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A two-dimensional warranty servicing strategy based on reduction in product failure intensity

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ABSTRACT

The cost of servicing a warranty depends, amongst other factors, on the type of repair performed under warranty. Although "all minimal repair" and "all replacement" policies are easy to implement and analyze, they are not always feasible and/or practical. Having a combination of different types of repair often leads to lower warranty servicing costs. In this article, to reduce the warranty servicing cost, we study a servicing strategy that involves performing imperfect repairs in place of some of the minimal repairs of an "all minimal repair" strategy; the effect of an imperfect repair is characterized by a drop in the conditional intensity function of the failure process. We consider both fixed and random degrees of repair. For a given type of product, we partition the warranty region so that the expected total warranty servicing cost is minimized. We provide a numerical illustration and a comparison with previously-studied repair–replacement strategies.

1. Introduction

An effective warranty servicing strategy is essential in reducing manufacturers' warranty servicing costs and increasing profits from product sales.

Repair strategies for repairable products are often categorized by the types of repair performed during the warranty period and instances when different repairs are performed. A repair is typically characterized by its "degree", which usually ranges from 0 (minimal repair) to 1 (perfect repair or replacement). A repair with a degree between 0 and 1 is an imperfect repair [1,2]. Minimal and perfect repairs are often viewed as special (extreme) cases of imperfect repair. Strategies with a single type of repair (e.g. "all minimal repair" or "all replacement" strategies) are often the simplest, but not always feasible and/or realistic options. Strategies that include a combination of various types of repair are expected to be more effective in reducing warranty servicing costs.

In this article, we study a modification of the "all minimal repair" strategy, where some of the minimal repairs are replaced by repairs of degree greater than zero, which can be random or pre-assigned. This strategy involves (optimally) partitioning the warranty region into a number of disjoint subregions and performing repairs of random degree in each subregion. This is a generalization of previously-suggested warranty servicing strategies in which the degrees of repair are pre-assigned. We also study an alternative model to the virtual age models, namely, an intensity reduction model, to model the effects of the imperfect repairs [3,4].

The outline of this article is as follows. In Section 2, we define the imperfect repair strategy and review other two-dimensional warranty servicing strategies. In Section 3, we provide formulation for the failure (imperfect repair) model and derive the distribution of the times to imperfect repair. In Section 4, we derive the expected total warranty servicing cost for the proposed strategy and provide an analysis of the model. In Section 5, we present a numerical example of the

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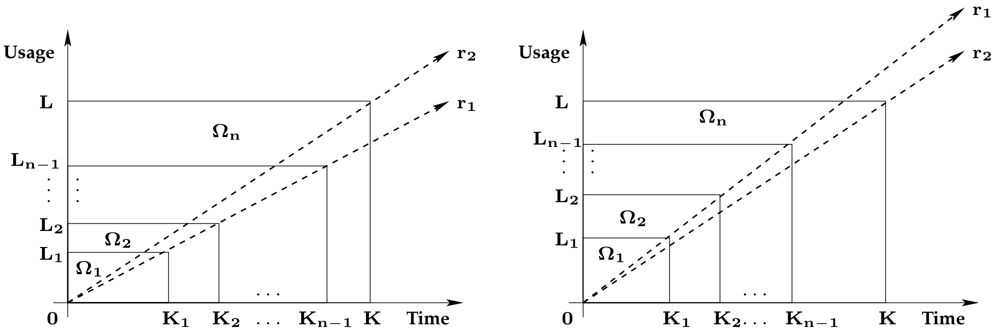


Fig. 1. Subregions of the warranty region: (left) $r_1 \leq r_2$; (right) $r_2 \leq r_1$.

proposed strategy and a comparison with the best strategy amongst previously-studied repair–replacement strategies. In Section 6, we conclude with a summary of our study and outline possible research directions.

2. The warranty servicing strategy

In the case of two-dimensional warranties, failures of the warranted product are assumed to be caused by an increase in the age and/or the usage of the product. Often the usage of the product is assumed to be a linear function of its age, thus, reducing the two-dimensional failure process to a one-dimensional one. In analyzing two-dimensional warranty strategies, using this one-dimensional approach is common practice, and we also adopt this approach for modeling failures (repairs) [1].

For a repairable product sold with a two-dimensional free-replacement warranty policy, we propose the following warranty servicing strategy.

2.1. Imperfect repair strategy

First, the rectangular warranty region $\Omega = [0, K) \times [0, L)$ is partitioned into n disjoint subregions. Here, K and L are the warranty time and usage limits (respectively), and the warranty expires when either limit is exceeded. For $i = 1, \ldots, n$, the partitioning is such that

$$\Omega_i = \{[0, K_i) \times [0, L_i)\} \setminus \{[0, K_{i-1}) \times [0, L_{i-1})\},$$

where $K_0 = L_0 = 0$, $K_n = K$ and $L_n = L$. For simplicity, the shapes of the subregions are governed by a single rate parameter r_1 ($r_1 > 0$), such that

$$\frac{L_1}{K_1} = \frac{L_2}{K_2} = \cdots = \frac{L_{n-1}}{K_{n-1}} = r_1,$$

and the warranty coverage (region) is defined by $L_n/K_n = L/K = r_2$ (see Fig. 1). In other words, the time limits defining the subregions depend on a single usage rate (parameter) and not on the individual customer usage rates.

Then, given the subregions $\Omega_1, \Omega_2, \ldots, \Omega_n$, the imperfect repair strategy is such that

(i) all repairs in the first (Ω_1) and last (Ω_n) subregions are minimal with cost $c_{\min}$;
(ii) the first repair in each of the intermediate subregions, $\Omega_2, \ldots, \Omega_{n-1}$, is imperfect with corresponding degrees $\delta_1, \ldots, \delta_{n-2}$ and cost c_{imp}, and all subsequent repairs in each of these subregions are minimal with cost $c_{\min}$.

At the start and towards the end of the warranty coverage, having only minimal repairs is reasonable, because it is unlikely for a new product to need major repair at the start of its life, and when the warranty is near its expiry, keeping the product in an operating condition (as opposed to attempting to improve its reliability) is sufficient, from the manufacturer's point of view. Away from the start and end, around the middle of the warranty coverage, imperfect repairs undo the product degradation to a degree and improve its reliability. However, having too many imperfect repairs can result in an undesirably higher servicing cost.

The optimal strategy, based on the partition (subregions) $\{\Omega_1^*, \Omega_2^*, \ldots, \Omega_n^*\}$, for a given type of product, is determined by minimizing the expected total warranty servicing cost over the warranty region Ω. This expected cost, say $E[C^{\Omega}(\psi_n)]$, is viewed as a function of the following n decision variables $\psi_n = (K_1, K_2, \ldots, K_{n-1}, r_1)$, which define the servicing strategy. Therefore, the objective is to minimize this expected cost by determining the optimal servicing strategy, i.e., optimal decision variables $\psi_n^* = (K_1^*, K_2^*, \ldots, K_{n-1}^*, r_1^*)$, such that $\psi_n^* = \arg\min_{\psi_n} E[C^{\Omega}(\psi_n)]$.

We denote this imperfect repair strategy by $\mathcal{S}_n^{\delta}$, where n is the number of subregions and $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{n-2})$ denotes the degrees of the imperfect repairs in the $n - 2$ intermediate subregions $\Omega_2, \ldots, \Omega_{n-1}$.

This strategy is designed to improve product reliability (thus reducing the number of future failures), while reducing the warranty servicing cost (when compared to repair–replacement strategies) by employing a combination of minimal and imperfect repairs. This strategy, as a generalization of repair–replacement strategies, provides more flexibility in terms of the degree of repairs within the warranty coverage.

In [5], a 3-subregion repair–replacement strategy is proposed, where all repairs are minimal except the first repair in the middle subregion Ω_2, which is perfect (replacement). In [6], the strategy in [5] is extended to an n-dimensional strategy where all repairs are minimal except the first repair in each of the intermediate subregions (Ω_2–Ω_{n-1}), which is perfect. In [7], the strategy in [5] is generalized to an imperfect repair strategy, where all repairs are minimal except the first repair in the middle subregion Ω_2 which is imperfect. In [8], these repair strategies are generalized to an n-subregion strategy, where all repairs are minimal except the first repair in each of the intermediate subregions (Ω_2–Ω_{n-1}), which is imperfect.

A number of *unrestricted* repair–replacement strategies have been proposed, where the shape of the subregions are not all governed by a single parameter; see [9–11]. In [10,11], non-rectangular subregions are considered, where optimal time limits for each customer (usage rate) generate a closed curve that defines the subregions of the two-dimensional warranty strategy. In [11], the first repair within this closed curve is perfect and all other repairs are minimal, whereas in [10] the first repair within this curve is imperfect and all other repairs are minimal. The effect of the usage rate on product degradation is modeled using the concept of accelerated failure time. For more on accelerated failure time modeling please refer to [12–14] (and the references therein). The reader is also referred to [15,16] for modeling warranty costs in two dimensions, failure time models, reliability assessment and two-dimensional warranty policies and analysis.

In this article, we generalize the strategy in [8] to account for random degrees of repair (not pre-assigned), and use an alternative framework (intensity reduction model) to model the effect of the imperfect repairs, for an arbitrary number of subregions of the warranty coverage partition.

2.1.1. *Fixed degrees of imperfect repair*

For a given type of product, the degrees of the imperfect repairs in the intermediate subregions could be preassigned (fixed) by the manufacturer, and the optimal servicing strategy is determined for a given fixed set of degrees of repair $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{n-2})$. The optimal servicing strategy can vary for different sets of degrees of repair.

2.1.2. *Random degrees of imperfect repair*

In some situations, the preassigned degrees of repair may not be adequate or appropriate to fix the faulty product. Hence, we also consider the case where the degrees of repair are random with estimated (empirical) distributions.

Two models for random degrees of repair are the (p, q) model proposed in [17] and the age-dependent $(p(t), q(t))$ model proposed in [18], where with probability p $(p(t))$ the repair is perfect (degree $\delta = 1$) and with probability $q = 1 - p$ $(q(t) = 1 - p(t))$ the repair is minimal (degree $\delta = 0$). For the case where repairs are categorized as minimal ($\delta = 0$), imperfect ($0 < \delta < 1$), and perfect ($\delta = 1$), one may consider the following probability structure: the repair performed on a failed product is minimal with probability $p_{\min}$, imperfect with probability p_{imp} and perfect with probability p_{per}. This discretization does not take into consideration the various degrees of imperfect repairs. Since the degree of repair is a continuous variable in [0, 1], having a density function seems appropriate.

Let D_i denote the degree of the i-th imperfect repair, with $\delta_i \in [0, 1]$ (this interval may or may not include the extremes 0 and 1, depending on the strategy), denoting its realization, for $i = 1, \ldots, n-2$. We assume that $D_1, \ldots, D_{n-2}$ all have the same distribution, and suggest the following two scenarios:

- The distribution of D_i, $i = 1, \ldots, n-2$, is independent of the age and usage of the product at failure, i.e., $D_1, \ldots, D_{n-2}$ are identically distributed, with probability density function $h(.)$.
- The distribution of D_i, $i = 1, \ldots, n-2$, depends on the age and usage of the product at failure, so they can be parametrized by the age t at failure and the usage rate r, i.e., the density function has the form $h(.; t, r)$.

The servicing strategy remains the same as in the case of having fixed degrees of repair, with the adjustment for random degrees of imperfect repair. Here, the optimal servicing strategy is determined for a given distribution of the degrees of repair.

3. Modeling imperfect repairs

We model the failure process using the one-dimensional approach to modeling failures in two dimensions. Here, the usage of the product is modeled as a linear function of its age, such that $U(t) = RA(t)$, where $U(t)$ and $A(t)$ are the usage and age of the product at time t and the usage rate R is a random variable with known distribution G (see [1] for details).

To model the effect of the imperfect repairs on the failure intensity of the process, we use a modification of the "intensity reduction" model proposed in [4], where the effect of an imperfect repair is characterized by a reduction in the conditional intensity function of the underlying failure process. At any time, the conditional intensity function of the process after an imperfect repair is between the conditional intensities after a minimal repair and a perfect repair. Also, we assume that: (1) a failure results in an immediate repair; (2) repairs are instantaneous.

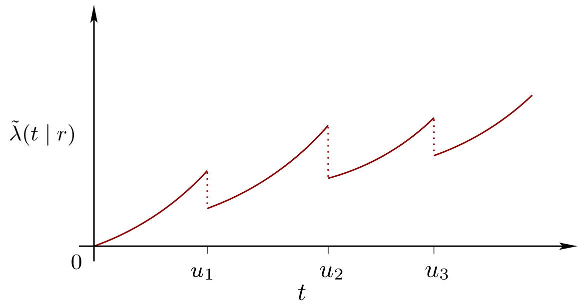


Fig. 2. Conditional intensity function following imperfect repairs.

3.1. The effect of imperfect repairs on the conditional intensity function

Let $\{\tilde{N}(t \mid r); t \geq 0\}$ denote the one-dimensional counting process conditional on $R = r$, and let $\tilde{\lambda}(t \mid r)$ denote the conditional intensity function of the process. We assume that, the effect of an imperfect repair is characterized by a drop in the conditional intensity function of the process. This model is such that, given the times and degrees of all previous imperfect repairs, after the i-th imperfect repair of degree δ_i at time u_i, the conditional intensity function becomes

$$\lambda_i(t \mid r) = \lambda_{i-1}(t \mid r) - \delta_i[\lambda_{i-1}(t \mid r) - \lambda(t - u_i \mid r)], \tag{1}$$

for $u_i < t \leq u_{i+1}$, where $\lambda(.|r)$ is the initial intensity function of the process. Here, the reduction in the intensity function is proportional to the degree of the imperfect repair. If the repair is minimal ($\delta_i = 0$), then

$$\lambda_i(t \mid r) = \lambda_{i-1}(t \mid r),$$

where $\lambda_{i-1}(t \mid r)$ is the intensity function after the $(i-1)$-th and before the i-th imperfect repair. If the repair is perfect ($\delta_i = 1$), then

$$\lambda_i(t \mid r) = \lambda(t - u_i \mid r),$$

which is the initial intensity function of the process at time $t - u_i$.

Therefore, conditional on $R = r$ and given the times to imperfect repair $u_1, u_2, \ldots, u_{n-2}$, and the degrees of these repairs $\delta_1, \delta_2, \ldots, \delta_{n-2}$, the conditional intensity function of the process $\{\tilde{N}(t \mid r); t \geq 0\}$ is given by

$$\tilde{\lambda}(t \mid r) = \begin{cases} \lambda(t \mid r), & 0 \leq t \leq u_1 \\ \lambda_i(t \mid r), & u_i < t \leq u_{i+1}, \quad \text{for } 1 \leq i \leq n-3. \\ \lambda_{n-2}(t \mid r), & u_{n-2} < t < \infty. \end{cases}$$

Fig. 2 depicts a possible shape of $\tilde{\lambda}(t \mid r)$.

The expected number of minimal repairs in any interval $(x, t]$ between two imperfect repairs at times u_i and u_{i+1}, i.e., for $u_i \leq x < t \leq u_{i+1}$, is given by

$$E[\tilde{N}(t \mid r) - \tilde{N}(x \mid r)] = \int_x^t \tilde{\lambda}(s \mid r)\, ds = \int_x^t \lambda_i(s \mid r)\, ds, \tag{2}$$

where $\lambda_i(s \mid r)$ is a function of the imperfect repair times $u_1, \ldots, u_i$ and their degrees of repair $\delta_1, \ldots, \delta_i$. Let $\Lambda_i(t \mid r)$ denote the expected number of failures in $(u_i, t]$, so that

$$\int_x^t \lambda_i(s \mid r)\, ds = \int_{u_i}^t \lambda_i(s \mid r)\, ds - \int_{u_i}^x \lambda_i(s \mid r)\, ds = \Lambda_i(t \mid r) - \Lambda_i(x \mid r), \tag{3}$$

denotes the expected number of minimal repairs in $(x, t]$. This expected number is conditional on the times to imperfect repair $u_1, u_2, \ldots, u_{n-2}$, and when the degrees of repair are random, it is also conditional on the degrees of the imperfect repairs $\delta_1, \delta_2, \ldots, \delta_{n-2}$.

3.2. The distribution of times to imperfect repair

The first failure in each of the intermediate subregions is followed by an imperfect repair. Therefore, the time to imperfect repair in a subregion is the time to first failure in that subregion.

For $l = 1, \ldots, n$, the interval $(K_{l-1}, K_l]$ corresponds to the l-th subregion Ω_l, and we will use the interval and subregion notations interchangeably. Let, conditional on $R = r$, $T_{K_l|r}$, $l = 1, 2, \ldots, n-2$, denote the time to first failure after K_l, in subregion Ω_{l+1}. If there has been at least one failure in each of the subregions $\Omega_2, \ldots, \Omega_l$, then $T_{K_l|r}$ is the time of the l-th imperfect repair. To derive the distribution of $T_{K_l|r}$, we must consider all imperfect repairs in previous subregions (before K_l), since they affect the conditional intensity function.

Let i denote the number of previous intermediate subregions in which at least one failure has occurred. When in Ω_{l+1}, we have $l-1$ previous intermediate subregions which are $\Omega_2, \ldots, \Omega_l$, and therefore, $i \in \{0, 1, \ldots, l-1\}$. Since the first repair in each of these subregions is imperfect, i is also the number of previous imperfect repairs performed.

When $i = 0$, the first failure after K_l is actually the first failure after K_1, and hence, the first imperfect repair since the start of the warranty. In other words, no failures have occurred in subregions $\Omega_2, \ldots, \Omega_l$, i.e., no failures in the interval $(K_1, K_l]$. This event can occur in $\binom{l-1}{0}$ ways.

When $0 < i \le l-1$, the previous imperfect repairs can be in any i of the $l-1$ previous intermediate subregions $\Omega_2, \ldots, \Omega_l$; this event occurs in $\binom{l-1}{i}$ ways. To generate the i corresponding subregions (subintervals), we define the following set:

$$J_{i,l} = \{\{j_1, \ldots, j_i\} : \{j_1, \ldots, j_i\} \subseteq \{2, \ldots, l\} \text{ and } j_1 < \cdots < j_i\}, \tag{4}$$

whose elements are the set of indexes that generate all possible combinations of size i, i.e., $(K_{j_1-1}, K_{j_1}], \ldots, (K_{j_i-1}, K_{j_i}]$, of the $l-1$ previous intermediate intervals $(K_1, K_2], \ldots, (K_{l-1}, K_l]$, in which the i previous imperfect repairs have been performed.

Now, by conditioning on the times to imperfect repair performed in the i subintervals, and then removing the conditioning, we account for all possible times to imperfect repair within the corresponding subintervals.

Hence, for given degrees of repair $\delta_1, \ldots, \delta_{l-1}$, the density function of $T_{K_l|r}$ is given by

$$\begin{aligned} f_{T_{K_l|r}}(t) &= f_{T_{K_1|r}}(t) + \sum_{i=1}^{l-1} \sum_{\forall\{j_1, j_2, \ldots, j_i\} \in J_{i,l}} \int_{K_{j_i-1}}^{K_{j_i}} \cdots \int_{K_{j_2-1}}^{K_{j_2}} \int_{K_{j_1-1}}^{K_{j_1}} \{\lambda_i(t \mid r) e^{-\{\Lambda_i(t|r) - \Lambda_i(K_{j_i}|r)\}} \\ &\quad \times \lambda_{i-1}(u_i \mid r) e^{-\{\Lambda_{i-1}(u_i|r) - \Lambda_{i-1}(K_{j_{i-1}}|r)\}} \\ &\quad \vdots \\ &\quad \times \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r) - \Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1)\}\, du_1\, du_2 \ldots du_i, \end{aligned} \tag{5}$$

where $\lambda_i(t \mid r)$ is the conditional intensity function following the i-th imperfect repair, and $e^{-\{\Lambda_i(t|r) - \Lambda_i(K_{j_i}|r)\}}$ is the probability that no failures have occurred in the interval $(K_{j_i}, t]$. In addition,

$$f_{T_{K_1|r}}(t) = \lambda(t \mid r) e^{-[\Lambda(t|r) - \Lambda(K_1|r)]}, \tag{6}$$

is the density component corresponding to $i = 0$, which is the density function of the time to the first imperfect repair [9, 5]. Each element of the form

$$\lambda_{i-1}(u_i \mid r) e^{-\{\Lambda_{i-1}(u_i|r) - \Lambda_{i-1}(K_{j_{i-1}}|r)\}}$$

corresponds to the event that {the imperfect repair after $K_{j_{i-1}}$ is at time $u_i \in (K_{j_i-1}, K_{j_i}]$, and no failures have occurred in the interval $(K_{j_{i-1}}, u_i)$}. The density function for $T_{K_l|r}$, $l = 1, \ldots, n-2$, has $\sum_{i=0}^{l-1} \binom{l-1}{i}$ summands.

When $l = 1$, the density function for the time to first imperfect repair, $T_{K_1|r}$, is given by Eq. (6). When $l = 2$, the density function for $T_{K_2|r}$ is given by

$$f_{T_{K_2|r}}(t) = f_{T_{K_1|r}}(t) + \int_{K_1}^{K_2} \lambda_1(t \mid r) e^{-\{\Lambda_1(t|r) - \Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1)\, du_1,$$

where the summands correspond to the cases where no imperfect repairs have been performed before K_2 ($i = 0$) and one imperfect repair has been performed before K_2 ($i = 1$) at time u_1, respectively. When $l = 3$, we have

$$\begin{aligned} f_{T_{K_3|r}}(t) &= f_{T_{K_1|r}}(t) + \int_{K_1}^{K_2} \lambda_1(t \mid r) e^{-\{\Lambda_1(t|r) - \Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1) du_1 + \int_{K_2}^{K_3} \lambda_1(t \mid r) e^{-\{\Lambda_1(t|r) - \Lambda_1(K_3|r)\}} f_{T_{K_1|r}}(u_1) du_1 \\ &\quad + \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left\{ \lambda_2(t \mid r) e^{-\{\Lambda_2(t|r) - \Lambda_2(K_3|r)\}} \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r) - \Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1\, du_2, \end{aligned}$$

where the first summand corresponds to $i = 0$ previous imperfect repairs, the two summands with the single integral correspond to $i = 1$ previous imperfect repair, performed either in $(K_1, K_2]$ or in $(K_2, K_3]$, and the last summand with the double integral corresponds to $i = 2$ previous imperfect repairs, one in each of the subintervals $(K_1, K_2]$ and $(K_2, K_3]$; and so on.

When the degrees of the imperfect repairs are random, we remove the conditioning on $D_i = \delta_i$, before removing the conditioning on u_i, using the density function $h(\delta_i; u_i, r)$ over the interval $(0, 1)$, for $i = 1, \ldots, l-1$.

Next, given the density function derived in (5), we derive the expected warranty servicing cost.

4. Analysis of the warranty servicing strategy

The expected total warranty servicing cost over the warranty region Ω is denoted by $E[C^{\Omega}(\psi_n)]$, where $\psi_n = (K_1, \ldots, K_{n-1}, r_1)$. We consider two cases: (A) $r_1 \le r_2$ with expected cost denoted by $E[C_A^{\Omega}(\psi_n)]$ and (B) $r_2 \le r_1$ with expected cost denoted by $E[C_B^{\Omega}(\psi_n)]$; see Fig. 1.

4.1. Case A: $r_1 \le r_2$

We divide the warranty region based on r_1 and r_2 and, conditional on the usage rate $R = r$, derive the expected warranty servicing costs for the following subcases:

$$(1)\ r \le r_1 \le r_2, \qquad (2)\ r_1 \le r \le r_2, \quad \text{and} \quad (3)\ r_1 \le r_2 \le r.$$

We denote the expected warranty servicing costs, conditional on $R = r$, for the three subcases by $E[C_r^{(1)}(\psi_n)]$, $E[C_r^{(2)}(\psi_n)]$, and $E[C_r^{(3)}(\psi_n)]$ respectively. Then the expected total warranty servicing cost for case A is given by

$$E[C_A^{\Omega}(\psi_n)] = \int_0^{r_1} E[C_r^{(1)}(\psi_n)]dG(r) + \int_{r_1}^{r_2} E[C_r^{(2)}(\psi_n)]dG(r) + \int_{r_2}^{\infty} E[C_r^{(3)}(\psi_n)]dG(r). \tag{7}$$

Each of the expected costs $E[C_r^{(j)}(\psi_n)],\ j = 1, 2, 3$, is the sum of the corresponding expected costs in the n subregions.

4.1.1. Subcase $r \le r_1 \le r_2$

We denote the cost of an imperfect repair by c_{imp} and the cost of a minimal repair by c_{min}. These costs can have any form. For simplicity, we assumed that these costs are constant. Later, we will adjust them to account for the degrees of repair.

The expected cost for a subregion is computed if there has been at least one failure in that subregion, otherwise, the cost is zero.

In the first subregion Ω_1, since there have been no previous imperfect repairs, the expected cost is

$$E[C_r^{\Omega_1}(\psi_n)] = c_{\text{min}} \Lambda(K_1 \mid r), \tag{8}$$

where $\Lambda(K_1 \mid r)$ is the expected number of minimal repairs in $(0, K_1]$.

When deriving the expected warranty servicing cost in all remaining subregions $\Omega_2, \ldots, \Omega_n$, we must consider all imperfect repairs performed in these and preceding subregions. Let $u_1, u_2, \ldots, u_{n-2}$ denote the realizations of the times of these imperfect repairs. Conditional on these times, we determine the expected warranty servicing cost in a subregion, and then using the components of the density functions derived earlier, we uncondition to derive the expected costs in the subregions.

In the $n-2$ intermediate subregions $\Omega_l,\ l = 2, \ldots, n-1$, the first repair is imperfect and all other repairs are minimal, thus making the conditional expected costs similar in pattern with the only distinction being the number of possible previous imperfect repairs. This conditional expected cost is given by

$$c_{\text{imp}} + c_{\text{min}} \Lambda_{i+1}(K_l \mid r),$$

where i denotes the number of previous imperfect repairs, and $\Lambda_{i+1}(K_l \mid r)$ is the expected number of minimal repairs in the subinterval $(u_{i+1}, K_l]$, and $i \in \{0, 1, \ldots, l-2\}$.

To derive the expected cost in Ω_l, we remove the conditioning on the times to imperfect repair $u_1, \ldots, u_{l-1}$, by using the set $J_{i,l-1},\ i = 1, \ldots, l-2$, defined in (4) and the corresponding density functions. Then, the expected cost becomes

$$\begin{aligned} E[C_r^{\Omega_l}(\psi_n)] = {} & \int_{K_{l-1}}^{K_l} [c_{\text{imp}} + c_{\text{min}} \Lambda_1(K_l \mid r)] f_{T_{K_1|r}}(u_1)\, du_1 + \int_{K_{l-1}}^{K_l} \left(\sum_{\forall \{j_1\} \in J_{1,l-1}} \int_{K_{j_1-1}}^{K_{j_1}} \Big\{ [c_{\text{imp}} + c_{\text{min}} \Lambda_2(K_l \mid r)] \right. \\ & \left. \times \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r) - \Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1) \Big\}\, du_1 \right) du_2 \\ & + \int_{K_{l-1}}^{K_l} \left(\sum_{\forall \{j_1, j_2\} \in J_{2,l-1}} \int_{K_{j_2-1}}^{K_{j_2}} \int_{K_{j_1-1}}^{K_{j_1}} \Big\{ [c_{\text{imp}} + c_{\text{min}} \Lambda_3(K_l \mid r)] \lambda_2(u_3 \mid r) e^{-\{\Lambda_2(u_3|r) - \Lambda_2(K_{j_2}|r)\}} \right. \\ & \left. \times \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r) - \Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1) \Big\}\, du_1\, du_2 \right) du_3 + \cdots \\ & + \int_{K_{l-1}}^{K_l} \int_{K_{l-2}}^{K_{l-1}} \cdots \int_{K_2}^{K_3} \int_{K_1}^{K_2} \Big\{ [c_{\text{imp}} + c_{\text{min}} \Lambda_{l-1}(K_l \mid r)] \end{aligned}$$

$$\times \lambda_{l-2}(u_{l-1} \mid r) e^{-\{\Lambda_{l-2}(u_{l-1}|r)-\Lambda_{l-2}(K_{l-1}|r)\}} \lambda_{l-3}(u_{l-2} \mid r) e^{-\{\Lambda_{l-3}(u_{l-2}|r)-\Lambda_{l-3}(K_{l-2}|r)\}}$$

$$\vdots$$

$$\left. \times \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r)-\Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1\, du_2 \ldots du_{l-2}\, du_{l-1}. \tag{9}$$

The first summand corresponds to the case where the number of previous imperfect repairs is $i = 0$, making the imperfect repair in $(K_{l-1}, K_l]$, the first imperfect repair. The second summand corresponds to the case $i = 1$, making the imperfect repair in $(K_{l-1}, K_l]$ the second imperfect repair. The sum is over the elements of $J_{1,l-1}$, each indicating the subregion (subinterval) in which the previous imperfect repair has been performed. The third summand corresponds to the case $i = 2$, making the imperfect repair in $(K_{l-1}, K_l]$ the third imperfect repair. Here, the sum is over the elements of $J_{2,l-1}$, each indicating the two subregions (subintervals) in which the two previous imperfect repairs have been performed, and so on. The last summand is for the case $i = l - 2$, which implies that there has been at least one failure in each of the previous subregions $\Omega_2, \ldots, \Omega_{l-1}$, and hence $l - 2$ previous imperfect repairs. This makes the imperfect repair in Ω_l the $(l-1)$-th.

Now, we have the expected cost for the $n - 2$ intermediate subregions. The pattern in this cost becomes clear when we look at specific subregions. For instance, the cost in Ω_2 is given by

$$E[C_r^{\Omega_2}(\psi_n)] = \int_{K_1}^{K_2} \left[c_{\mathrm{imp}} + c_{\min} \Lambda_1(K_2 \mid r)\right] f_{T_{K_1|r}}(u_1)\, du_1, \tag{10}$$

where $\Lambda_1(K_2 \mid r)$ is the number of minimal repairs in $(u_1, K_2]$. The cost in Ω_3 is given by

$$\begin{aligned} E[C_r^{\Omega_3}(\psi_n)] = {} & \int_{K_2}^{K_3} \left[c_{\mathrm{imp}} + c_{\min} \Lambda_1(K_3 \mid r)\right] f_{T_{K_1|r}}(u_1)\, du_1 \\ & + \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left\{ \left[c_{\mathrm{imp}} + c_{\min} \Lambda_2(K_3 \mid r)\right] \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r)-\Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1\, du_2 \end{aligned} \tag{11}$$

where $\Lambda_1(K_3 \mid r)$ and $\Lambda_2(K_3 \mid r)$ are the expected number of minimal repairs in the subintervals $(u_1, K_3]$ and $(u_2, K_3]$ respectively.

In the last subregion Ω_n, although there is no imperfect repair, the cost depends on the previous imperfect repairs. Therefore, the conditional expected warranty servicing cost in the last subregion Ω_n, for a given number of previous imperfect repairs, i, is

$$\begin{cases} c_{\min}\left[\Lambda(K \mid r) - \Lambda(K_{n-1} \mid r)\right], & i = 0 \\ c_{\min}\left[\Lambda_i(K \mid r) - \Lambda_i(K_{n-1} \mid r)\right], & 0 < i \le n-2 \end{cases}$$

when at least one failure has occurred in Ω_n. To derive the expected warranty servicing cost in the last subregion Ω_n, we remove the conditioning on $u_1, \ldots, u_{n-2}$ as follows

$$\begin{aligned} E[C_r^{\Omega_n}(\psi_n)] = {} & c_{\min}[\Lambda(K \mid r) - \Lambda(K_{n-1} \mid r)] e^{-\{\Lambda(K_{n-1}|r)-\Lambda(K_1|r)\}} \\ & + \sum_{\forall\{j_1\} \in J_{1,n-1}} \int_{K_{j_1-1}}^{K_{j_1}} \left\{ c_{\min}[\Lambda_1(K \mid r) - \Lambda_1(K_{n-1} \mid r)] e^{-\{\Lambda_1(K_{n-1}|r)-\Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1 \\ & + \sum_{\forall\{j_1, j_2\} \in J_{2,n-1}} \int_{K_{j_2-1}}^{K_{j_2}} \int_{K_{j_1-1}}^{K_{j_1}} \left\{ c_{\min}\{\Lambda_2(K \mid r) - \Lambda_2(K_{n-1} \mid r)\} \right. \\ & \left. \times e^{-\{\Lambda_2(K_{n-1}|r)-\Lambda_2(K_{j_2}|r)\}} \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r)-\Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1\, du_2 + \cdots \\ & + \int_{K_{n-2}}^{K_{n-1}} \int_{K_{n-3}}^{K_{n-2}} \cdots \int_{K_2}^{K_3} \int_{K_1}^{K_2} \left\{ c_{\min}\{\Lambda_{n-2}(K \mid r) - \Lambda_{n-2}(K_{n-1} \mid r)\} \right. \\ & \times \lambda_{n-3}(u_{n-2} \mid r) e^{-\{\Lambda_{n-3}(u_{n-2}|r)-\Lambda_{n-3}(K_{n-2}|r)\}} \lambda_{n-4}(u_{n-3} \mid r) e^{-\{\Lambda_{n-4}(u_{n-3}|r)-\Lambda_{n-4}(K_{n-3}|r)\}} \\ & \vdots \\ & \left. \times \lambda_1(u_2 \mid r) e^{-\{\Lambda_1(u_2|r)-\Lambda_1(K_2|r)\}} f_{T_{K_1|r}}(u_1) \right\} du_1\, du_2 \ldots du_{n-3}\, du_{n-2}, \end{aligned} \tag{12}$$

where, for $i = 1, 2, \ldots, n-2$, the probability $e^{-\{\Lambda_i(K_{n-1}|r)-\Lambda_i(K_{j_i}|r)\}}$ reflects the event that the last imperfect repair was performed in the subinterval $(K_{j_i-1}, K_{j_i}]$, i.e., no failures have occurred in the subregions between Ω_{j_i} and Ω_n.

Having derived the costs in the n subregions, we now derive the expected cost for subcase (1) of case A by adding the expected costs in the n subregions. Conditional on $R = r$, this cost is given by

$$\begin{aligned}
E[C_r^{(1)}(\psi_n)] &= E[C_r^{\Omega_1}(\psi_n)] + E[C_r^{\Omega_2}(\psi_n)] + \cdots + E[C_r^{\Omega_n}(\psi_n)] \\
&= c_{\min}\Lambda(K_1 \mid r) + c_{\min}[\Lambda(K \mid r) - \Lambda(K_{n-1} \mid r)]e^{-\{\Lambda(K_{n-1}|r)-\Lambda(K_1|r)\}} \\
&\quad + \sum_{l=2}^{n-1}\int_{K_{l-1}}^{K_l}\left\{\left[c_{\text{imp}} + c_{\min}\Lambda_1(K_l \mid r) + c_{\min}\{\Lambda_1(K \mid r) - \Lambda_1(K_{n-1} \mid r)\}\right.\right. \\
&\quad \left.\left.\times e^{-\{\Lambda_1(K_{n-1}|r)-\Lambda_1(K_l|r)\}}\right] f_{T_{K_1|r}}(u_1)\right\} du_1 \\
&\quad + \sum_{l=3}^{n-1}\sum_{i=1}^{l-2}\int_{K_{l-1}}^{K_l}\left(\sum_{\forall\{j_1,j_2,\ldots,j_i\}\in J_{i,l-1}}\int_{K_{j_i-1}}^{K_{j_i}}\cdots\int_{K_{j_2-1}}^{K_{j_2}}\int_{K_{j_1-1}}^{K_{j_1}}\{[c_{\text{imp}} + c_{\min}\Lambda_{i+1}(K_l \mid r)\right. \\
&\quad + c_{\min}\{\Lambda_{i+1}(K \mid r) - \Lambda_{i+1}(K_{n-1} \mid r)\}e^{-\{\Lambda_{i+1}(K_{n-1}|r)-\Lambda_{i+1}(K_l|r)\}}] \\
&\quad \times \lambda_i(u_{i+1} \mid r)e^{-\{\Lambda_i(u_{i+1}|r)-\Lambda_i(K_{j_i}|r)\}}\lambda_{i-1}(u_i \mid r)e^{-\{\Lambda_{i-1}(u_i|r)-\Lambda_{i-1}(K_{j_{i-1}}|r)\}} \\
&\quad \vdots \\
&\quad \left.\times \lambda_1(u_2 \mid r)e^{-\{\Lambda_1(u_2|r)-\Lambda_1(K_{j_1}|r)\}} f_{T_{K_1|r}}(u_1)\}\, du_1\, du_2 \ldots du_i\right) du_{i+1}.
\end{aligned} \tag{13}$$

When the degrees of repair are random, we remove the conditioning on δ_i before removing the conditioning on u_i, $i = 1, \ldots, n-2$.

The expected cost in this equation can be viewed as a function of the decision variables $K_1, K_2, \ldots, K_{n-1}$ and the warranty time limit K. Let

$$E[C_r^{(1)}(\psi_n)] \stackrel{\text{def}}{=} \xi(K_1, K_2, \ldots, K_{n-1}, K). \tag{14}$$

Then, the costs for subcases (2) and (3) of case A, and for case B (see Fig. 3) can be obtained as follows.

4.1.2. Subcase $r_1 \le r \le r_2$

For subcase (2) of case A, the warranty over the subregions $\Omega_1, \Omega_2, \ldots, \Omega_{n-1}$ will expire at time points

$$\tau_1 = \frac{L_1}{r}, \qquad \tau_2 = \frac{L_2}{r}, \ldots, \qquad \tau_{n-1} = \frac{L_{n-1}}{r},$$

due to exceeding the usage limits $L_1, L_2, \ldots, L_{n-1}$, respectively. Hence, the expected warranty cost for subcase (2) of case A becomes

$$E[C_r^{(2)}(\psi_n)] = \xi(\tau_1, \tau_2, \ldots, \tau_{n-1}, K),$$

where the time limits $K_1, \ldots, K_{n-1}$ in Eq. (13) have been replaced by $\tau_1, \ldots, \tau_{n-1}$, respectively; see Fig. 3: A-(2).

4.1.3. Subcase $r_1 \le r_2 \le r$

Similarly, the expected cost for subcase (3) of case A is given by

$$E[C_r^{(3)}(\psi_n)] = \xi(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau),$$

since the warranty over the entire region Ω expires at time $\tau = L/r$, instead of at K; see Fig. 3: A-(3).

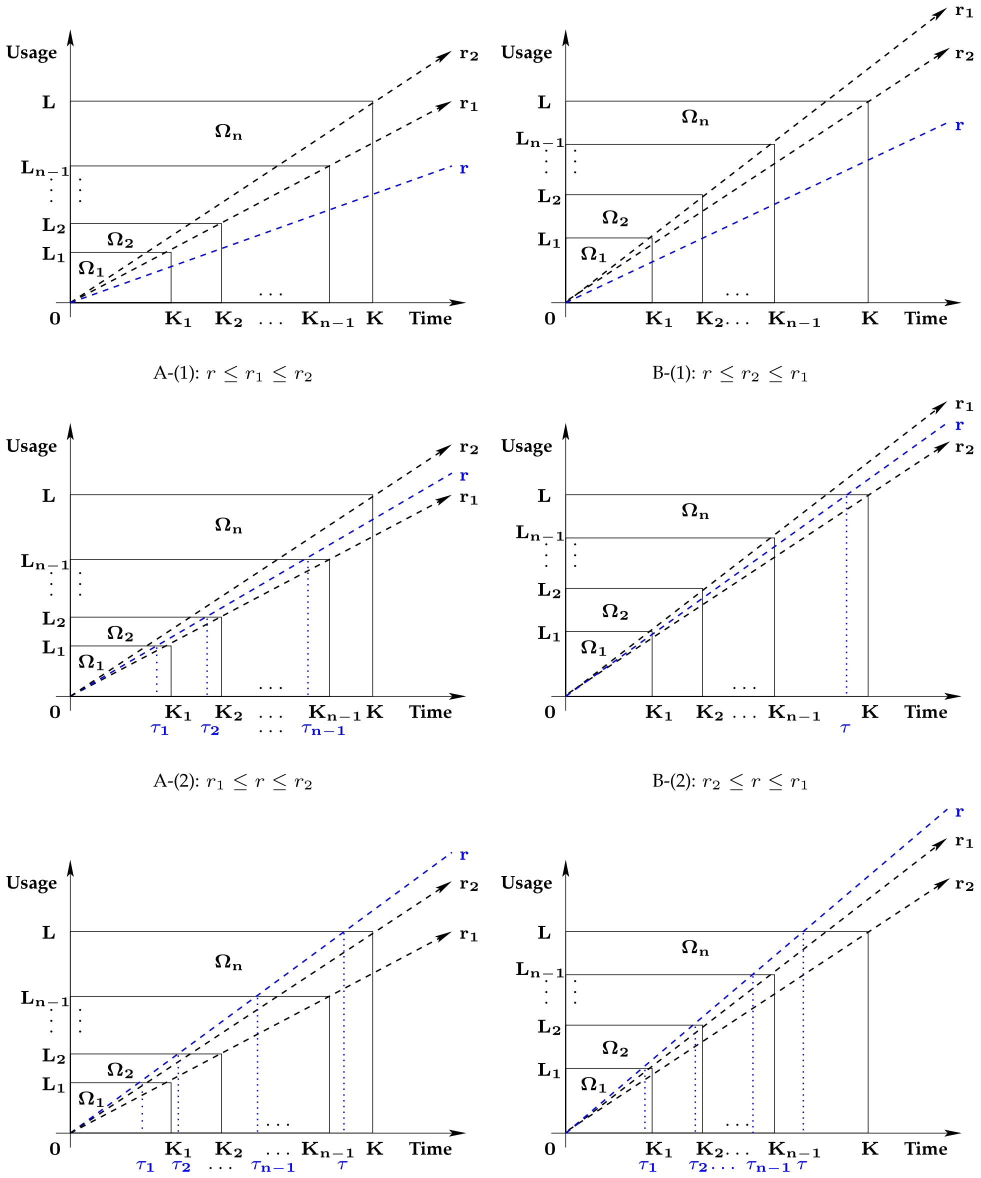
Next, we remove the conditioning on $R = r$, where R has distribution function $G(r)$, to get the expected total warranty servicing cost for case A, given in Eq. (7).

4.2. Case B: $r_2 \le r_1$

For case B, as for case A, we condition on the usage rate $R = r$ and derive the expected warranty servicing costs for the following subcases:

(1) $r \le r_2 \le r_1$, (2) $r_2 \le r \le r_1$, and (3) $r_2 \le r_1 \le r$.

Let $E[C_r^{(1)}(\psi_n)]$, $E[C_r^{(2)}(\psi_n)]$, and $E[C_r^{(3)}(\psi_n)]$ denote the expected warranty servicing costs, conditional on $R = r$, for the three subcases of case B respectively [6].



A-(1): $r \le r_1 \le r_2$ B-(1): $r \le r_2 \le r_1$

A-(2): $r_1 \le r \le r_2$ B-(2): $r_2 \le r \le r_1$

A-(3): $r_1 \le r_2 \le r$ B-(3): $r_2 \le r_1 \le r$

Fig. 3. Warranty time limits for the subcases: (left column) subcases of case A: $r_1 \le r_2$; (right column) subcases of case B: $r_2 \le r_1$.

Using (13), the expected warranty servicing costs for the subcases of case B become

$$E[C_r^{(1)}(\psi_n)] = \xi(K_1, K_2, \ldots, K_{n-1}, K),$$
$$E[C_r^{(2)}(\psi_n)] = \xi(K_1, K_2, \ldots, K_{n-1}, \tau),$$
$$E[C_r^{(3)}(\psi_n)] = \xi(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau).$$

Note that subcases (1) and (3) for both cases A and B are the same, and the only difference is in the expected costs for subcase (2) of both cases; see Fig. 3.

The expected total warranty servicing cost for case B is computed by unconditioning $R = r$:

$$E[C_B^{\Omega}(\psi_n)] = \int_0^{r_2} E[C_r^{(1)}(\psi_n)]\, dG(r) + \int_{r_2}^{r_1} E[C_r^{(2)}(\psi_n)]\, dG(r) + \int_{r_1}^{\infty} E[C_r^{(3)}(\psi_n)]\, dG(r). \tag{15}$$

When $\delta_1 = \delta_2 = \cdots = \delta_{n-2} = 1$, the expected cost derived here, reduces to the one derived in [6].

In [8], the chosen imperfect repair model is the virtual age model, where the effect of an imperfect repair is characterized by a reduction in the virtual age of the product while the intensity function remains untouched. Here, the effect of an imperfect repair is modeled by a drop in the conditional intensity function. Note that, when $\delta = 1$, the expected costs derived here are equal to the expected costs derived using the age reduction model used in [8]. When the number of imperfect repairs is small (say 1 or 2), the age reduction and the intensity reduction models result in similar costs. However, when the number of imperfect repairs increases, the differences in the expected costs are more pronounced. Hence, choosing the appropriate imperfect repair model is important.

4.3. Notes on the imperfect repair strategy

In deriving the expected costs, for simplicity and generalization purposes, we denoted the cost of an imperfect repair by c_{imp}. For a particular product, this cost can be replaced by an appropriate cost function. For instance, the cost of an imperfect repair could be modeled as a function of the time of repair and/or the degree of repair. One such cost function, suggested in [7], is

$$c_{\text{imp}} = c_{\min} + (c_{\text{per}} - c_{\min})(\rho\delta + (1-\rho)\delta^2),$$

where $c_{\min}$ and c_{per} are the costs of minimal and perfect repairs, ρ $(0 \le \rho \le 1)$ is a cost parameter, and δ $(0 \le \delta \le 1)$ is the degree of the imperfect repair. Another option, adapted from [19], is

$$c_{\text{imp}} = c_{\min} + (c_{\text{per}} - c_{\min})\delta\frac{u}{K},$$

where $u \in (0, K]$ is the time of the imperfect repair, δ is the degree of the repair, and K is the Warranty time limit.

Other forms of cost function can be defined, once the product and the estimated cost of its repair are known. The cost function can also be pro-rated to account for the amount of time the product has been in service.

5. An example

To numerically illustrate the imperfect repair strategy proposed in this article, we use the example from [5]. To make a fair comparison between the perfect repair (replacement) and imperfect repair options, the example used here is identical to the one used in [5,6].

As the number of subregions increases, the numerical optimization procedure (grid search) to find the minimum expected cost (and hence, the optimal strategy) becomes computationally intensive, and therefore, we illustrate the procedure for $n = 3$ and $n = 4$ subregions only. The results are then compared to previously reported 3-subregion and 4-subregion repair–replacement strategies applied to the same example.

5.1. Numerical example

As in [5], we consider an automobile component covered by a free-replacement warranty policy with time and usage limits $K = 2$ (2 years) and $L = 2$ (20 000 km), respectively, so that the rate parameter $r_2 = L/K = 1$.

For the 3-subregion strategy, the interval $(0, K]$ is divided into $(0, K_1]$, $(K_1, K_2]$ and $(K_2, K]$, and the first repair in $(K_1, K_2]$ is imperfect with degree δ_1.

For the 4-subregion strategy, the interval $(0, K]$ is divided into $(0, K_1]$, $(K_1, K_2]$, $(K_2, K_3]$ and $(K_3, K]$; the first repair in $(K_1, K_2]$ is imperfect with degree δ_1 and the first repair in $(K_2, K_3]$ is imperfect with degree δ_2. For simplicity (and comparison purposes), we take $\delta_1 = \delta_2 = \delta$, i.e., we take the degrees of the imperfect repairs in both intermediate subregions to be equal. Thus, when $\delta = 1$, this strategy becomes comparable to those in [5,6].

Cost of repair. Minimal repair, imperfect repair and perfect repair (replacement) costs are denoted by $c_{\min}$, c_{imp} and c_{per}, respectively, and it is assumed that $c_{\min} < c_{\text{imp}} < c_{\text{per}}$. The costs $c_{\min}$ and c_{per} are assumed to be constant, and c_{imp} is assumed to be proportional to the degree of the imperfect repair. The cost of a perfect repair is set to 1, so that

$$\frac{c_{\min}}{c_{\text{per}}} = c_{\min} \stackrel{\text{def}}{=} \mu \quad \text{and} \quad \frac{c_{\text{imp}}}{c_{\text{per}}} = c_{\text{imp}} \stackrel{\text{def}}{=} \delta.$$

This makes the degree of repair δ directly related to the cost of the corresponding imperfect repair.

Initial intensity function and usage categories. In this example, the initial intensity function of the failure process, conditional on $R = r$, is taken to be

$$\begin{aligned}\lambda(t \mid r) &= \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t^2 \\ &= 0.1 + 0.2r + (0.7 + 0.7r)t^2,\end{aligned}$$

and for unconditioning the consumer usage rate $R = r$, the following distributions are considered:

$$\begin{array}{lll} \text{Light}: & R \sim \text{uniform} & [0.1, 0.9] \\ \text{Medium}: & R \sim \text{uniform} & [0.7, 1.3] \\ \text{Heavy}: & R \sim \text{uniform} & [1.1, 2.9]. \end{array}$$

Table 1
Minimum expected warranty servicing costs for the three usage categories.

Ratios		Light usage			Medium usage			Heavy usage		
μ	δ	$E[C^{\Omega}(\psi_3^*)]$	$E[C^{\Omega}(\psi_4^*)]$	$\delta = 1$ or 0	$E[C^{\Omega}(\psi_3^*)]$	$E[C^{\Omega}(\psi_4^*)]$	$\delta = 1$ or 0	$E[C^{\Omega}(\psi_3^*)]$	$E[C^{\Omega}(\psi_4^*)]$	$\delta = 1$ or 0
0.1	0.2	0.3209	0.3218	**0.3200** ($\mathcal{S}_{MR}$)	0.3643	0.3649	**0.3637** ($\mathcal{S}_{MR}$)	0.1465	0.1470	**0.1460** ($\mathcal{S}_{MR}$)
	0.3	0.3218	0.3237		0.3649	0.3661		0.1470	0.1480	
	0.4	0.3227	0.3255		0.3655	0.3673		0.1475	0.1490	
	0.5	0.3236	0.3274		0.3661	0.3685		0.1480	0.1500	
	0.6	0.3245	0.3293		0.3667	0.3697		0.1485	0.1510	
	0.7	0.3254	0.3311		0.3673	0.3709		0.1490	0.1520	
	0.8	0.3263	0.3330		0.3679	0.3721		0.1495	0.1530	
	0.9	0.3272	0.3349		0.3685	0.3732		0.1500	0.1540	
0.2	0.3	**0.6260**	0.6262	0.6400 ($\mathcal{S}_{MR}$)	0.7033	**0.7032**	0.7274 ($\mathcal{S}_{MR}$)	0.2924	0.2928	**0.2919** ($\mathcal{S}_{MR}$)
	0.4	0.6416	0.6433		0.7281	0.7290		0.2929	0.2939	
	0.5	0.6425	0.6450		0.7289	0.7304		0.2934	0.2949	
	0.6	0.6434	0.6468		0.7296	0.7317		0.2939	0.2959	
	0.7	0.6443	0.6486		0.7301	0.7328		0.2944	0.2969	
	0.8	0.6451	0.6504		0.7307	0.7340		0.2949	0.2979	
	0.9	0.6460	0.6522		0.7313	0.7352		0.2954	0.2989	
0.3	0.4	0.8531	**0.8411**	0.9600 ($\mathcal{S}_{MR}$)	0.9579	**0.9390**	1.0894 ($\mathcal{S}_3$)	**0.4365**	0.4366	0.4379 ($\mathcal{S}_{MR}$)
	0.5	0.8885	0.8889		0.9908	0.9904		0.4387	0.4395	
	0.6	0.9180	0.9211		1.0189	1.0226		0.4392	0.4405	
	0.7	0.9410	0.9454		1.0431	1.0498		0.4397	0.4415	
	0.8	0.9569	0.9588		1.0635	1.0706		0.4402	0.4425	
	0.9	0.9623	0.9666		1.0791	1.0843		0.4407	0.4435	
0.4	0.5	1.0425	**1.0119**	1.1401 ($\mathcal{S}_3$)	1.1713	**1.1279**	1.2420 ($\mathcal{S}_3$)	0.5569	**0.5561**	0.5839 ($\mathcal{S}_{MR}$)
	0.6	1.0661	1.0592		1.1884	1.1753		0.5811	0.5815	
	0.7	1.0872	1.0885		1.2038	1.2043		0.5849	0.5859	
	0.8	1.1065	1.1106		1.2178	1.2231		0.5854	0.5869	
	0.9	1.1239	1.1308		1.2304	1.2388		0.5859	0.5879	
0.5	0.6	1.1978	**1.1496**	1.2258 ($\mathcal{S}_3$)	1.3463	**1.2800**	1.3390 ($\mathcal{S}_3$)	0.6630	**0.6594**	0.7299 ($\mathcal{S}_{MR}$)
	0.7	1.2072	1.1881		1.3459	1.3163		0.6943	0.6947	
	0.8	1.2141	1.2124		1.3440	1.3396		0.7164	0.7176	
	0.9	1.2211	1.2237		1.3417	1.3453		0.7286	0.7290	
0.6	0.7	1.3202	**1.2593**	1.2971 ($\mathcal{S}_3$)	1.4834	**1.4006**	1.4249 ($\mathcal{S}_3$)	0.7585	**0.7522**	0.8384 ($\mathcal{S}_3$)
	0.8	1.3132	1.2867		1.4642	1.4245		0.7908	0.7910	
	0.9	1.3062	1.3001		1.4449	1.4351		0.8172	0.8184	
0.7	0.8	1.4093	**1.3425**	1.3552 ($\mathcal{S}_4$)	1.5824	**1.4921**	1.4950 ($\mathcal{S}_4$)	0.8451	**0.8364**	0.9022 ($\mathcal{S}_3$)
	0.9	1.3862	1.3551		1.5447	1.5002		0.8757	0.8755	
0.8	0.9	1.4649	1.3999	**1.3952** ($\mathcal{S}_4$)	1.6433	1.5553	**1.5449** ($\mathcal{S}_4$)	0.9220	**0.9118**	0.9489 ($\mathcal{S}_4$)

In order to make a valid comparison between the imperfect repair strategy and the repair–replacement strategies, the parameters of the conditional intensity function (i.e., the θ_i) and the distribution of the three usage categories are identical to those used in previous works [5,6].

Optimization procedure. For different values of μ and δ ($\mu < \delta$) and for each of the three usage categories, we perform a grid search to find the optimal decision variables denoted by $\psi_3^* = (K_1^*, K_2^*, r_1^*)$ and $\psi_4^* = (K_1^*, K_2^*, K_3^*, r_1^*)$ that minimize the expected total warranty servicing costs $E[C^{\Omega}(\psi_3)]$ and $E[C^{\Omega}(\psi_4)]$, respectively. The variables K_i, are incremented in steps of 0.1 over the interval [0.1, 2.0), and the rate parameter r_1 is incremented in steps of 0.2 over the interval [0.2, 3.0]. This is the grid considered in [5,6].

5.2. Numerical results and analysis

The minimum expected total warranty servicing costs for the 3-subregion and the 4-subregion imperfect repair strategies for the three usage categories and for different cost ratios are presented in Table 1.

Columns 3–5 are the costs for the light usage category. For each (μ, δ) pair (in the first and second columns), the minimum expected cost $E[C^{\Omega}(\psi_3^*)]$ of the 3-subregion imperfect strategy is given in column 3 and the minimum expected cost $E[C^{\Omega}(\psi_4^*)]$ of the 4-subregion imperfect repair strategy is given in column 4. Presented in column 5 is the minimum of the expected servicing costs, and the corresponding repair strategy in brackets, among four previously-studied strategies: the "all minimal repair" strategy (denoted by $\mathcal{S}_{MR}$), the "all replacement strategy" (denoted by $\mathcal{S}_R$), the 3-subregion repair–replacement strategy in [5] (denoted by $\mathcal{S}_3$) and the 4-subregion repair–replacement strategy in [6] (denoted by $\mathcal{S}_4$). All these strategies are based on minimal repairs and replacements (i.e., when $\delta = 0$ or $\delta = 1$).

Columns 6–8 are the corresponding costs for the medium usage category and columns 9–11 are the corresponding costs for the heavy usage category.

Table 2
Optimal warranty servicing strategies for the three usage categories.

Usage category	μ	K_1^*	K_2^*	K_3^*	r_1^*	Expected cost	Strategy
Light	0.1	–	–	–	–	0.3200	$\mathcal{S}_{MR}$
	0.2	0.7	1.5	–	1.0	0.6260	$S_3^{0.3}$
	0.3	0.4	1.1	1.7	1.0	0.8411	$S_4^{0.4}$
	0.4	0.4	1.1	1.8	1.0	1.0119	$S_4^{0.5}$
	0.5	0.3	1.1	1.9	1.0	1.1496	$S_4^{0.6}$
	0.6	0.3	1.1	1.9	1.0	1.2593	$S_4^{0.7}$
	0.7	0.3	1.1	1.9	1.0	1.3425	$S_4^{0.8}$
	0.8	0.3	0.9	1.9	1.0	1.3952	S_4
	0.9	0.2	1.0	1.9	1.0	1.4292	S_4
Medium	0.1	–	–	–	–	0.3637	$\mathcal{S}_{MR}$
	0.2	0.6	0.8	1.5	1.0	0.7032	$S_4^{0.3}$
	0.3	0.4	1.1	1.8	1.0	0.9390	$S_4^{0.4}$
	0.4	0.4	1.1	1.9	1.0	1.1279	$S_4^{0.5}$
	0.5	0.3	1.1	1.9	1.0	1.2800	$S_4^{0.6}$
	0.6	0.3	1.1	1.9	1.0	1.4006	$S_4^{0.7}$
	0.7	0.3	1.1	1.9	1.0	1.4921	$S_4^{0.8}$
	0.8	0.3	0.9	1.9	1.0	1.5449	S_4
	0.9	0.3	1.0	1.9	1.0	1.5880	S_4
Heavy	0.1	–	–	–	–	0.1460	$\mathcal{S}_{MR}$
	0.2	–	–	–	–	0.2919	$\mathcal{S}_{MR}$
	0.3	1.1	1.6	–	0.8	0.4365	$S_3^{0.4}$
	0.4	0.6	1.0	1.6	1.0	0.5561	$S_4^{0.5}$
	0.5	0.4	1.0	1.7	1.0	0.5694	$S_4^{0.6}$
	0.6	0.3	1.0	1.8	1.0	0.7522	$S_4^{0.7}$
	0.7	0.3	1.0	1.9	1.0	0.8364	$S_4^{0.8}$
	0.8	0.3	0.9	1.9	1.0	0.9118	$S_4^{0.9}$
	0.9	0.2	0.9	1.9	1.0	0.9794	S_4

For each of the three usage categories, the minimum of the expected costs in each row is printed in boldface. The costs for the ''all replacement'' strategy for the light, medium and heavy usage categories are 1.4281, 1.5850 and 0.9902, respectively. This strategy does not appear in this table, because it is not among the optimal warranty servicing strategies here.

Note that for all three categories of usage, when μ (cost of a minimal repair) is low, either the ''all minimal repair'' strategy $\mathcal{S}_{MR}$ or the 3-subregion imperfect repair strategy with the lowest degree of repair costs the least. As the cost of a minimal repair increases, the strategy with the minimum expected cost is the 4-subregion imperfect repair strategy with the lowest degree of repair $\delta > \mu$. When the cost of a minimal repair is very high, for the light (columns 3–5) and medium (columns 6–8) categories, the 4-subregion repair–replacement policy proposed in [6] costs the least.

The summary of these results is presented in Table 2, along with the optimal partition of the warranty region for the strategy with the minimum expected cost (for different values of μ).

6. Conclusions

In this article, we use an intensity reduction approach to model the imperfect repairs, and under the specified warranty servicing strategy, identify the optimal strategy from the manufacturer's point of view. We provide a comparison of our results with results previously reported in the literature of warranty servicing strategies.

We conclude that the approach employed to model the imperfect repairs has a significant impact on the estimation of the expected warranty servicing cost for similar servicing strategies. Therefore, an appropriate model for the imperfect repairs is needed, so that the computed expected costs provide accurate estimations of the warranty servicing costs, which is useful to manufacturers in decision-making and developing servicing strategies.

This line of work can be extended in several ways. One possible extension is to model the usage rate as a stochastic process instead of a random variable. Another option for generalization is to use delayed (or accelerated) functions (as in [20]) to model the lifetime distribution of the product after the completion of an imperfect repair. Also, introducing non-zero repair times will make the models more realistic. Non-zero repair time models are very valuable if they are related to customer satisfaction or incurring high penalty costs for the time the product is down (e.g. when a plant assembly line is down).

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