only if $y \neq f(x)$. We denote by $K_{s,t}$ the complete bipartite graph with parts of sizes s and t. Let $M(s,t)$ denote the graph with vertex set a disjoint union of sets $X_1, X_2, \ldots, X_s$ each of size t, and with two vertices adjacent if and only if they belong to distinct sets X_i and X_j. We call $M(s,t)$ the *complete multipartite graph* with s parts each of size t. If Γ is a graph, the *line graph* $L(\Gamma)$ has as vertex set the edge set of Γ, and two edges of Γ are adjacent in $L(\Gamma)$ if they meet in a Γ-vertex.

In [13] the finite C-homogeneous graphs were classified.

Theorem 1. *(See [13].) A connected finite graph is C-homogeneous if and only if it is isomorphic to one of the following*: *complement of a complete matching, complete graph K_r ($r \geqslant 1$), complete multipartite $M(s,t)$ ($s, t \geqslant 2$), cycle C_n ($n \geqslant 5$), the line graph $L(K_{s,s})$ of a complete bipartite graph $K_{s,s}$ (where $s \geqslant 3$), Petersen's graph O_3, or $\square_5$ (the graph obtained by identifying antipodal vertices of the 5-dimensional cube Q_5).*

We may now state our main theorem.

Theorem 2. *A countable graph is C-homogeneous if and only if it is isomorphic to the disjoint union of a finite or countable number of copies of one of the following*:

(i) *a finite C-homogeneous graph*;
(ii) *a homogeneous graph*;
(iii) *the generic bipartite graph*;
(iv) *the complement of a complete matching*;
(v) *the line graph of a complete bipartite graph $K_{\aleph_0,\aleph_0}$*;
(vi) *a graph X_{κ_1,κ_2} with $\kappa_1, \kappa_2 \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$.*

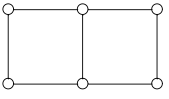
Lachlan and Woodrow's classification of countable homogeneous graphs is stated below in Theorem 3. It is easy to see that each of the graphs in the statement of Theorem 2 is C-homogeneous. The rest of this article will be devoted to proving that these are, in fact, the only examples.

2. Preliminaries

A *graph* Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a non-empty set that we call the *vertex set* and $E\Gamma$ is a set of 2-element subsets of $V\Gamma$ called the *edge set*. We often just write $x \in \Gamma$ for $x \in V\Gamma$. If $\{v, u\} \in E\Gamma$ we say that the vertices u and v are *adjacent*, and write $u \sim v$. If $A \subset V\Gamma$ we write $\langle A \rangle$ for the induced subgraph of Γ with vertex set A, and edge set the collection of 2-subsets of A which lie in $E\Gamma$. This is the only notion of subgraph in this paper, and we frequently identify A

with $\langle A\rangle$, and often write $A \subseteq \Gamma$ to mean that A is a subgraph of Γ. The *neighbourhood* of a vertex v, denoted $\Gamma(v)$, is the set of all vertices adjacent to v. The *valency* (or *degree*) of a vertex v is the cardinality of the set $\Gamma(v)$. A graph is called *regular* if all vertices have the same degree. A graph in which every pair of distinct vertices is adjacent is called a *complete* graph, and we call a graph with no edges an *independent set*. We use K_n and I_n to denote the complete graph, and independent set, on n vertices, respectively. For $k_1, k_2 \in \mathbb{N} \cup \{\aleph_0\}$ we use $k_1 \cdot K_{k_2}$ to denote k_1 disjoint copies of the complete graph K_{k_2}. A *walk* in a graph is a sequence of vertices such that adjacent terms in the sequence are adjacent in the graph. A *path* is a walk all of whose vertices are distinct. We use L_n to denote the graph with n vertices $\{1, 2, \ldots, n\}$ and $i \sim j$ if and only if $|i - j| = 1$. We call L_n the *line* with n vertices. Note that this terminology is slightly nonstandard, since the word line is usually used to denote a two-way infinite path in a graph. The *length* of a walk is defined to be one less than the length of the sequence of vertices defining that walk. In particular, the length of a path is simply the number of edges in the path. A graph is called *connected* if between any two vertices there exists at least one path. A graph is called *bipartite* if the vertex set may be partitioned into two disjoint sets X and Y such that no two vertices in X are adjacent, and no two vertices of Y are adjacent. In this case we say that Γ is bipartite with bipartition $X \cup Y$. A bipartite graph $\Gamma = X \cup Y$ is called a *complete matching* if there is a bijection $\pi : X \to Y$ such that $E\Gamma = \{\{x, \pi(x)\}\colon\ x \in X\}$, and Γ is called *complete bipartite* if every vertex in X is adjacent to every vertex of Y. A bipartite graph is called *semiregular* if any two vertices in X have the same degree as one another, and any two vertices of Y have the same degree as one another.

A *cycle* is a finite path with at least three vertices but where the first and last vertices are equal, and the only edges are those between adjacent vertices of the defining sequence; so a cycle with n vertices has n edges. A connected graph without any cycles is called a *tree*. We use C_n to denote the cycle with n vertices. Also we use $\square\!\square$, which we call the *two-squares* configuration, to denote the graph



In connected graphs there is a natural notion of *distance* where the distance $d(x, y)$ between x and y is defined to be the minimum length of a path with end-vertices x and y. For $i \in \mathbb{N}$ define $\Gamma_i(v) = \{w \in V\Gamma\colon\ d(w, v) = i\}$, so that in particular we have $\Gamma(v) = \Gamma_1(v)$. The *diameter* of a connected graph is the supremum (possibly infinite) of the set of distances between all pairs of distinct vertices.

A *partial automorphism* of a graph is an isomorphism between induced subgraphs of that graph. We denote by $\mathrm{Aut}(\Gamma)$ the group of all automorphisms of Γ. Given a subgroup G of $\mathrm{Aut}(\Gamma)$ and a vertex $x \in V\Gamma$ we write G_x for the stabilizer of x in G (that is the set of all elements of G that fix the vertex x). More generally, if $x_1, \ldots, x_n$ are vertices then $G_{x_1 \ldots x_n} := G_{x_1} \cap \cdots \cap G_{x_n}$. We use lower case Greek letters for automorphisms, and write the action on the right, so x^α denotes the image of the vertex x under the automorphism α.

Let Γ and Γ' be bipartite graphs with given bipartitions $X \cup Y$ and $X' \cup Y'$, respectively. A *partial isomorphism* between the bipartite graphs Γ and Γ' is a one-one partial map ϕ from $X \cup Y$ into $X' \cup Y'$ which preserves the bipartition (i.e. $\phi(X) \subseteq X'$ and $\phi(Y) \subseteq Y'$) and preserves edges and non-edges (i.e. $\{x, y\} \in E\Gamma$ if and only if $\{\phi(x), \phi(y)\} \in E\Gamma'$, for all $x \in X$ and $y \in Y$). A *partial automorphism* of Γ is a partial isomorphism from Γ to itself. A *homogeneous bipartite graph* is one that has the property that all partial automorphisms between finite subgraphs, which preserve the bipartition, extend to full automorphisms. Note that for a connected bipartite graph the bipartition is unique, so any graph-automorphism will respect the bipartition. More generally any graph-isomorphism between connected subgraphs (with at least two vertices) of a connected bipartite graph will respect the bipartition.

In our proof we will make use of the classification of the countable homogeneous graphs, and also the classification of homogeneous bipartite graphs. The classification of countable homogeneous graphs is a highly non-trivial result.

Theorem 3. *(See [21, Theorem 1].) Let Γ be a countable homogeneous graph. Then Γ is isomorphic to one of: the random graph, a disjoint union of complete graphs (or its complement), the generic K_n-free graph (or its complement, the generic co-K_n-free graph).*

For the definitions of the graphs appearing in the above theorem, and for details on how they are constructed, we refer the reader to [21] or [8]. Each countably infinite homogeneous graph is determined up to isomorphism by its collection of isomorphism types of finite induced subgraphs. For example, for the random graph this consists (up to isomorphism) of *all* finite graphs, and for the generic K_n-free graph, of *all* finite graphs which do not have K_n as an induced subgraph. The examples are built by 'Fraïssé amalgamation', and the proof of Lachlan and Woodrow is basically a classification of possible amalgamation classes.

We now summarise the properties of the graphs appearing in Theorem 3 that we will use later on. The random graph embeds every finite or countable graph as an induced subgraph i.e. the random graph is *universal*. Also, the random graph satisfies the following extension property:

> For any two finite disjoint sets U and V of vertices, there exists a vertex z joined to every vertex in U and to no vertex in V.

Similarly, the generic K_n-free graph embeds every finite or countable K_n-free graph, and satisfies the following extension property:

> For any two finite disjoint sets U and V of vertices, such that U does not embed K_{n-1} as an induced subgraph, there exists a vertex z joined to every vertex in U and to no vertex in V.

This tells us that given any finite induced subgraph X of the generic K_n-free graph, every possible one-vertex extension of X that is K_n-free, is realised inside the graph. The obvious dual statements to these hold in the generic co-K_n-free graph.

The classification of homogeneous bipartite graphs is straightforward and a one-page proof may be found in [14, Section 1].

Theorem 4. *(See [14, Section 1].) If Γ is a countable homogeneous bipartite graph then Γ is isomorphic to one of the: complete bipartite, empty bipartite (i.e. an independent set), complete matching, complement of complete matching, or the countable generic bipartite graph.*

For the rest of the paper, unless otherwise stated, Γ will denote a countably infinite connected C-homogeneous graph which is not locally finite. This last assumption is justified by the work of Gardiner [13] and Enomoto [6], which asserts that countable locally finite C-homogeneous graphs belong to the list of Theorem 2. The next lemma tells us that homogeneous graphs may be found inside C-homogeneous graphs.

Lemma 5. *The graph $\langle \Gamma(v) \rangle$ induced on the neighbourhood $\Gamma(v)$ of any vertex v of Γ is homogeneous.*

Proof. Let $\phi : A \to B$ be a partial isomorphism between finite induced subgraphs of $\Gamma(v)$. Then ϕ extends to an isomorphism $\hat{\phi} : A \cup \{v\} \to B \cup \{v\}$ between connected substructures of Γ by defining $\hat{\phi}(v) = v$. By C-homogeneity the isomorphism $\hat{\phi}$ extends to an automorphism α of Γ fixing v. Now α restricted to $\Gamma(v)$ is an automorphism of $\Gamma(v)$ extending ϕ. $\square$

Sometimes homogeneous bipartite graphs also arise naturally inside C-homogeneous graphs as the following lemma demonstrates.

Lemma 6. *Suppose that $\Gamma(v)$ is an independent set. Let $\{x, y\} \in E\Gamma$ and define $X = \Gamma(x) \setminus \{y\}$ and $Y = \Gamma(y) \setminus \{x\}$. Then the graph induced by $X \cup Y$, as a bipartite graph with bipartition $X \cup Y$, is a homogeneous bipartite graph.*

Proof. Let Δ denote the graph induced by $X \cup Y$. Let $\phi : A \to B$ be an isomorphism between finite induced subgraphs of Δ preserving the bipartition $X \cup Y$. Then ϕ extends to an isomorphism $\hat{\phi} : A \cup \{x, y\} \to B \cup \{x, y\}$ between connected substructures of Γ by defining $\hat{\phi}(x) = x$ and $\hat{\phi}(y) = y$. By C-homogeneity the isomorphism $\hat{\phi}$ extends to an automorphism α of Γ fixing both x and y. Now α restricted to $X \cup Y$ is an automorphism of the bipartite graph $X \cup Y$ extending ϕ. □

The next few lemmas will be used to find bounds on the diameter of Γ in certain circumstances.

Lemma 7. *Assume that Γ is not a tree. Then the following hold.*

(i) *If $C_n \leqslant \Gamma$ for some $n \geqslant 5$ then* $\mathrm{diam}(\Gamma) \leqslant \lfloor n/2 \rfloor$.
(ii) *If C_n is the smallest cycle embedding in Γ then $n \leqslant 6$.*

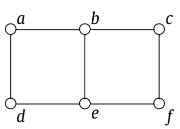
Proof. For part (i) suppose, for the sake of a contradiction, that $\mathrm{diam}(\Gamma) > \lfloor n/2 \rfloor$. Let $x, y \in \Gamma$ satisfy $d(x, y) = \lfloor n/2 \rfloor + 1$. Fix some copy C of C_n in Γ and partition this cycle into two edge disjoint paths $C = C' \cup C''$ where C' has $\lfloor n/2 \rfloor + 1$ edges and C'' has the remaining $n - (\lfloor n/2 \rfloor + 1)$ edges. By C-homogeneity there is an automorphism mapping C' to a path of length $\lfloor n/2 \rfloor + 1$ with end-vertices x and y. But now the image of C'' under the same automorphism is a path from x to y but it has length $n - (\lfloor n/2 \rfloor + 1) \leqslant \lfloor n/2 \rfloor$ (since $n \geqslant 5$) which contradicts $d(x, y) = \lfloor n/2 \rfloor + 1$.

For part (ii) suppose, seeking a contradiction, that the smallest cycle that embeds is C_n where $n \geqslant 7$. Fix a vertex $v \in V\Gamma$ and let a, b, c be distinct elements of $\Gamma(v)$; they will be non-adjacent as Γ is triangle-free. By C-homogeneity the path (b, v, c) extends to a copy $(v, b, b_1, \ldots, b_k, c)$ of C_n. Since $n \geqslant 7$ it follows that $k \geqslant 4$ and therefore a is not adjacent to b_i for all $1 \leqslant i \leqslant k$ (since any such edge would create a cycle in Γ shorter than C_n itself). Now by C-homogeneity there is an automorphism α satisfying

$$\langle c, v, b, b_1, b_2, \ldots, b_{k-1} \rangle^{\alpha} = \langle a, v, b, b_1, b_2, \ldots, b_{k-1} \rangle.$$

Note that the vertex b_k^{α} does not belong to $B = \{v, a, b, c, b_1, b_2, \ldots, b_k\}$ since b_k^{α} is adjacent both to a and to b_{k-1}, and none of the vertices in B have this property. Let $D = \langle a, v, c, b_k, b_{k-1}, b_k^{\alpha} \rangle$. If b_k^{α} is adjacent to c then $\langle a, v, c, b_k^{\alpha} \rangle \cong C_4$ which is a contradiction. If $b_k^{\alpha} \nsim c$ but $b_k^{\alpha} \sim b_k$ then $\langle a, v, c, b_k, b_k^{\alpha} \rangle \cong C_5$ a contradiction. Finally if $b_k^{\alpha} \nsim c$ and $b_k^{\alpha} \nsim b_k$ then $\langle a, v, c, b_k, b_{k-1}, b_k^{\alpha} \rangle \cong C_6$, which is again a contradiction. □

Lemma 8. *If Γ embeds the graph*:



then $\mathrm{diam}(\Gamma) \leqslant 3$.

Proof. If $\mathrm{diam}(\Gamma) \geqslant 4$ then we could find x and y in Γ with $d(x, y) = 4$. A path of length 4 between these vertices would induce a line with 5 vertices whose end vertices are at distance 4 in the graph. On the other hand in the above configuration $\langle d, a, b, c, f \rangle$ is an induced line with 5 vertices whose end-vertices are at distance 2 in the graph. This contradicts C-homogeneity. □

Given an induced subgraph E of Γ, by an *extension* $\overline{E}$ of E we mean an induced subgraph $\overline{E}$ of Γ such that E is a subset of $\overline{E}$. The following straightforward observation gives a sufficient condition for a C-homogeneous graph to be homogeneous. It will be used frequently in what follows.

Lemma 9. *If for any isomorphism $\phi : E_1 \to E_2$ between finite induced subgraphs of Γ there exist connected extensions $\overline{E_1}$, $\overline{E_2}$ such that ϕ extends to an isomorphism $\overline{\phi} : \overline{E_1} \to \overline{E_2}$, then Γ is a homogeneous graph.*

Proof. Any isomorphism $\phi : E_1 \to E_2$ between finite induced subgraphs extends to $\bar{\phi} : \overline{E_1} \to \overline{E_2}$, an isomorphism between finite connected induced subgraphs, which then extends to an automorphism by C-homogeneity. Thus ϕ extends to an automorphism and Γ is homogeneous. □

We will now work through the proof of the main theorem. Our strategy is to use Lemma 5, in conjunction with Theorem 3 applied to the neighbourhood. For each possibility of $\Gamma(v)$ we want to determine all possibilities for Γ. We begin with the easiest cases and move on to those that are more difficult towards the end.

3. Proof of the main theorem I: neighbourhood isomorphic to random, generic K_n-free (or its complement), or complete multipartite graph

In this section we shall prove the following result.

Theorem 10. *Let Γ be a countable connected C-homogeneous graph. If the neighbourhood of a vertex of Γ is isomorphic to the random, co-K_n-free, K_n-free, or complete multipartite graph, then Γ is homogeneous.*

We deal with each possibility for the neighbourhood in turn.

$\Gamma(v)$ is the random graph

Let R denote the countable random graph.

Lemma 11. *If $\Gamma(v) \cong R$ then Γ is homogeneous (and therefore $\Gamma \cong R$).*

Proof. By Lemma 9, in order to prove that Γ is homogeneous it is sufficient to prove that for any finite induced subgraph E of Γ there exists $v \in V\Gamma$ such that $E \subseteq \Gamma(v)$. Let E be a finite induced subgraph of Γ. Since Γ is connected, there is a finite connected induced subgraph F of Γ with $E \subseteq F$. Let $u \in V\Gamma$ be arbitrary. Since $\Gamma(u) \cong R$ there is $F' \subseteq \Gamma(u)$ with $F' \cong F$. Now by C-homogeneity there is an automorphism α with $F'^{\alpha} = F$ and so $E \subseteq F \subseteq \Gamma(u^{\alpha})$. Hence Γ is homogeneous and so $\Gamma \cong R$ by inspection of the list of homogeneous graphs in Theorem 3. □

$\Gamma(v)$ isomorphic to a complete multipartite graph

Suppose that $\Gamma(v) \cong M(s,t)$ the complete multipartite graph with t parts each of size s where $s, t \in (\mathbb{N} \setminus \{0\}) \cup \{\aleph_0\}$. If a graph is connected and the neighbourhood of every vertex is isomorphic to K_r for some $r \geqslant 2$ then the entire graph must, clearly, be a complete graph, and so we may suppose that $s > 1$. Also, when $t = 1$, $M(s,1)$ is an independent set and this comes under a different case (see Section 4), so suppose that $s, t \geqslant 2$. In this case the following lemma ensures that $\Gamma \cong M(s, t+1)$. Versions of it are well known (see [12, Lemma 6 and 8] for example), but for completeness we give a proof.

Lemma 12. *Let Γ be a connected graph, and suppose that for each $v \in V\Gamma$, $\Gamma(v) \cong M(s,t)$ with $s, t > 1$. Then $\Gamma \cong M(s, t+1)$.*

Proof. Suppose $v \in V\Gamma$, and the parts in the multipartite partition of $\Gamma(v)$ are $\{A_i\colon i \in I\}$. For distinct $i, j \in I$, pick $a_1 \in A_i$ and $a_2 \in A_j$. Then let $\{v\} \cup B$ be the part containing v in the complete multipartite partition of $\Gamma(a_1)$. Thus $\{v\} \cup B$ is an independent set, whose vertices are joined to all members of $\bigcup\{A_k\colon k \in I\} \setminus A_i$. It follows that $\Gamma(a_2) \setminus \Gamma(v) \supseteq \Gamma(a_1) \setminus \Gamma(v)$. Reversing a_1, a_2, we see $\Gamma(a_2) \setminus \Gamma(v) = \Gamma(a_1) \setminus \Gamma(v)$. Likewise, if $y \in B$ then $\Gamma(y) \setminus \Gamma(a_2) = \Gamma(a_1) \setminus \Gamma(a_2)$. Hence as B is an independent set, $\Gamma(y)$ is contained in (and in fact equals) $\Gamma(v)$. It follows by connectedness that Γ is complete multipartite with parts $\{B \cup \{v\},\ A_i\colon i \in I\}$. □

$\Gamma(v)$ is the complement of the generic K_n-free graph ($n \geqslant 3$)

Since the complement of a pentagon is a pentagon, in this case we know that Γ embeds a pentagon and therefore by Lemma 7(i), Γ has diameter 2 in this case.

Lemma 13. *If $I_m \subseteq \Gamma$ with $m < n$ then there exists $u \in V\Gamma$ such that $I_m \subseteq \Gamma(u)$.*

Proof. The result holds for $m = 2$ since Γ has diameter 2. Let m be the smallest integer for which the result fails. Let $I_m = \{x_1, \ldots, x_m\}$. By minimality there exist u_1, u_2 such that u_1 is adjacent to every vertex in $I_m \setminus \{x_m\}$, and u_2 is adjacent to every vertex in $I_m \setminus \{x_1\}$. Now $Y = \langle I_m \cup \{u_1, u_2\} \rangle$ does not embed I_{m+1} and, since $m < n$, does not embed I_n. Therefore Y is co-K_n-free, connected (as $m > 2$), and embeds in a neighbourhood, and thus I_m has a common neighbour by C-homogeneity. □

Lemma 14. *If $E \subseteq \Gamma$ is co-K_n-free there exists $u \in \Gamma$ such that $E \subseteq \Gamma(u)$.*

Proof. Let $F \subseteq \Gamma$ be a minimal counterexample. By Lemma 13 we may assume that F is not an independent set. Therefore we may choose $f \in F$ so that f is adjacent to at least one other vertex of F. Now by minimality there is a vertex $u' \in \Gamma$ adjacent to every vertex in $F \setminus \{f\}$, and by the choice of f we know that $\langle F \cup \{u'\} \rangle$ is a connected graph. Now $\langle F \cup \{u'\} \rangle$ is also co-K_n-free because any copy of I_n would need to contain u'. Therefore by C-homogeneity there exists $u \in V\Gamma$ such that $\langle F \cup \{u'\} \rangle \subseteq \Gamma(u)$. □

Now we use the lemmas above to prove the main proposition for this subsection.

Proposition 15. *Let Γ be a countable connected C-homogeneous graph such that $\Gamma(v)$ is co-K_n-free for some $n \geqslant 3$. Then Γ is homogeneous.*

Proof. Suppose, for the sake of a contradiction, that Γ is not homogeneous, and choose E not satisfying the conditions of Lemma 9 and with the minimum number of connected components with respect to this; here 'not satisfying the conditions of Lemma 9' means that there is some isomorphism $\phi : E \to F$ with no extension to an isomorphism between finite connected subgraphs of Γ. Let $E_1, \ldots, E_k$ be the connected components of E.

Case 1. $k < n$.

Choose $E' \subseteq E$ with the following properties:

(i) $E' \cap E_i \neq \emptyset$ for $1 \leqslant i \leqslant k$;
(ii) E' is maximal co-K_n-free.

Now by Lemma 14 there exists u such that $E' \subseteq \Gamma(u)$ and by maximality $\Gamma(u) \cap E = E'$. But now $\langle E \cup \{u\} \rangle$ is connected, and any other induced subgraph isomorphic to E may be connected together in this same way, which is a contradiction to the original choice of E.

Case 2. $k \geqslant n$.

In this case let $E' \subseteq E$ with the following properties:

(i) $E' \cap E_i \neq \emptyset$ for $1 \leqslant i \leqslant n-1$;
(ii) E' is maximal co-K_n-free.

Then by Lemma 14 there exists u such that $E' \subseteq \Gamma(u)$ and $\Gamma(u) \cap (E_1 \cup \cdots \cup E_{n-1}) = E'$. In addition to this, we know that $\Gamma(u)$ does not intersect any of $\{E_n, E_{n+1}, \ldots, E_k\}$ by (i) and the fact that $\Gamma(u)$

is co-K_n-free. Now define $F_0 = \langle \{u\} \cup E_1 \cup \cdots \cup E_{n-1} \rangle$. Then $\langle F_0 \cup E_n \cup \cdots \cup E_k \rangle$ has strictly fewer connected components than E and therefore by minimality satisfies the conditions of Lemma 9. It follows that any induced subgraph of Γ isomorphic to E may be connected together in this same way, which is a contradiction to the original choice of E. $\square$

We remark that there is no countable homogeneous graph satisfying Proposition 15; so, by the proposition, this case does not arise.

$\Gamma(v)$ is the generic K_r-free graph ($r \geqslant 3$)

In this case we prove that Γ must be homogeneous, and hence, by inspection of the list of countable homogeneous graphs, must be the generic K_{r+1}-free graph.

First observe that $\Gamma(v)$ embeds the pentagon, so by Lemma 7(i), $\mathrm{diam}(\Gamma) = 2$. The proof follows rapidly from Lemma 17, and for that we first need the following.

Lemma 16. *Let $X = K_l \cup K_m \subseteq \Gamma$, a disjoint union of complete graphs where $l, m \leqslant r - 1$. Then there exists a finite subset Y of Γ such that $\langle X \cup Y \rangle$ is connected and K_r-free.*

Proof. The problem splits into two cases depending on the value of r.

Case 1. $r > 3$. Choose l and m so that $K_l \cup K_m$ is minimal with respect to not extending. Since $\mathrm{diam}(\Gamma) = 2$ we may suppose, without loss of generality, that $m \geqslant 2$. Let $M = K_l \cup K_m \setminus \{y\}$ where $y \in K_m$. Let $x \in K_l$ and $z \in K_m \setminus \{y\}$. By minimality, M can be connected in a K_r-free way, so by C-homogeneity M lies in a neighbourhood. Indeed, M has an extension $\overline{M}$ that is connected and K_r-free. Also, for any vertex w of Γ the neighbourhood $\Gamma(w)$ is isomorphic to the generic K_r-free graph which embeds all countable K_r-free graphs, and so in particular has an induced subgraph isomorphic to $\overline{M}$. Now applying C-homogeneity we conclude that here exists a vertex w' with $M \subseteq \overline{M} \subseteq \Gamma(w')$. Now since $\Gamma(w')$ is isomorphic to the generic K_r-free graph, it follows by the extension property that there exists $u \in \Gamma(w')$ such that $\Gamma(u) \cap M = \{x, z\}$. Now since $r > 3$, regardless of whether or not $u \sim y$, $\langle K_l \cup K_m \cup \{u\} \rangle$ is K_r-free and connected, which is a contradiction.

Case 2. $r = 3$ (*so $\Gamma(v)$ is generic triangle-free*). Again, we may assume $l + m > 2$. If $K_l = \{x_1\}$ and $K_m = \{y_1, y_2\}$ then $\{x_1, y_1\}$ has a common neighbour u since $\mathrm{diam}(\Gamma) = 2$. If $u \nsim y_2$ then $\langle x_1, u, y_1, y_2 \rangle$ is connected and triangle-free. On the other hand, if $u \sim y_2$ then $\{x_1, y_1, y_2\} \subset \Gamma(u)$, so can be connected in a triangle-free way within $\Gamma(u)$. Dually we can deal with the case that $l = 2$ and $m = 1$. Therefore we may assume that $l = m = 2$.

Suppose that the lemma is false. Write $ab|cd$ if $\langle \{a, b, c, d\} \rangle$ is the disjoint union of two edges $\{a, b\}$ and $\{c, d\}$. By assumption there exist such quadruples which do not lie in any neighbourhood, since otherwise they can be connected in a triangle-free way within a neighbourhood. Call such quadruples *bad*, and *good* if they do lie in a neighbourhood. By the last paragraph, if $ab|cd$ is a bad quadruple then there is $v \notin \{a, b, c, d\}$ joined to b, c, d but not a (and likewise for any other choice of one of the four points); and this does not hold for good quadruples, as otherwise we could connect up good and bad quadruples in the same way, contrary to C-homogeneity.

Claim 1. If $ab|cd$ is bad then there is $e \notin \{a, b, c, d\}$ joined to a and b and not to c or d.

Proof of claim. As $ab|cd$ is bad there is u joined to b, c, d but not a. Then, as $\Gamma(u)$ is generic triangle-free, by the extension property there is $f \in \Gamma(u)$ joined to d but not b, c. Then $f \nsim a$, as otherwise $\langle a, b, f, d, c \rangle$ is connected triangle-free.

Now $ab|df$ is a bad quadruple. Indeed, if not, then (working in a neighbourhood containing a, b, d, f) there is v joined to b, f and not to a, d. Whether or not $v \sim c$, the graph on $\{a, b, v, c, d, f\}$ is connected triangle-free, a contradiction.

It follows that there is v joined to b, d, f and not a. Then $v \sim c$, as otherwise $\{a, b, c, d, v\}$ is connected triangle-free. Inside $\Gamma(v)$, pick e joined to b, f but not to c, d. Then e is joined to a, as otherwise $\{a, b, c, d, e, f\}$ is connected triangle-free. This e satisfies the claim. $\square$

Given the claim, suppose first that $be|cd$ is bad. Then there is w joined to b, e, d but not c. As $ab|cd$ is bad, it follows that w is joined to a (since otherwise $\langle a, b, w, c, d\rangle$ would be triangle-free and so would map into a neighbourhood), so $\{a, b, e, w\}$ is complete, which is a contradiction.

Thus, $be|cd$ is good, and likewise $ae|cd$ is good. Choose w joined to b, c, d. Then $w \sim e$, as $be|cd$ is good, and hence $w \sim a$, as $ae|cd$ is good. Thus, again, $\{a, b, e, w\}$ carries a complete graph, a contradiction. $\square$

Lemma 17. *If $A \subseteq \Gamma$ and $\langle A\rangle$ is K_r-free, then there exists a finite subset X of Γ such that $\langle A \cup X\rangle$ is connected and K_r-free.*

Proof. If $\langle A\rangle$ is connected, then $X = \emptyset$ is a solution. Thus, we may suppose that A has at least two components, and we first show by induction on the number of vertices that if A has exactly two components then the result holds. So suppose A has exactly two components, E_1 and E_2. If E_1 and E_2 are each complete graphs then the result follows by Lemma 16, so we may suppose without loss of generality that E_2 is not a complete graph. Let $e, f \in E_2$ be non-adjacent vertices. We may suppose that $E_2 \setminus \{f\}$ is connected; to see this, for example, it is an easy exercise to find a spanning tree for E_2 such that some vertex f of E_2 which is not joined in Γ to all of E_2 is a leaf, and then deletion of this vertex leaves $E_2 \setminus \{f\}$ connected. Let $A' = \langle A\rangle \setminus \{f\}$. By inductive hypothesis there is a finite set Z such that $\langle A' \cup Z\rangle$ is connected and K_r-free. It follows by C-homogeneity that $\langle A' \cup Z\rangle$ embeds in a neighbourhood. Indeed, for any vertex w of Γ the neighbourhood $\Gamma(w)$ of w is isomorphic to the generic K_r-free graph. Since the generic K_r-free graph embeds all countable K_r-free graphs it follows that $\Gamma(w)$ embeds a subgraph isomorphic to $\langle A' \cup Z\rangle$. Since $\langle A' \cup Z\rangle$ is connected we may apply C-homogeneity to deduce that there is a vertex w' in Γ with $A' \subseteq A' \cup Z \subseteq \Gamma(w')$. Let $g \in E_1$ be arbitrary. Since $\Gamma(w')$ is isomorphic to the generic K_r-free graph it follows from the extension property that there is a vertex $u \in \Gamma(w')$ such that $\Gamma(u) \cap A' = \{g, e\}$. Since e is not adjacent to f it follows that $\langle A \cup \{u\}\rangle$ is connected and K_r-free, completing the inductive step.

We now assume the result when A has two components, and prove it for the case when A has components $E_1, \ldots, E_s$, with $s > 2$. We may assume inductively that the lemma holds for proper subgraphs of A. Pick $b \in E_s$. Let $A' = A \setminus \{b\}$. By induction there is a finite set Y such that $\langle A' \cup Y\rangle$ is connected and K_r-free. Arguing as in the previous paragraph, by C-homogeneity this implies there exists a vertex y' of Γ such that $A' \subseteq A' \cup Y \subseteq \Gamma(y')$. Since $\Gamma(y')$ is isomorphic to the generic K_r-free graph it follows by the extension property that there exists a vertex c in $\Gamma(y')$ that is adjacent to exactly one vertex of E_i for each $i < s$, and adjacent to no vertices of $E_s \setminus \{b\}$. If $c \sim b$ then $A \cup \{c\}$ is connected and K_r-free, and so we are done. On the other hand, if $c \nsim b$ then $\langle A \cup \{c\}\rangle$ has two components, namely $E_1 \cup \cdots \cup E_{s-1} \cup \{c\}$ and E_s, and is K_r-free, and so A satisfies the lemma by the previous paragraph (the two-component case). $\square$

Proposition 18. *Let Γ be a countable connected C-homogeneous graph. If $\Gamma(u)$ is isomorphic to the generic K_n-free graph with $n \geqslant 3$ then Γ is homogeneous (and so is isomorphic to the generic K_{n+1}-free graph).*

Proof. Let $E_1, E_2 \subseteq \Gamma$ and let $\phi : E_1 \to E_2$ be an isomorphism. If E_1 is connected then ϕ extends to an automorphism by C-homogeneity, so suppose that E_1 is not connected. Let E_1' be a maximal K_r-free induced subgraph of E_1, noting that E_1' must intersect every connected component of E_1. Let E_2' be the image of E_1' under ϕ. By Lemma 17, and C-homogeneity, the set E_1' has a common neighbour u in Γ. Also, by maximality of E_1' it follows that $u \nsim w$ for all $w \in E_1 \setminus E_1'$. Similarly the set E_2' has a common neighbour v and v is not adjacent to any vertex in $E_2 \setminus E_2'$. Therefore $\hat{\phi} : \langle u, E_1\rangle \to \langle v, E_2\rangle$ extending ϕ by $\hat{\phi}u = v$ is an isomorphism of connected substructures and so extends to an automorphism. Therefore the map ϕ also extends.

The final assertion follows by inspection of the list of countable homogeneous graphs. $\square$

4. Proof of the main theorem II: neighbourhood isomorphic to disjoint union of complete graphs

In contrast to the previous section, in the cases dealt with in this section we shall see that the graph Γ need not be homogeneous.

$\Gamma(v)$ isomorphic to a disjoint union of at least two non-trivial complete graphs

Let $\Gamma(v) \cong k_1 \cdot K_{k_2}$, that is, k_1 disjoint copies of K_{k_2} where $k_1, k_2 \in \mathbb{N} \cup \{\aleph_0\}$. We will deal separately with the special case that $\Gamma(v)$ is an independent set (i.e. the case $k_2 = 1$). Also when $k_1 = 1$, $\Gamma(v)$ is complete and, as observed in the paragraph immediately before Lemma 12, this implies that Γ itself is complete. Therefore we suppose that $k_1, k_2 \geqslant 2$. In this case we shall prove that either Γ is isomorphic to X_{k_1,k_2} where one of the parameters is infinite, or Γ is isomorphic to the line graph $L(K_{\aleph_0,\aleph_0})$ of the complete bipartite graph with countably infinite parts. This case divides into two parts depending on whether or not a square embeds into Γ. The first task in this case is to prove Corollary 26, that, except in the case when $\Gamma \cong L(K_{\aleph_0,\aleph_0})$, the square does not embed into Γ. Fix an edge $\{x, y\}$ in $E\Gamma$. Let $X = \Gamma(x) \setminus (\{y\} \cup \Gamma(y))$ and $Y = \Gamma(y) \setminus (\{x\} \cup \Gamma(x))$.

Lemma 19. *Suppose Γ embeds a square. For any $x' \in X$ and $y' \in Y$ if $x' \nsim y'$ then there exists $y'' \in \Gamma(x') \cap Y$ such that $y'' \sim y'$.*

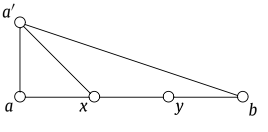
Proof. Since a square embeds there exists $y_1 \in \Gamma(y) \cap \Gamma(x')$ such that $\langle x', x, y, y_1 \rangle$ is a square. Since $k_2 \geqslant 2$ there exists $y_2 \sim y_1$ with $y_2 \in Y$. Now $y_2 \nsim x'$ since if $y_2 \sim x'$ then $\{x', y_1, y\}$ would all belong to the same connected component of $\Gamma(y_2)$ implying that $x' \sim y$, which is a contradiction. Now by C-homogeneity there exists $\alpha \in \mathrm{Aut}\,\Gamma$ such that $(x', x, y, y_2)^\alpha = (x', x, y, y')$. The element y_1^α belongs to Y and is adjacent both to y' and to x'. Therefore the element y_1^α satisfies the requirements of the lemma. $\square$

Lemma 20. *Suppose Γ embeds a square. Let X' and Y' be connected components of X and Y, respectively. Then there is a bijection $f : X' \to Y'$ such that for all $x' \in X'$, $y' \in Y'$, $x' \sim y'$ if and only if $y' = f(x')$.*

Proof. Note that X', Y' are complete graphs. Let $x' \in X'$. It follows from Lemma 19 that x' is adjacent to at least one vertex in Y'. By considering the neighbourhood of y' we conclude that x' is adjacent to no more than one vertex of Y'. This gives an injection f from X' to Y'. By Lemma 19 it follows that this map is onto, completing the proof of the lemma. $\square$

Lemma 21. *Suppose Γ embeds a square. Then* $\mathrm{diam}(\Gamma) \leqslant 2$.

Proof. Suppose that $\mathrm{diam}(\Gamma) \geqslant 3$ and let $a, b \in V\Gamma$ be at distance 3, as depicted.



It follows from Lemma 19 that there is a vertex a' joined, as in the configuration, to a, x, b but not y, and so $d(a, b) = 2$, which is a contradiction. $\square$

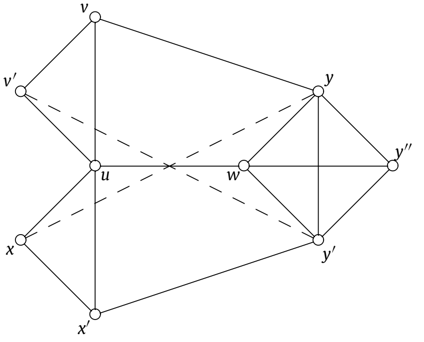
Now we shall consider three cases depending on the values of k_1 and k_2.

Case 1. $k_1 > 2$ and $k_2 > 2$.

Lemma 22. *If $k_1 > 2$ and $k_2 > 2$ then Γ does not embed a square.*

Proof. Suppose, for the sake of a contradiction, that Γ does embed a square. Let $u \in V\Gamma$ and let $v, w, x \in \Gamma(u)$ each be in distinct copies of K_{k_2} of $\Gamma(u)$ (i.e. $\langle v, w, x\rangle \cong I_3$). Let $y \in \Gamma(w) \setminus \{u\}$, let V be the connected component of $\Gamma(u)$ containing v, let X be the connected component of $\Gamma(u)$ containing x, and let Y be the connected component of $\Gamma(w)$ that contains y. Note that w is not adjacent to any member of $X \cup V$, so $Y \cap (X \cup V) = \emptyset$.

Let $f : V \to Y$ and $g : Y \to X$ be the bijections given by Lemma 20 (with uw replacing xy). We may suppose $f(v) = y$ and $g(y) = x$. Pick $v' \in V \setminus \{v\}$, and put $y' = f(v')$ and $x' = g(y')$. Let $y'' \in Y \setminus \{y, y'\}$. The following diagram illustrates the situation.



Now by C-homogeneity there is an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ fixing the vertices in the set $\{v, v', u, w, y''\}$ and interchanging x and x'. Since $x \sim y \sim v$ it follows that $x^\psi \sim y^\psi \sim v^\psi$ which implies that $x' \sim y^\psi \sim v$. But since ψ fixes w and y'' it follows that $y^\psi \in \Gamma(w) \cap \Gamma(y'')$. Thus $x' \sim y^\psi$ implies that $y^\psi = y'$ while $v \sim y^\psi$ implies that $y^\psi = y$. Therefore $y = y'$, which is a contradiction. $\square$

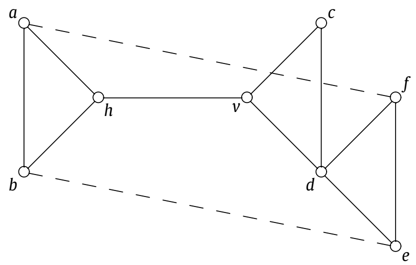
Case 2. $k_2 = 2$ and $k_1 = \aleph_0$.

Again, under the assumption that squares embed, we show this case cannot occur.

Lemma 23. *Assume Γ embeds a square. Then C_5 embeds into Γ as an induced subgraph.*

Proof. Fix x, and neighbours u, u', v, v', with u joined to u' and v joined to v'. Let y be another neighbour of x not adjacent to either u or to v (so $\{u, v, y\}$ is an independent set in the subgraph induced by $\Gamma(x)$). Consider an automorphism α fixing x,u,u' and y, and swapping v and v'. Consider a further joined pair a, a', with $\{a, a'\} \subseteq \Gamma(y) \setminus \Gamma(x)$. By Lemma 20, we may suppose that $\langle u, a, a', u'\rangle$ and $\langle v, a, a', v'\rangle$ are squares. Now α can neither fix nor swap a, a'. If $(a, a')^\alpha = (b, b')$ with $\{a, a'\} \cap \{b, b'\} = \emptyset$, then we have squares $\langle v, b', b, v'\rangle$, $\langle u, b, b', u'\rangle$, and so a pentagon $\langle a, v, b', b, u\rangle$. $\square$

Now we shall use the fact that a pentagon exists, along with the other facts of this case, to arrive at a contradiction. Fix v, and let $\{c, d\}$ be an edge in its neighbourhood. Let $h \in \Gamma(v)$ with $h \notin \{c, d\}$ and let $\{a, b\}$ be an edge in the neighbourhood of h, where a and b are at distance 2 from v. Moreover let $\{e, f\}$ be an edge in the neighbourhood of d, with e and f in $\Gamma_2(v)$, and $\{e, f\} \cap \{a, b\} = \emptyset$. Since a pentagon embeds we may choose a, b, c, d, e, f, h so that $a \sim f$. Now by considering the structure of $\Gamma(b)$, we see $b \nsim f$. Likewise, $a \nsim e$. Hence, by considering an automorphism taking $ahvde$ to $bhvdf$, we see that $b \sim e$ (compare the proof of Lemma 20). The diagram below illustrates the situation.



Now $b \nsim d$, as $\Gamma(d) \cong \aleph_0 \cdot K_2$. By Lemma 20 we must therefore have $b \sim c$ and $a \sim d$. But now $\langle a, f, e\rangle$ violates the structure of $\Gamma(d)$.

Case 3. $k_1 = 2$ and $k_2 = \aleph_0$.

Let Γ be a connected C-homogeneous graph such that $\Gamma(u) \cong K_{\aleph_0} \cup K_{\aleph_0}$, and such that a square embeds into Γ. By Lemma 21, Γ has diameter 2.

Lemma 24. *Each induced line L_3 of Γ extends uniquely to a square.*

Proof. Suppose otherwise and let $\langle a, b, e\rangle$ be a line that extends to two distinct squares $\langle a, b, e, c\rangle$ and $\langle a, b, e, d\rangle$. Since the neighbourhood of a has only two connected components it follows that $c \sim d$. But then a, d and e all belong to the same connected component of $\langle \Gamma(c)\rangle$ which is a contradiction since $a \nsim e$. □

We now use this lemma to prove the following result.

Proposition 25. *Let Γ be a connected C-homogeneous graph such that $\Gamma(u) \cong K_{\aleph_0} \cup K_{\aleph_0}$ and a square embeds into Γ. Then Γ is isomorphic to $L(K_{\infty,\infty})$, the line graph of the countable complete bipartite graph with infinite parts.*

Proof. By Lemma 21, Γ has diameter 2. The graph $L(K_{\aleph_0,\aleph_0})$ can be represented in the following way. Take two countably infinite sets C and D. Define the vertex set to be $C \times D$ and two distinct vertices are adjacent if their first or second coordinates coincide. It is easy to see that this graph is isomorphic to $L(K_{\aleph_0,\aleph_0})$.

Fix a vertex $v \in V\Gamma$. Let A and B denote the two connected components of $\Gamma(v)$. By Lemma 24 for every $a \in A$ and $b \in B$ the line $\langle a, v, b\rangle$ extends uniquely to a square $\langle a, v, b, v(a, b)\rangle$. Clearly $v(a, b) \notin A \cup B \cup \{v\}$. Also, if $w \notin (A \cup B \cup \{v\})$ is joined to $a \in A$ then as wav lies in a square, $w = v(a, b)$ for some $b \in B$. Thus, since $\operatorname{diam}(\Gamma) = 2$ it follows that $\Gamma = \{v\} \cup A \cup B \cup \{v(a, b)\colon\ a \in A,\ b \in B\}$. Also the elements $v(a, b)$ are all distinct in the sense that

$$v(a, b) = v(a', b') \quad \Rightarrow \quad a = a' \quad \text{and} \quad b = b'.$$

Indeed, if $a \neq a'$ then v, a' and $v(a, b)$ all belong to the same connected component of $\langle \Gamma(a)\rangle$ which contradicts the fact that $v \nsim v(a, b)$. Similarly $b \neq b'$ leads to a contradiction.

Claim 1. For distinct ordered pairs $(a, b), (c, d) \in A \times B$ we have

$$v(a, b) \sim v(c, d) \quad \Leftrightarrow \quad a = c \quad \text{or} \quad b = d.$$

Proof of claim. For the forward implication suppose, for the sake of a contradiction, that $a \neq c$ and $b \neq d$, but that $v(a, b) \sim v(c, d)$. By considering the neighbourhood $\Gamma(v(a, b))$ since $a \nsim b$ we must have either $a \sim v(c, d)$ or $b \sim v(c, d)$. But if $a \sim v(c, d)$ then $v(a, d) = v(c, d)$ implying $a = c$, a contradiction. Similarly $b \sim v(c, d)$ would imply $v(c, b) = v(c, d)$, again a contradiction.

For the converse implication, suppose without loss of generality that $a = c$ and $b \neq d$. By considering the neighbourhood of $a = c$ it is immediate that $v(a,b)$ is adjacent to $v(c,d)$. □

Returning to the proof of the proposition, the claim completely determines the structure of the graph Γ, and we conclude that $\Gamma \cong L(K_{\aleph_0,\aleph_0})$. Indeed, let C, D as in the first paragraph of the proof, with $L(K_{\aleph_0,\aleph_0})$ having vertex set $C \times D$, let $c \in C$ and $d \in D$, and fix bijections $f_1 : A \to C \setminus \{c\}$ and $f_2 : B \to D \setminus \{d\}$. Identify v with (c,d), $a \in A$ with $(f_1(a), d)$ and $b \in B$ with $(c, f_2(b))$ and $v(a,b)$ with $(f_1(a), f_2(b))$. □

Corollary 26. *If Γ is C-homogeneous and $\Gamma(v) \cong k_1 \cdot K_{k_2}$ where $k_1, k_2 \geqslant 2$ then either $\Gamma \cong L(K_{\aleph_0,\aleph_0})$, or Γ does not embed a square.*

Using this corollary we can now prove the second main result of this subsection.

Theorem 27. *If Γ is connected, C-homogeneous, a square does not embed into Γ, and $\Gamma(v) \cong k_1 \cdot K_{k_2}$ where $k_1, k_2 \geqslant 2$ then $\Gamma \cong X_{k_1,k_2}$.*

Proof. Clearly it is sufficient to prove that the only cycles that Γ embeds are triangles. Suppose otherwise and let C_n be the smallest non-triangle cycle that embeds into Γ. Since we are assuming a square does not embed it follows that $n \geqslant 5$. Fix a copy $\langle x, y, y', b_1, b_2 \ldots, b_{n-4}, x' \rangle$ of C_n in Γ. Let X be the connected component of $\Gamma(x')$ that contains x, and let Y be the connected component of $\Gamma(y')$ containing y. Clearly $X \cap Y = \emptyset$. Let $x_1 \in X \setminus \{x\}$ and $y_1 \in Y \setminus \{y\}$. By considering the neighbourhoods $\Gamma(x)$ and $\Gamma(y)$ we see that $x_1 \nsim y$ and $y_1 \nsim x$. Therefore, since no square embeds into Γ, it follows that $x_1 \nsim y_1$. Since y_1 was an arbitrary member of $Y \setminus \{y\}$ it follows that x_1 is not adjacent to any member of Y.

Since no square embeds we also have $y_1 \nsim x'$ and $x_1 \nsim y'$. In addition to this, for all j we have $y_1 \nsim b_j$, and $x_1 \nsim b_j$ (since any such edge would create a cycle C_m with $m < n$ contradicting the minimality of C_n). Therefore by C-homogeneity there is an automorphism α such that

$$(y_1, y', b_1, b_2, \ldots, b_{n-4}, x', x)^\alpha = (y_1, y', b_1, b_2, \ldots, b_{n-4}, x', x_1).$$

But $x \sim y$ implies $x_1 = x^\alpha \sim y^\alpha$ where, since α fixes y' and y_1, $y^\alpha \in Y$. This contradicts the fact that x_1 is not adjacent to any member of Y. □

$\Gamma(v)$ an independent set

The case where $\Gamma(v)$ is an independent set is special. It is the most difficult case, partly because of the large family of examples, namely the following: the triangle-free graph, bipartite graphs (complete, generic bipartite, or complement of a complete matching), and trees. It is in this case that we make use of the classification of homogeneous bipartite graphs. The following lemma will be used several times in this section.

Lemma 28. *Suppose that $\Gamma(v)$ is an independent set. If for all $k \in \mathbb{N}$ and for every I_k in Γ there is a vertex u such that $I_k \subseteq \Gamma(u)$ then Γ is a homogeneous graph.*

By inspection of the list of homogeneous graphs, the only possibilities in this case are the triangle free graph and the complete bipartite graph.

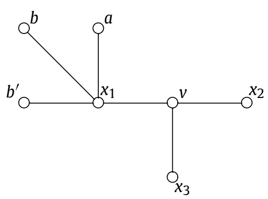
Proof. Let A be a finite induced subgraph of Γ. Let A' be a maximal independent subset of A. By assumption there exists u such that $A' \subseteq \Gamma(u)$. By maximality of A' we conclude that $\Gamma(u) \cap A = A'$. Also by maximality of A' the graph $\langle u \cup A \rangle$ is connected. Now any other induced subgraph isomorphic to A may be connected together in this same way and by Lemma 9 it follows that Γ is a homogeneous graph. □

For the rest of this section we shall suppose that Γ is not a tree, and that each $\Gamma(v)$ is an independent set. Let C_n be the smallest cycle that embeds into Γ. We know by Lemma 7 that $n \in \{4,5,6\}$. We also know that if $n = 5$ then $\mathrm{diam}(\Gamma) \leqslant 2$. Also, if $n = 6$ then $\mathrm{diam}(\Gamma) \leqslant 3$. We shall now rule out $n = 5, 6$ as possibilities.

Lemma 29. *If C_n is the smallest cycle that embeds into Γ then $n \neq 5$.*

Proof. For a contradiction suppose that C_5 embeds in Γ but C_3 and C_4 do not. Fix $v \in V\Gamma$ and consider $\Gamma_2(v) = \{w \in V\Gamma\colon\ d(v,w) = 2\}$. Let $\{x_i\colon\ i \in I\}$ be the neighbours of v, and define sets $X_i = \Gamma(x_i) \setminus \{v\}$. We may suppose $I = \mathbb{N}$. Since no square embeds it follows that $\Gamma_2(v)$ is a disjoint union of the sets X_i $(i \in I)$. By C-homogeneity for all $i \neq j$ the graph $\langle X_i \cup X_j \rangle$ is a homogeneous bipartite graph; indeed, any isomorphism between finite bipartite subgraphs that preserves the parts of the bipartition extends to an isomorphism of finite connected subgraphs of Γ fixing v, x_i, x_j. Therefore by Theorem 4, $\langle X_i \cup X_j \rangle$ is one of: complete bipartite, generic bipartite, empty, complete matching, complement of complete matching. Since we are in the case $n = 5$, $\langle X_i \cup X_j \rangle$ must be isomorphic to a complete matching.

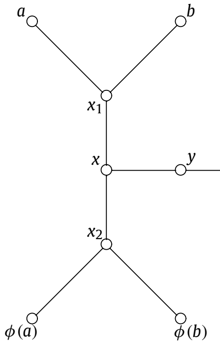
Consider X_1, X_2 and X_3. There is a bijection $\phi : X_1 \to X_1$ given by composing the bijection, arising from the complete matching, from X_1 to X_2 with the bijection from X_2 to X_3 and then with the bijection from X_3 back to X_1. Let $a \in X_1$. Since Γ does not embed a triangle we must have $\phi(a) \neq a$. Now we claim that any automorphism of Γ fixing all of $\{v, a, x_1, x_2, x_3\}$ must also fix $b := \phi(a)$. Indeed, let $\alpha \in \mathrm{Aut}\,\Gamma$ be such an automorphism. Clearly α preserves ϕ. Since $\phi(a) = b$ it follows that $\phi(a^\alpha) = b^\alpha$ and therefore $b = \phi(a) = \phi(a^\alpha) = b^\alpha$ as claimed. Now if we let b' be another vertex in X_1 then there is no automorphism fixing all of $\{v, x_1, x_2, x_3, a\}$ and interchanging b and b'. The following diagram illustrates the situation.



This contradicts C-homogeneity and completes the proof of the lemma. $\square$

Lemma 30. *If C_n is the smallest cycle that embeds into Γ then $n \neq 6$.*

Proof. The argument is similar to that of Lemma 29 above. For a contradiction suppose that C_6 is the smallest cycle which embeds in Γ. Fix an edge $\{x, y\}$ in the graph Γ, let $\{x_i\colon\ i \in I\} = \Gamma(x) \setminus \{y\}$ and let $\{y_i\colon\ i \in I\} = \Gamma(y) \setminus \{x\}$ where for simplicity we take $I = \mathbb{N}$. Also define $X_i = \Gamma(x_i) \setminus \{x\}$ and $Y_i = \Gamma(y_i) \setminus \{y\}$ for each $i \in I$. Our assumptions on cycles ensure that distinct sets X_i and X_j are disjoint with no edges between them, and that for all i, j the sets X_i and Y_j are disjoint. By C-homogeneity for all $i, j \in I$ the graph $\langle X_i \cup Y_j \rangle$ is a homogeneous bipartite graph and, because we are in the case $n = 6$, it must be isomorphic to a complete matching. There is a map $\phi : X_1 \to X_2$ given by composing the bijection from X_1 to Y_1, and that from Y_1 to X_2, given by the complete matchings. Fix $a, b \in X_1$. The situation is illustrated in the diagram below.



By C-homogeneity there is an automorphism α fixing $\{x, y, x_1, x_2, y_1, a, b\}$ and interchanging $\phi(a)$ and $\phi(b)$. Let $a' \in Y_1$ be the unique neighbour of a in Y_1. So by definition of ϕ we have $a \sim a' \sim \phi(a)$. Applying α to $a \sim a' \sim \phi(a)$ gives $a^\alpha \sim (a')^\alpha \sim \phi(a)^\alpha$, so $\phi(a)^\alpha$ is the image of a^α under the map ϕ given by composing the matchings $\langle X_1 \cup Y_1 \rangle$ and $\langle X_2 \cup Y_1 \rangle$. This shows $\phi(a)^\alpha = \phi(a^\alpha)$, and we conclude $\phi(b) = \phi(a)^\alpha = \phi(a^\alpha) = \phi(a)$ which is a contradiction since ϕ is a bijection. $\square$

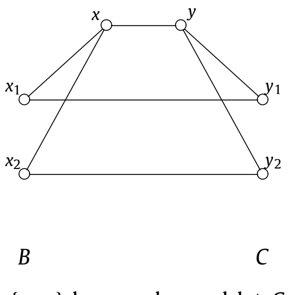
Thus we are left only with the possibility that $n = 4$, so let us suppose that $n = 4$. Let $\{x, y\}$ be an edge and define $C = \Gamma(y) \setminus \{x\}$ and $B = \Gamma(x) \setminus \{y\}$. By Lemma 6, $\langle C \cup B \rangle$ is a homogeneous bipartite graph. Let $\Delta = \Delta(\Gamma)$ denote this graph noting that, since Γ embeds a square, Δ contains at least one edge. Now by Theorem 4, Δ must be isomorphic to either the generic bipartite, complete matching, complement of complete matching, or complete bipartite graph.

First we find a diameter bound for Γ.

Lemma 31. *If* $n = 4$ *then* $\operatorname{diam}(\Gamma) \leqslant 3$.

Proof. We go through the possibilities for Δ.

Case 1. Δ isomorphic to a complete matching.



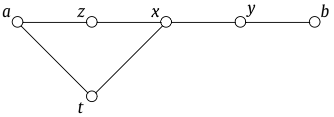
As above, let $\{x, y\}$ be an edge and let $C = \Gamma(y) \setminus \{x\}$ and $B = \Gamma(x) \setminus \{y\}$. Let $x_i \in B$ and $y_i \in C$ with $i \in \{1, 2\}$ and $x_i \sim y_i$. It follows that the subgraph induced by $\{x, y, x_1, x_2, y_1, y_2\}$ is isomorphic to $\square\!\square$ and so $\operatorname{diam}(\Gamma) \leqslant 3$ by Lemma 8.

Case 2. Δ isomorphic to generic bipartite graph.

Every bipartite graph embeds as a subgraph of $\langle B \cup C \rangle$, in particular $\square\!\square$ embeds and again by applying Lemma 8 we deduce $\operatorname{diam}(\Gamma) \leqslant 3$.

Case 3. Δ isomorphic to a complete bipartite or complement of complete matching.

Assume that $\mathrm{diam}(\Gamma) \geqslant 4$ and let $a, b \in V\Gamma$ be at distance 4. Let a, z, x, y, b be a path of length 4 from a to b. Extend azx to a square $azxt$ noting that $t \notin \{y, b\}$. This configuration is illustrated below.



Since $\Delta = \langle B \cup C\rangle$ is in this case assumed to be either complete bipartite or the complement of a complete matching it follows that either $z \sim b$ or $t \sim b$, contradicting the assumption that $d(a, b) = 4$. $\square$

In conclusion, thus far in this subsection we have proved the following.

Corollary 32. *Assume that Γ is C-homogeneous, that $\Gamma(v)$ is an independent set, and that Γ is not a tree. Then Γ embeds C_4, and* $\mathrm{diam}(\Gamma) \leqslant 3$.

We now aim to prove the following result.

Proposition 33. *Let Γ be a countable connected C-homogeneous graph. If $\Gamma(v)$ is an independent set, and Γ is not a tree* (*so Γ embeds C_4 and* $\mathrm{diam}(\Gamma) \leqslant 3$) *then either Γ is homogeneous or Γ is the generic bipartite graph or the* (*bipartite*) *complement of a complete matching.*

The rest of this section will be devoted to proving this proposition, so we work under its assumptions. We break the argument up according to the possibilities for the homogeneous bipartite graph Δ on $B \cup C$.

Case 1. Δ is a complete matching.

We show that this case does not occur. Let $f : B \to C$ denote the matching (so each $u \in B$ is joined to $f(u)$). First observe that each path of length 2 lies on a unique square: this is clear for example for the path yxu where $u \in B$, as the only such square is $yxuf(u)$. Now let $(p_i\colon\ i \in \mathbb{N})$ list B. For distinct $i, j \in \mathbb{N}$, there is r_{ij} such that $p_i x p_j r_{ij}$ is a square. Clearly $r_{ij} \notin \Delta \cup \{x, y\}$. If i, j, k are distinct then the following hold.

(a) $r_{ij} \neq r_{ik}$. Otherwise; $xp_i r_{ij}$ lies on two squares, through p_j and p_k;
(b) $r_{ij} \nsim r_{ik}$. Otherwise there is a triangle $r_{ij} p_i r_{ik}$;
(c) $r_{ij} \sim f(p_k)$. Indeed, if not, then by C-homogeneity there is an automorphism α such that $(x, p_i, r_{ij}, p_k, r_{jk})^\alpha = (x, p_i, r_{ij}, p_k, f(p_k))$. This automorphism fixes p_j (the unique element completing a square with $xp_i r_{ij}$), but this is impossible as $p_j \sim r_{jk}$ and $p_j \nsim f(p_k)$.

Now the path $f(p_1) y f(p_2)$ lies on two squares (in fact, infinitely many), namely squares through r_{34}, r_{35}. This is a contradiction.

Case 2. Δ is the complement of a complete matching.

We claim in this case that Γ itself is homogeneous bipartite, and is the complement of a complete matching. Let $f : B \to C$ be the bijection giving the matching; that is, each $w \in B$ is joined to all elements of C except $f(w)$. Pick $u \in B$. Then $\Gamma(u)$ contains all but one point (namely $f(u)$) of $\Gamma(y)$. It follows by transitivity on paths of length 2 that:

$(*)$ for any $u, v \in V\Gamma$ with $d(u, v) = 2$, each of $\Gamma(u) \setminus \Gamma(v)$, $\Gamma(v) \setminus \Gamma(u)$ has size 1.

Pick $u \in B$ and $v \in C$ with $u \nsim v$. Then by $(*)$ there is $c \in \Gamma(u) \setminus \Gamma(y)$ and $b \in \Gamma(v) \setminus \Gamma(x)$. Now by $(*)$, for each $u' \in B$, $u' \sim c$ and for each $v' \in C$, $v' \sim b$. By $(*)$ and connectedness, the vertex set of Γ is $\{x, y, b, c\} \cup B \cup C$. By $(*)$ applied to (b, y), $b \sim c$. Thus, Γ is the complement of a complete matching, with parts $B' := B \cup \{y, b\}$ and $C' := C \cup \{x, c\}$ and with a bijection $f' : B' \to C'$ extending f, with $f'(y) = c$ and $f'(b) = x$.

Case 3. Δ is a complete bipartite graph.

We claim that $\Gamma = \Delta \cup \{x, y\}$. This implies Γ is complete bipartite with partition $(B \cup \{y\}) \cup (C \cup \{x\})$.

To see this, suppose there is $r \notin \Delta \cup \{x, y\}$, with say $r \sim a \in B$. Let α be an automorphism with $(a, x)^\alpha = (x, y)$. Then $r^\alpha \in B$, $y^\alpha \in C$, so $r^\alpha \sim y^\alpha$, contradicting that $r \nsim y$.

Case 4. Δ is generic bipartite.

This case arises if Γ is the generic triangle-free graph or the generic bipartite graph. These are separated according to diameter (diameter 2 and 3, respectively).

Lemma 34. *Suppose that Δ is generic bipartite and* $\mathrm{diam}(\Gamma) = 2$. *Then Γ is the generic triangle-free graph.*

Proof. It suffices, by inspection of the list of homogeneous graphs, to show that Γ is homogeneous, and for this we apply Lemma 28. So we prove by induction on m that if $\{x_1, \ldots, x_m\}$ is an independent set then there is $u \in V\Gamma$ with $x_1, \ldots, x_m \in \Gamma(u)$. As $\mathrm{diam}(\Gamma) = 2$, we may assume $m > 2$. By induction, there is y_1 joined to $x_1, \ldots, x_{m-1}$, and y_2 joined to $x_2, \ldots, x_m$. As Γ is triangle-free, $y_1 \nsim y_2$, so $\langle x_1, \ldots, x_m, y_1, y_2\rangle$ is a connected bipartite graph so embeds via $\alpha \in \mathrm{Aut}(\Gamma)$ into Δ, by C-homogeneity. The images $x_1^\alpha, \ldots, x_m^\alpha$ of the x_i lie in one part of Δ, so have a common neighbour v in the other part of Δ. Then $x_1, \ldots, x_m \in \Gamma(v^{\alpha^{-1}})$. $\square$

Lemma 35. *Suppose that Δ is generic bipartite and Γ has diameter* 3. *Let $X = \{v\} \cup \Gamma_2(v)$, and $Y = \Gamma_1(v) \cup \Gamma_3(v)$. Then $X \cup Y$ is a bipartition for the graph Γ.*

Proof. By definition v is not adjacent to any vertex from $\Gamma_2(v)$. Similarly, if $a \in \Gamma(v)$ and $b \in \Gamma_3(v)$ then $a \nsim b$, since if $a \sim b$ then $d(v, b) < 3$, a contradiction. Also, if $a, b \in \Gamma_2(v)$ and $a \sim b$ then it would follow that C_5 embeds into Γ, which in turn would imply that $\mathrm{diam}(\Gamma) \leqslant 2$, a contradiction.

The only remaining possibility is that $a, b \in \Gamma_3(v)$ and $a \sim b$. We shall now show that this also leads to a contradiction.

Claim 1. There exist $y_1 \in \Gamma_1(v)$ and $x_1 \in \Gamma_2(v)$ such that $y_1 \nsim x_1$.

Proof of claim. Suppose otherwise. Then $\langle \Gamma_1(v) \cup \Gamma_2(v)\rangle$ would be a complete bipartite graph. Pick $a' \in X$ with $a' \sim a$, and $b' \in X$ with $b' \sim b$. Since $\mathrm{diam}(\Gamma) = 3$ it follows that $a', b' \in \Gamma_2(v)$. Then if $z \in \Gamma(v)$ we have $z \sim a'$ and $z \sim b'$. It follows that $\langle a, a', z, b', b\rangle$ is a pentagon, but by Lemma 7(i) this is impossible since Γ has diameter 3. $\square$

Now let y_1, x_1 be as in the claim. Pick $y_2 \in \Gamma(v) \cap \Gamma(x_1)$, and $x_2 \in \Gamma(y_1) \cap \Gamma(y_2) \setminus \{v\}$ (which exists since we are assuming a square embeds). Let $\alpha \in \mathrm{Aut}(\Gamma)$ with $(x_2, y_2)^\alpha = (x, y)$, so $x_1^\alpha, v^\alpha \in C$ and $y_1^\alpha \in B$. As Δ is generic bipartite there is $y' \in B$ joined to x_2^α, x_1^α but not to v^α (or to y_2^α or y_1^α), and then $y_3 := y'^{\alpha^{-1}}$ is joined to x_2, x_1 but not to v, y_1, y_2. Thus, $L := \langle v, y_1, x_2, y_3, x_1\rangle$ is a line of length 5 with $d(v, x_1) = 2$.

Also, let v, s, t, a be a path of length three from v to a. Then $L' = \langle v, s, t, a, b\rangle$ is a line with 5 vertices and such that the first and last vertices are at distance 3 in the graph. But this contradicts C-homogeneity since no automorphism sends L to L'. $\square$

Thus, by Corollary 32, to complete the analysis when Δ is generic bipartite, and hence to complete the proof of Theorem 2, it remains to prove the following.

Lemma 36. *Suppose that Δ is generic bipartite and* $\mathrm{diam}(\Gamma) = 3$. *Then Γ is generic bipartite.*

Proof. By the last lemma, Γ is bipartite with bipartition $V\Gamma = X \cup Y$.

Let $X' \subseteq X$ and $Y' \subseteq Y$ be finite subsets, and let $X' = U \cup V$ be a partition of X'. We shall show there is $w \in Y$ joined to all members of U and no members of V. The same argument applies with X and Y reversed, and this ensures Γ is generic bipartite.

By considering how the sets X, Y were defined, we see that X', Y' can be extended to X'' and Y'' so that $\langle X'' \cup Y'' \rangle$ is connected. By C-homogeneity there is $\alpha \in \mathrm{Aut}(\Gamma)$ such that $(X'' \cup Y'')^\alpha \subset \Delta$. By connectedness of $\langle X'' \cup Y'' \rangle$, X''^α and Y''^α lie in different parts of Δ. Thus, as Δ is generic bipartite, there is $w' \in \Delta$, not in the part of Δ containing $(X'')^\alpha$, joined to all members of U^α and no members of V^α. Then $w := w'^{\alpha^{-1}}$ lies in Y and is joined to all members of U and no members of V. $\square$

5. Connected-homogeneous partial orders

One may consider the notion of C-homogeneity but for other kinds of relational structure, for example, for partial orders or more generally for digraphs. The result for posets is not too hard, in fact it is far easier than the result above for graphs, and we include a proof in this section. The corresponding result for digraphs looks more difficult.

The countable homogeneous partially ordered sets were classified in [26] by Schmerl. We shall now classify the countable C-homogeneous posets. In contrast to what happens for graphs above, for posets when weakening homogeneous to C-homogeneous we obtain no new connected examples. The proof is much shorter than the one above for graphs, and correspondingly, Schmerl's classification of countable homogeneous posets is much shorter than the corresponding result for graphs. The notion of connectedness for posets is as in the next section: $(P, <)$ is connected if for any distinct $x, y \in P$ there are $x = x_0, x_1, \ldots, x_k = y$ such that for each $i = 0, \ldots, k-1$, $x_i < x_{i+1}$ or $x_{i+1} < x_i$.

Theorem 37. *A countable poset is connected-homogeneous if and only if it is isomorphic to a disjoint union of a countable number of isomorphic copies of some homogeneous countable poset.*

We begin by quoting Schmerl's result. Let $1 \leqslant n \leqslant \aleph_0$. Let $(A_n, <)$ denote the antichain with n elements. Let $B_n = A_n \times \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers, with the ordering $(a, p) < (b, q)$ if and only if $a = b$ and $p < q$. Let $C_n = B_n$ but with a different ordering to B_n where $(a, p) < (b, q)$ if and only if $p < q$. Finally let D denote the universal countable homogeneous partially ordered set; that is, the Fraïssé limit of the amalgamation class consisting of all finite partial orders.

Theorem 38. *(See [26, Main Theorem].) Let $(H, <)$ be a countable partially ordered set. Then $(H, <)$ is homogeneous if and only if it is isomorphic to one of the following*:

(i) $(A_n, <)$, $1 \leqslant n \leqslant \aleph_0$,
(ii) $(B_n, <)$, $1 \leqslant n \leqslant \aleph_0$,
(iii) $(C_n, <)$, $2 \leqslant n \leqslant \aleph_0$,
(iv) $(D, <)$.

Clearly in order to prove Theorem 37 it is sufficient to prove that any countable connected C-homogeneous poset is in fact homogeneous. As in the case of graphs above we will use angled brackets to denote induced substructures. If $a < b$ we write (a, b) for the substructure induced by the points in the interval $\{x\colon a < x < b\}$; there is no clash below with ordered-pairs notation.

Let H be a countable connected C-homogeneous poset. The following lemma is an immediate consequence of the definitions.

Lemma 39. *For all $x \in H$ the posets $x^\uparrow = \langle z \in H\colon z > x \rangle$ and $x^\downarrow = \langle z \in H\colon z < x \rangle$ are both homogeneous. Moreover for any $x, y \in H$ with $x < y$ the interval $(x, y) = \{z\colon x < z < y\}$ is homogeneous.*

Also by C-homogeneity, given any $x, y \in H$ we have $x^{\uparrow} \cong y^{\uparrow}$. Let H_u denote this poset. Similarly define H_d to be isomorphic to $x^{\downarrow}$ and H_i to be isomorphic to some interval (x, y) (noting that all such intervals are isomorphic by C-homogeneity). We now consider the various possibilities for the posets H_u, H_d and H_i, in each case determining the possibilities for H.

Lemma 40. *If any of H_u, H_d, or H_i is isomorphic to D then H is homogeneous (in which case $H \cong D$).*

Proof. First suppose that $H_u \cong D$. Let $E \subseteq H$ be a finite subset. Then since H is connected there is a finite subset $\bar{E}$ of H containing E such that $\langle \bar{E} \rangle$ is connected. Let $x \in H$ be arbitrary. Since $x^{\uparrow} \cong D$ it follows that $\langle \bar{E} \rangle$ embeds into $x^{\uparrow}$ and by C-homogeneity it follows that there exists an element x' such that $x' < \langle \bar{E} \rangle$ and thus $x' < E$. But now it follows that isomorphisms between induced substructures isomorphic to E always extend to automorphisms, since they may all be connected together in the same way. Since E was arbitrary it follows that H is a homogeneous poset.

The cases of $H_d \cong D$ or $H_i \cong D$ are dealt with using almost identical arguments to the one above. □

From now on, assume none of H_u, H_i, H_d is isomorphic to D.

Lemma 41. *$H_i \ncong B_n$ ($H_u \ncong B_n$, $H_d \ncong B_n$) for any $n \geqslant 2$.*

Proof. Suppose that $H_i \cong B_n$. Let $a, b \in H$ with $a < b$. Then $(a, b) \cong B_n$. Choose $a' < b'$ in a copy of $\mathbb{Q}$ in (a, b). Then by C-homogeneity $(a, b) \cong (a', b')$ which is a contradiction.

The proofs for H_u and H_d are essentially the same as the one above. □

Lemma 42. *If $H_i \cong C_n$ for some $n \geqslant 2$ then $H_u \cong H_d \cong C_n$.*

Proof. Let $a, b \in H$ with $a < b$. Then $a^{\uparrow}$ contains (a, b) as an induced substructure. It follows that $a^{\uparrow} \cong C_m$ for some $m \geqslant n$. To see that $m = n$ fix an antichain A of size n in (a, b). If $m > n$ this antichain would extend to an antichain of size $n + 1$ in H with the new element necessarily lying below b (since $a^{\uparrow} \cong C_m$). But this would contradict the fact that $(a, b) \cong C_n$. □

Using the same approach we may also prove the following.

Lemma 43. *If $H_i \cong \mathbb{Q}$ then $H_u \cong H_d \cong \mathbb{Q}$.*

Lemma 44. *If $H_i \cong \mathbb{Q}$ then $H \cong \mathbb{Q}$.*

Proof. By Lemma 43 we know that $H_u \cong H_d \cong \mathbb{Q}$. Let $x, y \in H$ and suppose that x is incomparable to y. Let π be a path of minimal length from x to y. So $\pi = (z_0, z_1, \ldots, z_{k-1}, z_k)$ with $z_0 = x$, $z_k = y$, and z_i comparable to z_j if and only if $|i - j| = 1$. If $z_1 < z_2$ then since ${z_1}^{\uparrow} \cong \mathbb{Q}$ it follows that x is comparable to z_1, which is a contradiction. Dually, $z_1 > z_2$ leads to a contradiction. We conclude that H is a countable 2-transitive chain and is therefore isomorphic to $\mathbb{Q}$. □

Therefore, the only remaining possibility is that H_i, H_u and H_d are all isomorphic to C_n for some $n \geqslant 2$.

Proposition 45. *If $H_i \cong C_n$ for some $n \geqslant 2$ then $H \cong C_n$.*

Proof. From Lemma 42 it follows that $H_i \cong H_u \cong H_d \cong C_n$. We now use this to prove that H is homogeneous.

Claim 1. Any 2-element antichain in H has a common upper bound and a common lower bound.

Proof of claim. First we note that the statement in the claim is equivalent to the claim that every 2-element antichain has a common upper bound *or* common lower bound. This is because since $n \geqslant 2$, if a two element antichain has a common upper bound then by C-homogeneity it embeds in a copy of C_n so automatically has a common lower bound also (and vice-versa).

Suppose that the result fails, and let $\{x, y\}$ be an antichain that has no upper bound (and hence no lower bound) and chosen to be the minimal distance apart with respect to this property. Let π be a minimal length path from x to y, and let y' be the penultimate term in the sequence (so y' is comparable to y). Then x and y' is an antichain that by minimality has a common upper bound m and common lower bound l. It follows, since y' is comparable with y, that m is a common upper bound or l is a common lower bound for the antichain $\{x, y\}$, which is a contradiction. □

Claim 2. Any finite antichain A of H has a common upper bound.

Proof of claim. Suppose that the result is not true and let A be a minimal finite antichain for which it fails. By the previous claim we know that $|A| > 2$. Let $a, a' \in A$, let u be an upper bound for $A \setminus \{a\}$, and v be an upper bound for $A \setminus \{a'\}$. These exist by minimality. Now u and v cannot be comparable, since if they were one of them would serve as an upper bound for the whole of A. Therefore u and v have a common upper bound by Claim 1, and that upper bound serves as an upper bound for A, a contradiction. □

Claim 3. Any finite subset F of H has a common upper bound.

Proof of claim. Suppose the result fails and let F be minimal with respect to not having an upper bound. If F is an antichain we are done by the previous case so we may suppose that there exist $x, y \in F$ with $x > y$. But now by induction $F \setminus \{y\}$ has a common upper bound which serves as an upper bound for F also, a contradiction. □

Returning to the proof of the proposition, it is now clear that H is homogeneous because any two isomorphic finite induced substructures may be connected together in the same way (by adjoining an upper bound) and then by C-homogeneity any partial isomorphism extends. □

6. Concluding remarks

One obvious question is whether we really need the full power of C-homogeneity to prove the main result for graphs above? For example, does there exist some $k \in \mathbb{N}$ such that if $\mathrm{Aut}\,\Gamma$ is homogeneous on induced subgraphs with diameter $\leqslant k$ then Γ is C-homogeneous? For most (but not all) arguments in this paper a small value of k suffices.

A related idea to that of C-homogeneity, perhaps originally due to Cameron, is that of distance homogeneous graphs. These are countable graphs that are homogeneous in a language which encodes the distance, in the graph, by binary relations; that is, for each positive integer t there is a binary predicate P_t such that $P_t(x, y)$ holds if and only if $d(x, y) = t$. All the graphs listed in Theorem 2, apart from line graphs of complete bipartite graphs, are distance homogeneous. Thus, the notions of C-homogeneity, and distance homogeneity, are related and lie somewhere in the spectrum of conditions between homogeneity, and distance transitivity. As a starting point for investigating distance homogeneous graphs, note that if Γ is distance-homogeneous, then $\Gamma(v)$ is homogeneous, since non-adjacent pairs in $\Gamma(v)$ are at distance 2. There are results of Moss on distance-homogeneous graphs in [23] and [24].

For arbitrary first order relational structures, over a relational language L, there are natural notions of dimension and hence connectedness and diameter. We say $d(x, y) = k$ if k is least such that there are $x_0 = x, x_1, \ldots, x_k = y$ such that each successive pair $\{x_i, x_{i+1}\}$ lies in a tuple satisfying a relation of L. If L is a finite relational language and M is a countable connected C-homogeneous L-structure of finite diameter, in the natural sense, then there is a function f on the natural numbers such that each substructure of size k lies in a connected substructure of size at most $f(k)$. Hence, $\mathrm{Aut}(M)$ has finitely many orbits on M^k, so, by the Ryll-Nardzewski Theorem (see [25], or Theorem 7.3.1 of [15]), M is ω-categorical; that is, any countable L-structure which satisfies the same first-order sentences of L is isomorphic to M. On the other hand, if M is connected of infinite diameter, then $\mathrm{Aut}(M)$ has infinitely many orbits on pairs, so M is not ω-categorical. A possible goal is to show that in the infinite diameter case, M has a treelike structure. In the finite bounded diameter case, the theory of Lachlan [20] will apply, and the *finite* examples should fall into finitely many infinite families, coordinatised by dimensions, and finitely many sporadics.

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References

[1] P.J. Cameron, Oligomorphic Permutation Groups, Cambridge University Press, Cambridge, 1990.
[2] P.J. Cameron, A census of infinite distance-transitive graphs, Discrete Math. 192 (1–3) (1998) 11–26, Discrete metric spaces, Villeurbanne, 1996.
[3] G.L. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments, Mem. Amer. Math. Soc. 131 (621) (1998), xiv+161.
[4] M. Droste, H.D. Macpherson, On k-homogeneous posets and graphs, J. Combin. Theory Ser. A 56 (1) (1991) 1–15.
[5] M. Droste, M. Giraudet, H.D. Macpherson, N. Sauer, Set-homogeneous graphs, J. Combin. Theory Ser. B 62 (1) (1994) 63–95.
[6] H. Enomoto, Combinatorially homogeneous graphs, J. Combin. Theory Ser. B 30 (2) (1981) 215–223.
[7] P. Erdős, J. Spencer, Probabilistic Methods in Combinatorics, Probab. Math. Statist., vol. 17, Academic Press, New York, London, 1974.
[8] D.M. Evans, Examples of $\aleph_0$-categorical structures, in: Automorphisms of First-Order Structures, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994, pp. 33–72.
[9] R. Fraïssé, Sur certaines relations qui généralisent l'ordre des nombres rationnels, C. R. Acad. Sci. Paris 237 (1953) 540–542.
[10] R. Fraïssé, Theory of Relations, Stud. Logic Found. Math., vol. 145, North-Holland Publishing Co., Amsterdam, 2000, revised edition, with an appendix by Norbert Sauer.
[11] A. Gardiner, Homogeneous graphs, J. Combin. Theory Ser. B 20 (1) (1976) 94–102.
[12] A. Gardiner, Homogeneous graphs stability, J. Austral. Math. Soc. Ser. A 21 (3) (1976) 371–375.
[13] A. Gardiner, Homogeneity conditions in graphs, J. Combin. Theory Ser. B 24 (3) (1978) 301–310.
[14] M. Goldstern, R. Grossberg, M. Kojman, Infinite homogeneous bipartite graphs with unequal sides, Discrete Math. 149 (1–3) (1996) 69–82.
[15] W. Hodges, Model Theory, Encyclopedia Math. Appl., vol. 42, Cambridge University Press, Cambridge, 1993.
[16] J. Hubicka, J. Nesetril, Finite presentation of homogeneous graphs, posets and Ramsey classes, Israel J. Math. 149 (2005) 21–44.
[17] A.A. Ivanov, Bounding the diameter of a distance-regular graph., Dokl. Akad. Nauk SSSR 271 (4) (1983) 789–792.
[18] A.S. Kechris, V.G. Pestov, S. Todorcevic, Fraïssé limits Ramsey theory, and topological dynamics of automorphism groups, Geom. Funct. Anal. 15 (2005) 106–189.
[19] A.H. Lachlan, Countable homogeneous tournaments, Trans. Amer. Math. Soc. 284 (2) (1984) 431–461.
[20] A.H. Lachlan, On countable stable structures which are homogeneous for a finite relational language, Israel J. Math. 49 (1–3) (1984) 69–153.
[21] A.H. Lachlan, R.E. Woodrow, Countable ultrahomogeneous undirected graphs, Trans. Amer. Math. Soc. 262 (1) (1980) 51–94.
[22] H.D. Macpherson, Infinite distance transitive graphs of finite valency, Combinatorica 2 (1) (1982) 63–69.
[23] L.S. Moss, The universal graphs of fixed finite diameter, in: Graph Theory, Combinatorics, and Applications, vol. 2, Kalamazoo, MI, 1988, Wiley Intersci. Publ., New York, 1991, pp. 923–937.
[24] L.S. Moss, Distanced graphs., Discrete Math. 102 (3) (1992) 287–305.
[25] C. Ryll-Nardzewski, On the categoricity in power $\leqslant \aleph_0$, Bull. Pol. Acad. Sci. Math. 7 (1959) 545–548 (unbound insert).
[26] J.H. Schmerl, Countable homogeneous partially ordered sets, Algebra Universalis 9 (3) (1979) 317–321.