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Delay-dependent exponential stability for a class of neural networks with time delays☆

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Abstract

This paper is concerned with the exponential stability of a class of delayed neural networks described by nonlinear delay differential equations of the neutral type. In terms of a linear matrix inequality (LMI), a sufficient condition guaranteeing the existence, uniqueness and global exponential stability of an equilibrium point of such a kind of delayed neural networks is proposed. This condition is dependent on the size of the time delay, which is usually less conservative than delay-independent ones. The proposed LMI condition can be checked easily by recently developed algorithms solving LMIs. Examples are provided to demonstrate the effectiveness and applicability of the proposed criteria.

Keywords: Delay-dependent conditions; Global exponential stability; Linear matrix inequality; Neural networks; Neutral systems; Time-delay systems

1. Introduction

In the past decades, neural networks have received a great deal of interest due to their extensive applications in image processing, quadratic optimization, fixed-point computation, and other areas [3,8,9,15].

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Among the different proposed neural network schemes, cellular neural network and Hopfield neural network are the two important types, which have been widely studied in the literature. Now, it has been shown that applications of neural networks rely heavily on the dynamical behaviors of the networks. Therefore, stability analysis for neural networks has been investigated and various approaches have been proposed; see, e.g., [10,12,18] and the references therein.

In implementations of artificial neural networks, time delays are unavoidable due to finite switching speeds of the amplifiers. The existence of time delays may cause oscillations and instability of neural networks. Therefore, it is important to investigate the stability of delayed neural networks. In [1,6,11,16,19], global asymptotic stability conditions for different classes of delayed neural networks were proposed under some assumptions. In the design of neural networks, however, one is not only interested in global stability, but also in some other performances. Particularly, it is often desirable to have a neural network that converges fast enough in order to achieve fast response. Considering this, many researchers have studied the exponential stability analysis problem for delayed neural networks and a great number of results on this topic have been reported in the literature; see, e.g., [2,5,7,14,20] and the references therein.

In this paper, we consider a class of neural networks with time delays described by a nonlinear delay differential equation of neutral type. Attention is focused on the derivation of global exponential stability for such a class of delayed neural networks. In terms of a linear matrix inequality (LMI), a sufficient condition for global exponential stability is proposed; this condition is delay-dependent; that is, the condition depends on the size of time delays. It is worth pointing out that delay-dependent stability conditions are usually less conservative than delay-independent ones, especially in the case when the delay size is small [13]. Also, it should be noted that the proposed LMI condition can be checked numerically very efficiently by resorting to recently developed interior-point methods, and no tuning of parameters will be involved [4]. This is in contrast to the stability results in [1,2,6,7,11,16,19], which are often difficult to check. The maximum bound for the time delay which ensures that the delayed neural network is globally exponentially stable can be obtained by solving a quasi-convex optimization problem. Finally, we provide examples to demonstrate the effectiveness and applicability of the proposed method.

Notation. Throughout this paper, for real symmetric matrices X and Y, the notation $X \geqslant Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). The superscript "T" represents the transpose. We use $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote the minimum and maximum eigenvalue of a real symmetric matrix, respectively. The notation $\|x\|$ denotes a vector norm defined by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ where x is a vector, while $\|A\|$ denotes a matrix norm defined by $\|A\| = (\lambda_{\max}(A^{\mathrm{T}}A))^{1/2}$ where A is a matrix. $\rho(\cdot)$ denotes the spectral radius of a matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. Problem formulation

Consider the following class of delayed neural networks described by a nonlinear neutral delay differential equation:

$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij1} g_j(u_j(t)) + \sum_{j=1}^{n} w_{ij2} g_j(u_j(t-\tau)) + \sum_{j=1}^{n} d_{ij} \dot{u}_j(t-\tau) + \mathscr{I}_i, \tag{1}$$

$$u_i(t) = \phi_i(t), \quad -\tau \leqslant t \leqslant 0, \tag{2}$$

where $i = 1, 2, \ldots, n$, and n denotes the number of neurons in a neural network; $u_i(t)$ denotes the state of the ith neuron at time t; g_j is the activation function of the jth neuron; the scalar $a_i > 0$ is the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t; w_{ij1}, w_{ij2}, d_{ij}, $i, j = 1, 2, \ldots, n$, are known scalars; the scalar $\tau > 0$ represents the transmission delay; $\mathscr{I}_i$ is the external bias on the ith neuron at time t. The functions $\phi_i(t)$ denote the initial conditions.

Denote

$$u(t) = [u_1(t) \quad u_2(t) \quad \cdots \quad u_n(t)]^{\mathrm{T}},$$

$$g(u(t)) = [g_1(u_1(t)) \quad g_2(u_2(t)) \quad \cdots \quad g_n(u_n(t))]^{\mathrm{T}},$$

$$\mathscr{I} = [\mathscr{I}_1 \quad \mathscr{I}_2 \quad \cdots \quad \mathscr{I}_n]^{\mathrm{T}},$$

$$\phi(t) = [\phi_1(t) \quad \phi_2(t) \quad \cdots \quad \phi_n(t)]^{\mathrm{T}}.$$

Then, (1) and (2) can be re-written as

$$\dot{u}(t) = -Au(t) + W_1 g(u(t)) + W_2 g(u(t-\tau)) + D\dot{u}(t-\tau) + \mathscr{I}, \tag{3}$$

$$u(t) = \phi(t), \quad -\tau \leqslant t \leqslant 0, \tag{4}$$

where

$$A = \mathrm{diag}(a_1, a_2, \ldots, a_n) > 0, \quad W_1 = \{w_{ij1}\}, \quad W_2 = \{w_{ij2}\}, \quad D = \{d_{ij}\}.$$

Throughout the paper we assume that the activation function satisfies the following assumption.

Assumption 1 (*Arik [1]*). The activation function $g(u)$ is bounded and satisfies

$$0 \leqslant \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \leqslant \sigma_i, \quad i = 1, 2, \ldots, n, \tag{5}$$

for any $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$, where $\sigma_i > 0$ for $i = 1, 2, \ldots, n$.

Definition 1. A vector $u^* \in \mathbb{R}^n$ is said to be an equilibrium point of system (3) if it satisfies

$$-Au^* + (W_1 + W_2) g(u^*) + \mathscr{I} = 0.$$

By Assumption 1, it can be seen that there exists an equilibrium u^* for DCNN (4) [6]. Now, let

$$x(t) = u(t) - u^*. \tag{6}$$

Then, it is easy to see that system (3) can be transformed to

$$\dot{x}(t) = -Ax(t) + W_1 f(x(t)) + W_2 f(x(t-\tau)) + D\dot{x}(t-\tau), \tag{7}$$

where

$$x(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^{\mathrm{T}}$$

is the state vector of the transformed system, and

$$f(x(t)) = [f_1(x_1(t)) \quad f_2(x_2(t)) \quad \cdots \quad f_n(x_n(t))]^{\mathrm{T}}$$

with

$$f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*), \quad i = 1, 2, \ldots, n.$$

Then, it is easy to see that

$$f_i(0) = 0, \quad i = 1, 2, \ldots, n$$

and $f_i(\cdot)$ satisfies (5), that is,

$$0 \leqslant \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \leqslant \sigma_i, \quad i = 1, 2, \ldots, n. \tag{8}$$

We will also need the following definition.

Definition 2. System (7) is said to be exponentially stable if there exist scalars $k > 0$ and $\gamma > 0$ such that for every solution $x(t)$ of (7),

$$\|x(t)\| \leqslant \gamma \mathrm{e}^{-kt} \sup_{-\tau \leqslant \theta \leqslant 0} \{\|x(\theta)\|, \|\dot{x}(\theta)\|\}.$$

The problem to be addressed in this paper is to develop delay-dependent conditions such that the delayed neural network in (7) is globally exponentially stable. More specifically, for a given scalar $\bar{\tau} > 0$, our purpose is to determine whether the system in (7) is globally exponentially stable for any delay $0 < \tau \leqslant \bar{\tau}$.

3. Main results

We first introduce the following lemma, which will be used in the proof of our main results.

Lemma 1 (*Xu et al.* [17]). *Let $\mathscr{D}$, $\mathscr{S}$ and $\mathscr{P}$ be real matrices of appropriate dimensions with $\mathscr{P} > 0$. Then for any vectors x and y with appropriate dimensions,*

$$2x^{\mathrm{T}} \mathscr{D} \mathscr{S} y \leqslant x^{\mathrm{T}} \mathscr{D} \mathscr{P} \mathscr{D}^{\mathrm{T}} x + y^{\mathrm{T}} \mathscr{S}^{\mathrm{T}} P^{-1} \mathscr{S} y.$$

Now, we are in a position to give an exponential stability condition for the delayed neural network in (7).

Theorem 1. *The origin of the delayed neural network in* (7) *is the unique equilibrium point and is globally exponentially stable for any delay $0 < \tau \leqslant \bar{\tau}$ if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$ and* *two diagonal matrices* $S>0$ *and* $Y>0$ *such that the following LMI holds*:

$$\begin{bmatrix} -PA-AP+A\tilde{Q}A & (P-A\tilde{Q})W_1+\frac{1}{2}S\Sigma & (P-A\tilde{Q})W_2+Y & (P-A\tilde{Q})D & 0 \\ W_1^{\mathrm{T}}(P-\tilde{Q}A)+\frac{1}{2}\Sigma S & W_1^{\mathrm{T}}\tilde{Q}W_1+Q_1-S & W_1^{\mathrm{T}}\tilde{Q}W_2 & W_1^{\mathrm{T}}\tilde{Q}D & 0 \\ W_2^{\mathrm{T}}(P-\tilde{Q}A)+Y & W_2^{\mathrm{T}}\tilde{Q}W_1 & W_2^{\mathrm{T}}\tilde{Q}W_2-2Y\Sigma^{-1}-Q_1 & W_2^{\mathrm{T}}\tilde{Q}D & -\bar{\tau}Y \\ D^{\mathrm{T}}(P-\tilde{Q}A) & D^{\mathrm{T}}\tilde{Q}W_1 & D^{\mathrm{T}}\tilde{Q}W_2 & D^{\mathrm{T}}\tilde{Q}D-Q_2 & 0 \\ 0 & 0 & -\bar{\tau}Y & 0 & -\bar{\tau}Q_3 \end{bmatrix}<0, \tag{9}$$

where

$$\Sigma=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_n), \tag{10}$$

$$\tilde{Q}=Q_2+\bar{\tau}Q_3 \tag{11}$$

and $\sigma_i>0$, $i=1,2,\ldots,n$ *are given in Assumption* 1.

Proof. To show the global exponential stability of an equilibrium point of (7), we first note that (9) implies

$$D^{\mathrm{T}}Q_2D-Q_2<0.$$

Therefore,

$$\rho(D)<1. \tag{12}$$

Considering (9), it is easy to see that there exists a scalar δ simultaneously satisfying

$$0<\delta<\min\left\{\frac{1}{16\|A\|^4},\frac{1}{16\|W_1\|^4},\frac{1}{16\|W_2\|^4}\right\} \tag{13}$$

and

$$\begin{bmatrix} -PA-AP+A\tilde{Q}A & (P-A\tilde{Q})W_1+\frac{1}{2}S\Sigma & (P-A\tilde{Q})W_2+Y & (P-A\tilde{Q})D & 0 \\ W_1^{\mathrm{T}}(P-\tilde{Q}A)+\frac{1}{2}\Sigma S & W_1^{\mathrm{T}}\tilde{Q}W_1+Q_1-S & W_1^{\mathrm{T}}\tilde{Q}W_2 & W_1^{\mathrm{T}}\tilde{Q}D & 0 \\ W_2^{\mathrm{T}}(P-\tilde{Q}A)+Y & W_2^{\mathrm{T}}\tilde{Q}W_1 & W_2^{\mathrm{T}}\tilde{Q}W_2-2Y\Sigma^{-1}-Q_1 & W_2^{\mathrm{T}}\tilde{Q}D & -\bar{\tau}Y \\ D^{\mathrm{T}}(P-\tilde{Q}A) & D^{\mathrm{T}}\tilde{Q}W_1 & D^{\mathrm{T}}\tilde{Q}W_2 & D^{\mathrm{T}}\tilde{Q}D-Q_2 & 0 \\ 0 & 0 & -\bar{\tau}Y & 0 & -\bar{\tau}Q_3 \end{bmatrix} + \begin{bmatrix} (\delta+\sqrt{\delta})I & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\delta}I & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\delta}I & 0 & 0 \\ 0 & 0 & 0 & 4\delta\|D\|^2I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}<0. \tag{14}$$

It can be verified that (14) can be rewritten as

$$\begin{bmatrix} -PA-AP & PW_1+\frac{1}{2}S\Sigma & PW_2+Y & PD & 0 \\ W_1^{\mathrm{T}}P+\frac{1}{2}\Sigma S & Q_1-S & 0 & 0 & 0 \\ W_2^{\mathrm{T}}P+Y & 0 & -2Y\Sigma^{-1}-Q_1 & 0 & -\bar{\tau}Y \\ D^{\mathrm{T}}P & 0 & 0 & -Q_2 & 0 \\ 0 & 0 & -\bar{\tau}Y & 0 & -\bar{\tau}Q_3 \end{bmatrix} + \begin{bmatrix} -A \\ W_1^{\mathrm{T}} \\ W_2^{\mathrm{T}} \\ D^{\mathrm{T}} \\ 0 \end{bmatrix} \tilde{Q} \begin{bmatrix} -A \\ W_1^{\mathrm{T}} \\ W_2^{\mathrm{T}} \\ D^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} (\delta+\sqrt{\delta})I & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\delta}I & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\delta}I & 0 & 0 \\ 0 & 0 & 0 & 4\delta\|D\|^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0.$$

Then, it follows that for any $0 < \tau \leqslant \bar{\tau}$ we have

$$\Psi + \mathrm{diag}((\delta+\sqrt{\delta})I, \sqrt{\delta}I, \sqrt{\delta}I, 4\delta\|D\|^2 I, 0) < 0, \tag{15}$$

where

$$\Psi = \begin{bmatrix} -PA-AP & PW_1+\frac{1}{2}S\Sigma & PW_2+Y & PD & 0 \\ W_1^{\mathrm{T}}P+\frac{1}{2}\Sigma S & Q_1-S & 0 & 0 & 0 \\ W_2^{\mathrm{T}}P+Y & 0 & -2Y\Sigma^{-1}-Q_1 & 0 & -\tau Y \\ D^{\mathrm{T}}P & 0 & 0 & -Q_2 & 0 \\ 0 & 0 & -\tau Y & 0 & -\tau Q_3 \end{bmatrix} + \begin{bmatrix} -A \\ W_1^{\mathrm{T}} \\ W_2^{\mathrm{T}} \\ D^{\mathrm{T}} \\ 0 \end{bmatrix} \hat{Q} \begin{bmatrix} -A \\ W_1^{\mathrm{T}} \\ W_2^{\mathrm{T}} \\ D^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}}, \tag{16}$$

with

$$\hat{Q} = Q_2 + \tau Q_3. \tag{17}$$

Now, define a Lyapunov–Krasovskii functional candidate for system (7) as

$$V(x_t) = x(t)^{\mathrm{T}}Px(t) + V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{18}$$

where

$$x_t = x(t+\theta), \quad -\tau \leqslant \theta \leqslant 0, \tag{19}$$

$$V_1(x_t) = \int_{t-\tau}^{t} f(x(\alpha))^{\mathrm{T}} Q_1 f(x(\alpha))\,\mathrm{d}\alpha, \tag{20}$$

$$V_2(x_t) = \int_{t-\tau}^{t} \dot{x}(\alpha)^{\mathrm{T}} Q_2 \dot{x}(\alpha)\,\mathrm{d}\alpha, \tag{21}$$

$$V_3(x_t) = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{x}(\alpha)^{\mathrm{T}} Q_3 \dot{x}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{22}$$

Then, the time-derivative of $V(x_t)$ along the solution of (7) gives

$$\begin{aligned}\dot{V}(x_t) = {} & 2x(t)^{\mathrm{T}} P[-Ax(t) + W_1 f(x(t)) + W_2 f(x(t-\tau)) + D\dot{x}(t-\tau)] \\ & + \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t),\end{aligned} \tag{23}$$

where

$$\dot{V}_1(x_t) = f(x(t))^{\mathrm{T}} Q_1 f(x(t)) - f(x(t-\tau))^{\mathrm{T}} Q_1 f(x(t-\tau)), \tag{24}$$

$$\dot{V}_2(x_t) = \dot{x}(t)^{\mathrm{T}} Q_2 \dot{x}(t) - \dot{x}(t-\tau)^{\mathrm{T}} Q_2 \dot{x}(t-\tau), \tag{25}$$

$$\dot{V}_3(x_t) = \tau \dot{x}(t)^{\mathrm{T}} Q_3 \dot{x}(t) - \int_{t-\tau}^{t} \dot{x}(\alpha)^{\mathrm{T}} Q_3 \dot{x}(\alpha)\, \mathrm{d}\alpha. \tag{26}$$

By the Newton–Leibniz formula, it is easy to see that

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(\alpha)\, \mathrm{d}\alpha. \tag{27}$$

It then follows from (23)–(27) that

$$\begin{aligned}\dot{V}(x_t) = {} & 2x(t)^{\mathrm{T}} P[-Ax(t) + W_1 f(x(t)) + W_2 f(x(t-\tau)) + D\dot{x}(t-\tau)] \\ & + f(x(t))^{\mathrm{T}} Q_1 f(x(t)) - f(x(t-\tau))^{\mathrm{T}} Q_1 f(x(t-\tau)) + \dot{x}(t)^{\mathrm{T}} \hat{Q} \dot{x}(t) \\ & - \int_{t-\tau}^{t} \dot{x}(\alpha)^{\mathrm{T}} Q_3 \dot{x}(\alpha)\, \mathrm{d}\alpha - \dot{x}(t-\tau)^{\mathrm{T}} Q_2 \dot{x}(t-\tau) \\ = {} & \frac{1}{\tau} \int_{t-\tau}^{t} \{2x(t)^{\mathrm{T}} P[-Ax(t) + W_1 f(x(t)) + W_2 f(x(t-\tau)) + D\dot{x}(t-\tau)] \\ & + f(x(t))^{\mathrm{T}} Q_1 f(x(t)) - f(x(t-\tau))^{\mathrm{T}} Q_1 f(x(t-\tau)) + \dot{x}(t)^{\mathrm{T}} \hat{Q} \dot{x}(t) - \dot{x}(\alpha)^{\mathrm{T}} \tau Q_3 \dot{x}(\alpha) \\ & + 2f(x(t-\tau))^{\mathrm{T}} Y[x(t) - x(t-\tau)] - 2f(x(t-\tau))^{\mathrm{T}} \tau Y \dot{x}(\alpha) \\ & - f(x(t))^{\mathrm{T}} S f(x(t)) + f(x(t))^{\mathrm{T}} S f(x(t)) - \dot{x}(t-\tau)^{\mathrm{T}} Q_2 \dot{x}(t-\tau)\}\, \mathrm{d}\alpha,\end{aligned} \tag{28}$$

where $\hat{Q}$ is given in (17). Taking into account (8), we can deduce

$$-f(x(t-\tau))^{\mathrm{T}} Y x(t-\tau) \leqslant -f(x(t-\tau))^{\mathrm{T}} Y \Sigma^{-1} f(x(t-\tau)), \tag{29}$$

$$f(x(t))^{\mathrm{T}} S f(x(t)) \leqslant f(x(t))^{\mathrm{T}} S \Sigma x(t). \tag{30}$$

Therefore, by (28)–(30), it can be shown that

$$\dot{V}(x_t) \leqslant \frac{1}{\tau} \int_{t-\tau}^{t} \xi(t,\alpha)^{\mathrm{T}} \Psi \xi(t,\alpha)\, \mathrm{d}\alpha, \tag{31}$$

where Ψ is given in (16), and

$$\xi(t,\alpha) = [x(t)^{\mathrm{T}} \quad f(x(t))^{\mathrm{T}} \quad f(x(t-\tau))^{\mathrm{T}} \quad \dot{x}(t-\tau)^{\mathrm{T}} \quad \dot{x}(\alpha)^{\mathrm{T}}]^{\mathrm{T}}.$$

By (15) and (31), we have

$$\dot{V}(x_t) \leqslant -(\delta+\sqrt{\delta})\|x(t)\|^2 - \sqrt{\delta}\|f(x(t))\|^2 - \sqrt{\delta}\|f(x(t-\tau))\|^2 - 4\delta\|D\|^2\|\dot{x}(t-\tau)\|^2. \quad (32)$$

Thus, by Theorem 1.6 in Ref. [13, p. 129], it follows from (12) and (32) that the delayed neural network in (7) is globally asymptotically stable. To show the global exponential stability of the delayed neural network in (7), we further note that from (7) and (8), it follows that

$$\begin{aligned}\|\dot{x}(t)\|^2 &= \|-Ax(t)+W_1 f(x(t))+W_2 f(x(t-\tau))+D\dot{x}(t-\tau)\|^2\\ &\leqslant 4\|A\|^2\|x(t)\|^2+4\|W_1\|^2\|f(x(t))\|^2+4\|W_2\|^2\|f(x(t-\tau))\|^2+4\|D\|^2\|\dot{x}(t-\tau)\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}-\|\dot{x}(t-\tau)\|^2 \leqslant &-\frac{1}{4\|D\|^2}\|\dot{x}(t)\|^2+\frac{\|A\|^2}{\|D\|^2}\|x(t)\|^2+\frac{\|W_1\|^2}{\|D\|^2}\|f(x(t))\|^2\\ &+\frac{\|W_2\|^2}{\|D\|^2}\|f(x(t-\tau))\|^2. \end{aligned} \quad (33)$$

Now, considering (18), it is easy to see that

$$\begin{aligned}V(x_t) \leqslant\ &\lambda_{\max}(P)\|x(t)\|^2+\lambda_{\max}(Q_1)\sigma^2\int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha\\ &+\lambda_{\max}(Q_2)\int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha+\lambda_{\max}(Q_3)\int_{-\tau}^{0}\int_{t+\beta}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta, \end{aligned} \quad (34)$$

where

$$\sigma=\max_{i=1,\ldots,n}\{\sigma_i\}.$$

Note that

$$\int_{-\tau}^{0}\int_{t+\beta}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta=\int_{t-\tau}^{t}\int_{-\tau}^{\alpha-t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\beta\,\mathrm{d}\alpha\leqslant\tau\int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha. \quad (35)$$

Then, it follows from (34) and (35) that

$$\begin{aligned}V(x_t) \leqslant\ &\lambda_{\max}(P)\|x(t)\|^2+\lambda_{\max}(Q_1)\sigma^2\int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha\\ &+(\lambda_{\max}(Q_2)+\tau\lambda_{\max}(Q_3))\int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\\ \leqslant\ &a\left(\|x(t)\|^2+\int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha+\int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\right), \end{aligned} \quad (36)$$

where

$$a=\max\{\lambda_{\max}(P),\lambda_{\max}(Q_1)\sigma^2,\lambda_{\max}(Q_2)+\tau\lambda_{\max}(Q_3)\}.$$

By using (32), (33) and (36), we have that for any scalar $\theta > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\theta t}V(x_t)) &= \mathrm{e}^{\theta t}[\theta V(x_t) + \dot{V}(x_t)]\\ &\leqslant \mathrm{e}^{\theta t}\left[\theta a\left(\|x(t)\|^2 + \int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha + \int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\right)\right.\\ &\quad - (\delta+\sqrt{\delta})\|x(t)\|^2 - \sqrt{\delta}\|f(x(t))\|^2 - \sqrt{\delta}\|f(x(t-\tau))\|^2\\ &\quad \left. - 4\delta\|D\|^2\|\dot{x}(t-\tau)\|^2\right]\\ &\leqslant \mathrm{e}^{\theta t}\left[\theta a\left(\|x(t)\|^2 + \int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha + \int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\right)\right.\\ &\quad - \delta\|x(t)\|^2 - \delta\|\dot{x}(t)\|^2 + (4\delta\|A\|^2 - \sqrt{\delta})\|x(t)\|^2 + (4\delta\|W_1\|^2 - \sqrt{\delta})\\ &\quad \left.\times\ \|f(x(t))\|^2 + (4\delta\|W_2\|^2 - \sqrt{\delta})\|f(x(t-\tau))\|^2\right].\end{aligned}$$

Using this and (13) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\theta t}V(x_t)) \leqslant \mathrm{e}^{\theta t}&\left[(\theta a - \delta)\|x(t)\|^2 + \theta a\int_{t-\tau}^{t}\|x(\alpha)\|^2\,\mathrm{d}\alpha\right.\\ &\left.+\,\theta a\int_{t-\tau}^{t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha - \delta\|\dot{x}(t)\|^2\right].\end{aligned}\tag{37}$$

Integrating both sides of (37) from 0 to $T > 0$ results in

$$\begin{aligned}\mathrm{e}^{\theta T}V(x_T) - V(x_0) \leqslant (\theta a - \delta)\int_0^T \mathrm{e}^{\theta t}\|x(t)\|^2\,\mathrm{d}t + \theta a\int_0^T\int_{t-\tau}^{t}\mathrm{e}^{\theta t}\|x(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}t\\ +\ \theta a\int_0^T\int_{t-\tau}^{t}\mathrm{e}^{\theta t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}t - \delta\int_0^T \mathrm{e}^{\theta t}\|\dot{x}(t)\|^2\,\mathrm{d}t.\end{aligned}\tag{38}$$

Observe that

$$\begin{aligned}\int_0^T\int_{t-\tau}^{t}\mathrm{e}^{\theta t}\|x(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}t &= \int_{-\tau}^{0}\int_{0}^{\alpha+\tau}\mathrm{e}^{\theta t}\|x(\alpha)\|^2\,\mathrm{d}t\,\mathrm{d}\alpha\\ &\quad + \int_0^{T-\tau}\int_{\alpha}^{\alpha+\tau}\mathrm{e}^{\theta t}\|x(\alpha)\|^2\,\mathrm{d}t\,\mathrm{d}\alpha + \int_{T-\tau}^{T}\int_{\alpha}^{T}\mathrm{e}^{\theta t}\|x(\alpha)\|^2\,\mathrm{d}t\,\mathrm{d}\alpha\\ &\leqslant \tau\int_{-\tau}^{T}\mathrm{e}^{\theta(\alpha+\tau)}\|x(\alpha)\|^2\,\mathrm{d}\alpha\\ &= \tau\mathrm{e}^{\theta\tau}\int_{-\tau}^{0}\mathrm{e}^{\theta\alpha}\|x(\alpha)\|^2\,\mathrm{d}\alpha + \tau\mathrm{e}^{\theta\tau}\int_0^T \mathrm{e}^{\theta\alpha}\|x(\alpha)\|^2\,\mathrm{d}\alpha\end{aligned}\tag{39}$$

and

$$\int_0^T\int_{t-\tau}^{t}\mathrm{e}^{\theta t}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha\,\mathrm{d}t \leqslant \tau\mathrm{e}^{\theta\tau}\int_{-\tau}^{0}\mathrm{e}^{\theta\alpha}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha + \tau\mathrm{e}^{\theta\tau}\int_0^T \mathrm{e}^{\theta\alpha}\|\dot{x}(\alpha)\|^2\,\mathrm{d}\alpha.\tag{40}$$

Then, from (38) to (40), we obtain

$$\mathrm{e}^{\theta T} V(x_T) \leqslant (\theta a - \delta + \theta a \tau \mathrm{e}^{\theta\tau}) \int_0^T \mathrm{e}^{\theta t} \|x(t)\|^2 \, \mathrm{d}t + (-\delta + \theta a \tau \mathrm{e}^{\theta\tau}) \int_0^T \mathrm{e}^{\theta t} \|\dot{x}(t)\|^2 \, \mathrm{d}t$$
$$+ \, \theta a \tau \mathrm{e}^{\theta\tau} \int_{-\tau}^0 \mathrm{e}^{\theta t} \|x(t)\|^2 \, \mathrm{d}t + \theta a \tau \mathrm{e}^{\theta\tau} \int_{-\tau}^0 \mathrm{e}^{\theta t} \|\dot{x}(t)\|^2 \, \mathrm{d}t + V(x_0). \tag{41}$$

Now, choose $\theta > 0$ satisfying

$$\theta a - \delta + \theta a \tau \mathrm{e}^{\theta\tau} = 0.$$

This together with (41) implies

$$\mathrm{e}^{\theta T} V(x_T) \leqslant \theta a \tau \mathrm{e}^{\theta\tau} \int_{-\tau}^0 \mathrm{e}^{\theta t} \|x(t)\|^2 \, \mathrm{d}t + \theta a \tau \mathrm{e}^{\theta\tau} \int_{-\tau}^0 \mathrm{e}^{\theta t} \|\dot{x}(t)\|^2 \, \mathrm{d}t + V(x_0).$$

By this and (36), we have

$$V(x_T) \leqslant 2\mathrm{e}^{-\theta T} (a + a\tau + \theta a \tau^2 \mathrm{e}^{\theta\tau}) \sup_{-\tau \leqslant \theta \leqslant 0} \{\|x(\theta)\|^2, \|\dot{x}(\theta)\|^2\}. \tag{42}$$

Using this and noting (18), we obtain

$$\|x(T)\| \leqslant \mu \mathrm{e}^{-kT} \sup_{-\tau \leqslant \theta \leqslant 0} \{\|x(\theta)\|, \|\dot{x}(\theta)\|\},$$

where

$$\mu = \sqrt{\frac{2a(1 + \tau + \theta\tau^2 \mathrm{e}^{\theta\tau})}{\lambda_{\min}(P)}}, \quad k = \frac{\theta}{2}.$$

Finally, by Definition 2 and (42), it is easy to see an equilibrium point of the delayed neural network in (7) is globally exponentially stable, which further implies that the origin of the delayed neural network in (7) is the unique equilibrium point. This completes the proof. □

Remark 1. Theorem 1 provides a sufficient condition for the global exponential stability of the delayed neural network in (7). This condition is delay-dependent, therefore, such a condition will be less conservative than delay-independent ones. It is also worth pointing out that the LMI condition in Theorem 1 can be checked efficiently numerically by recently developed algorithms in solving LMIs, and no tuning of parameters is required [4].

The maximum bound for the time delay $\bar{\tau}$ in the context of Theorem 1 can be computed by solving a quasi-convex optimization problem in P, Q_1, Q_2, Q_3, S, Y and $\bar{\tau}$, which is given as follows:

$$\begin{array}{ll} \text{maximize} & \bar{\tau} \\ \text{subject to} & \bar{\tau} > 0,\ P > 0,\ Q_1 > 0,\ Q_2 > 0,\ Q_3 > 0, \end{array}$$
$$\text{diagonal matrices } S > 0,\ Y > 0, \text{ and } (9).$$

We now provide an example to show the effectiveness of the result in Theorem 1.

Example 1. Consider a delayed neural network in (7) with parameters as

$$A=\begin{bmatrix}2.7644 & 0 & 0\\ 0 & 1.0185 & 0\\ 0 & 0 & 10.2716\end{bmatrix},\quad W_1=\begin{bmatrix}0.2651 & -3.1608 & -2.0491\\ 3.1859 & -0.1573 & -2.4687\\ 2.0368 & -1.3633 & 0.5776\end{bmatrix},$$

$$W_2=\begin{bmatrix}-0.7727 & -0.8370 & 3.8019\\ 0.1004 & 0.6677 & -2.4431\\ -0.6622 & 1.3109 & -1.8407\end{bmatrix},\quad D=\begin{bmatrix}0.2076 & 0.0631 & 0.3915\\ -0.0780 & 0.3106 & 0.1009\\ -0.2763 & 0.1416 & 0.3729\end{bmatrix}.$$

In this example, we assume

$$\sigma_1=0.1019,\quad \sigma_2=0.3419,\quad \sigma_3=0.0633.$$

By Theorem 1, it can be found that this delayed neural network is globally exponentially stable for all $0<\tau\leqslant 1.0344$. In the case when $\tau=1.0344$, we use the Matlab LMI Control Toolbox to solve the LMI in (9), and obtain the solution as follows:

$$P=\begin{bmatrix}30.1640 & 9.9581 & -23.1018\\ 9.9581 & 34.5003 & -9.7952\\ -23.1018 & -9.7952 & 89.7873\end{bmatrix},\quad Q_1=\begin{bmatrix}190.0885 & -85.9188 & 206.2056\\ -85.9188 & 217.2680 & 48.9076\\ 206.2056 & 48.9076 & 336.8813\end{bmatrix},$$

$$Q_2=\begin{bmatrix}7.8223 & -1.9300 & -0.5617\\ -1.9300 & 14.3633 & 3.7471\\ -0.5617 & 3.7471 & 4.1243\end{bmatrix},\quad Q_3=\begin{bmatrix}2.5392 & 3.4567 & -2.8379\\ 3.4567 & 10.7688 & -6.4549\\ -2.8379 & -6.4549 & 5.8970\end{bmatrix},$$

$$S=\begin{bmatrix}832.2551 & 0 & 0\\ 0 & 832.2551 & 0\\ 0 & 0 & 832.2551\end{bmatrix},\quad Y=\begin{bmatrix}22.8971 & 0 & 0\\ 0 & 22.8971 & 0\\ 0 & 0 & 22.8971\end{bmatrix}.$$

Now, consider a special case with $D=0$; that is, the delayed neural network in (7) reduces to [1]:

$$\dot{x}(t)=-Ax(t)+W_1f(x(t))+W_2f(x(t-\tau)). \tag{43}$$

Then, by Theorem 1, we have the following result.

Corollary 1. *The origin of the delayed neural network in* (7) *is the unique equilibrium point and is globally exponentially stable for any delay* $0<\tau\leqslant\bar{\tau}$ *if there exist matrices* $P>0$, $Q_1>0$, $Q_2>0$ *and two diagonal matrices* $S>0$ *and* $Y>0$ *such that the following LMI holds*:

$$\begin{bmatrix}-PA-AP+\bar{\tau}AQ_2A & (P-\bar{\tau}AQ_2)W_1+\frac{1}{2}S\Sigma & (P-\bar{\tau}AQ_2)W_2+Y & 0\\ W_1^{\mathrm{T}}(P-\bar{\tau}Q_2A)+\frac{1}{2}\Sigma S & \bar{\tau}W_1^{\mathrm{T}}Q_2W_1+Q_1-S & \bar{\tau}W_1^{\mathrm{T}}Q_2W_2 & 0\\ W_2^{\mathrm{T}}(P-\bar{\tau}Q_2A)+Y & \bar{\tau}W_2^{\mathrm{T}}Q_2W_1 & \bar{\tau}W_2^{\mathrm{T}}Q_2W_2-2Y\Sigma^{-1}-Q_1 & -\bar{\tau}Y\\ 0 & 0 & -\bar{\tau}Y & -\bar{\tau}Q_2\end{bmatrix}<0, \tag{44}$$

where Σ *is given in* (10).

To show the reduced conservatism of Corollary 1, we provide the following example.

Example 2. Consider a delayed cellular neural network in (43) with parameters

$$A = \begin{bmatrix} 1.1110 & 0 & 0 \\ 0 & 2.0592 & 0 \\ 0 & 0 & 2.9015 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -3.3497 & 0.0899 & 0.3752 \\ -3.8446 & 4.8332 & 1.8954 \\ -0.8733 & -0.6198 & -1.0514 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -3.3468 & 0 & 0 \\ 0 & -4.7770 & 0 \\ 0 & 0 & 3.5245 \end{bmatrix}.$$

We also assume

$$\sigma_1 = 0.2051, \quad \sigma_2 = 0.2342, \quad \sigma_3 = 0.1593.$$

Then, it can be verified that the delay-dependent conditions of Theorems 2 and 6 in [2] cannot be satisfied for any $\tau > 0$. Thus, they cannot provide any results on the maximum allowed delay $\bar{\tau}$. However, by Corollary 1, it is computed that the maximum allowed delay $\bar{\tau} = 1.9382$. Therefore, Corollary 1 is less conservative than the delay-dependent results in [2].

4. Conclusions

This paper has investigated the exponential stability of a class of delayed neural networks described by nonlinear delay differential equations of neutral type. In terms of an LMI, a delay-dependent condition has been proposed, which ensures the existence, uniqueness of an equilibrium point and its global exponential stability. This LMI condition can be checked easily by resorting to recently developed standard algorithms solving LMIs. It has been shown that the maximum bound for the time delay which ensures that the delayed neural network is globally exponentially stable can be obtained by solving a quasi-convex optimization problem, which involves no tuning of parameters. Examples have been provided to demonstrate the effectiveness of the proposed method.

References

[1] S. Arik, Global asymptotic stability of a larger class of neural networks with constant time delay, Phys. Lett. A 311 (2003) 504–511.
[2] S. Arik, An analysis of exponential stability of delayed neural networks with time varying delays, Neural Networks 17 (2004) 1027–1031.
[3] V.S. Borkar, K. Soumyanatha, An analog scheme for fixed point computation—Part I: theory, IEEE Trans. Circuits and Systems I 44 (1997) 351–355.
[4] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
[5] J. Cao, Q. Li, On the exponential stability and periodic solutions of delayed cellular neural networks, J. Math. Anal. Appl. 252 (2000) 50–64.
[6] J. Cao, D. Zhou, Stability analysis of delayed cellular neural networks, Neural Networks 11 (1998) 1601–1605.
[7] T.-G. Chu, An exponential convergence estimate for analog neural networks with delay, Phys. Lett. A 283 (2001) 113–118.
[8] L.O. Chua, L. Yang, Cellular neural networks: applications, IEEE Trans. Circuits and Systems 35 (1988) 1273–1290.
[9] A. Cichocki, R. Unbehauen, Neural Networks for Optimization and Signal Processing, Wiley, Chichester, 1993.
[10] M.W. Hirsch, Convergent activation dynamics in continuous time networks, Neural Networks 2 (1989) 331–349.

[11] M. Joy, On the global convergence of a class of functional differential equations with applications in neural network theory, J. Math. Anal. Appl. 232 (1999) 61–81.
[12] E. Kaszkurewicz, A. Bhaya, On a class of globally stable neural circuits, IEEE Trans. Circuits and Systems I 41 (1994) 171–174.
[13] V.B. Kolmanovskii, A.D. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 1992.
[14] Z. Liu, L. Liao, Existence and global exponential stability of periodic solution of cellular neural networks with time-varying delays, J. Math. Anal. Appl. 290 (2004) 247–262.
[15] A.N. Michel, D. Liu, Qualitative Analysis and Synthesis of Recurrent Neural Networks, Marcel Dekker, New York, 2002.
[16] P. Van Den Driessche, X. Zou, Global attractivity in delayed Hopfield neural network models, SIAM J. Appl. Math. 6 (1998) 1878–1890.
[17] S. Xu, T. Chen, J. Lam, Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time delays, IEEE Trans. Automat. Control 48 (2003) 900–907.
[18] S. Xu, J. Lam, D.W.C. Ho, Y. Zou, Global robust exponential stability analysis for interval recurrent neural networks, Phys. Lett. A 325 (2004) 124–133.
[19] J. Zhang, Global stability analysis in delayed cellular neural networks, Comput. Math. Appl. 45 (2003) 1707–1720.
[20] Q. Zhang, X. Wei, J. Xu, Global exponential convergence analysis of delayed neural networks with time-varying delays, Phys. Lett. A 38 (2003) 537–544.