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Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications

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ABSTRACT

In this paper, we shall establish some Hadamard-type and Bullen-type inequalities for Lipschitzian functions and give several applications for special means.

1. Introduction

Throughout this paper, let $L \geq 0$ and $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hadamard's inequality [1].

See [2–14], the results of which are the generalization, improvement and extension of the famous integral inequality (1.1).

Recently, Tseng et al. [9] have established the following Hadamard-type inequality which refines the inequality (1.1).

Theorem A. *Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then we have the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

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The third inequality in (1.2) is known in the literature as Bullen's inequality.

In what follows we recall the following definition.

Definition 1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called an L-Lipschitzian function on the interval I of real numbers if

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in I$.

Dragomir et al. [5] and Matić and Pečarić [8] established the following Hadamard-type inequalities for Lipschitzian functions.

Theorem B. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an L-Lipschitzian function on the interval I of real numbers and $a, b \in I$. Then, we have the following inequalities*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{L(b-a)}{4} \tag{1.3}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{L(b-a)}{4}. \tag{1.4}$$

In this paper, we shall establish some Hadamard-type and Bullen-type inequalities for Lipschitzian functions and give several applications for special means.

2. Hadamard-type inequalities for Lipschitzian functions

Throughout this section, let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq b$ in I and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function.

In the next theorem, let $\alpha \in [0, 1]$, $V = (1-\alpha)a + \alpha b$, and define V_α as follows:

(1) If $a \leq V \leq A \leq B \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 - (A-V)^2 + (B-V)^2 + (b-B)^2.$$

(2) If $a \leq A \leq V \leq B \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 + (V-A)^2 + (B-V)^2 + (b-B)^2.$$

(3) If $a \leq A \leq B \leq V \leq b$, then

$$V_\alpha(A, B) = (A-a)^2 + (V-A)^2 + (b-B)^2 - (V-B)^2.$$

Theorem 1. *Let $A, B, \alpha, V, V_\alpha$ and the function f be defined as above. Then we have the inequality*

$$\left|\alpha f(A) + (1-\alpha)f(B) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{LV_\alpha(A, B)}{2(b-a)}. \tag{2.1}$$

Proof. Using the hypothesis of f, we have the following inequality

$$\begin{aligned}\left|\alpha f(A) + (1-\alpha)f(B) - \frac{1}{b-a}\int_a^b f(x)dx\right| &= \frac{1}{b-a}\left|\int_a^V [f(A) - f(x)]\,dx + \int_V^b [f(B) - f(x)]\,dx\right| \\ &\leq \frac{1}{b-a}\left[\int_a^V |f(A) - f(x)|\,dx + \int_V^b |f(B) - f(x)|\,dx\right] \\ &\leq \frac{L}{b-a}\left[\int_a^V |A-x|\,dx + \int_V^b |B-x|\,dx\right].\end{aligned} \tag{2.2}$$

Now, using simple calculations, we obtain the following identities $\int_a^V |A-x|\,dx$ and $\int_V^b |B-x|\,dx$.

(1) If $a \leq V \leq A \leq B \leq b$, then we have

$$\int_a^V |A-x|\,dx = \frac{(A-a)^2 - (A-V)^2}{2} \quad \text{and} \quad \int_V^b |B-x|\,dx = \frac{(B-V)^2 + (b-B)^2}{2}.$$

(2) If $a \le A \le V \le B \le b$, then we have

$$\int_a^V |A-x|\,dx = \frac{(A-a)^2+(V-A)^2}{2} \quad \text{and} \quad \int_V^b |B-x|\,dx = \frac{(B-V)^2+(b-B)^2}{2}.$$

(3) If $a \le A \le B \le V \le b$, then we have

$$\int_a^V |A-x|\,dx = \frac{(A-a)^2+(V-A)^2}{2} \quad \text{and} \quad \int_V^b |B-x|\,dx = \frac{(b-B)^2-(V-B)^2}{2}.$$

Using the inequality (2.2) and the above identities $\int_a^V |A-x|\,dx$ and $\int_V^b |B-x|\,dx$, we derive the inequality (2.1). This completes the proof. □

Under the assumptions of Theorem 1, we have the following corollaries and remarks:

Corollary 1. (1) *In Theorem 1, let $\lambda \in \left[\frac{1}{2}, 1\right]$, $A = \lambda a + (1-\lambda) b$ and $B = (1-\lambda) a + \lambda b$. Then, we have the inequality*

$$\left|\alpha f(\lambda a + (1-\lambda) b) + (1-\alpha) f((1-\lambda) a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{LM(\alpha,\lambda)(b-a)}{2} \tag{2.3}$$

where

$$M(\alpha,\lambda) = \begin{cases} 2(1-\lambda)^2 + (\lambda-\alpha)^2 - (1-\alpha-\lambda)^2 & \text{as } \alpha \le 1-\lambda \\ 2(1-\lambda)^2 + (\alpha+\lambda-1)^2 + (\lambda-\alpha)^2 & \text{as } 1-\lambda \le \alpha \le \lambda \\ 2(1-\lambda)^2 + (\alpha+\lambda-1)^2 - (\alpha-\lambda)^2 & \text{as } \lambda \le \alpha. \end{cases} \tag{2.4}$$

(2) *In Theorem 1, let $A = B$. Then, we have the inequality*

$$\left|f(A) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(A-a)^2+(b-A)^2}{2(b-a)}L. \tag{2.5}$$

Corollary 2. *We have the following weighted Hadamard-type inequalities for Lipschitzian functions.*

(1) *In the inequality (2.1), let $A = a$, $B = b$. Then*

$$\left|\alpha f(a) + (1-\alpha) f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{\alpha^2+(1-\alpha)^2}{2}L(b-a). \tag{2.6}$$

(2) *In the inequality (2.5), let $\alpha \in [0,1]$, $A = \alpha a + (1-\alpha) b$. Then*

$$\left|f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{\alpha^2+(1-\alpha)^2}{2}L(b-a). \tag{2.7}$$

Remark 1. (1) In the inequality (2.6), let $\alpha = 1/2$. Then the inequality (2.6) reduces to the inequality (1.3).
(2) In the inequality (2.7), let $\alpha = 1/2$. Then the inequality (2.7) reduces to the inequality (1.4).

Remark 2. In the inequality (2.3), let $\alpha = 1/2$ and $\lambda = 3/4$. Then, we have the inequality

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{L(b-a)}{8} \tag{2.8}$$

which is the second inequality of (1.2) for L-Lipschitzian functions.

3. Bullen-type inequalities for Lipschitzian functions

Throughout this section, let I be an interval in $\mathbb{R}$, $a \le A \le B \le C \le b$ in I and let $f: I \to \mathbb{R}$ be an L-Lipschitzian function.

In the next theorem, let $\alpha+\beta+\gamma = 1$ $(\alpha,\beta,\gamma \in [0,1])$, $V_1 = (1-\alpha) a + \alpha b$, $V_2 = \gamma a + (\alpha+\beta) b$, and define $V_{\alpha,\beta,\gamma}$ as follows:

(1) If $V_1 \le V_2 \le A \le B \le C$, then

$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 - (A-V_1)^2 + (B-V_1)^2 - (B-V_2)^2 + (C-V_2)^2 + (b-C)^2.$$

(2) If $V_1 \le A \le V_2 \le B \le C$, then

$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 - (A-V_1)^2 + (B-V_1)^2 - (B-V_2)^2 + (C-V_2)^2 + (b-C)^2.$$

(3) If $V_1 \le A \le B \le V_2 \le C$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 - (A-V_1)^2 + (B-V_1)^2 + (V_2-B)^2 + (C-V_2)^2 + (b-C)^2.$$
(4) If $V_1 \le A \le B \le C \le V_2$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 - (A-V_1)^2 + (B-V_1)^2 + (V_2-B)^2 + (b-C)^2 - (V_2-C)^2.$$
(5) If $A \le V_1 \le V_2 \le B \le C$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (B-V_1)^2 - (B-V_2)^2 + (C-V_2)^2 + (b-C)^2.$$
(6) If $A \le V_1 \le B \le V_2 \le C$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (B-V_1)^2 + (V_2-B)^2 + (C-V_2)^2 + (b-C)^2.$$
(7) If $A \le V_1 \le B \le C \le V_2$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (B-V_1)^2 + (V_2-B)^2 + (b-C)^2 - (V_2-C)^2.$$
(8) If $A \le B \le V_1 \le V_2 \le C$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (V_2-B)^2 - (V_1-B)^2 + (C-V_2)^2 + (b-C)^2.$$
(9) If $A \le B \le V_1 \le C \le V_2$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (V_2-B)^2 - (V_1-B)^2 + (b-C)^2 - (V_2-C)^2.$$
(10) If $A \le B \le C \le V_1 \le V_2$, then
$$V_{\alpha,\beta,\gamma}(A,B,C) = (A-a)^2 + (V_1-A)^2 + (V_2-B)^2 - (V_1-B)^2 + (b-C)^2 - (V_2-C)^2.$$

Theorem 2. *Let $A, B, C, \alpha, \beta, \gamma, V_1, V_2, V_{\alpha,\beta,\gamma}$ and the function f be defined as above. Then we have the inequality*
$$\left|\alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{LV_{\alpha,\beta,\gamma}(A,B,C)}{2(b-a)}. \tag{3.1}$$

Proof. Using the hypothesis of f, we have the following inequality
$$\begin{aligned}
&\left|\alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&= \frac{1}{b-a}\left|\int_a^{V_1}[f(A)-f(x)]\,dx + \int_{V_1}^{V_2}[f(B)-f(x)]\,dx + \int_{V_2}^{b}[f(C)-f(x)]\,dx\right| \\
&\le \frac{1}{b-a}\left[\int_a^{V_1}|f(A)-f(x)|\,dx + \int_{V_1}^{V_2}|f(B)-f(x)|\,dx + \int_{V_2}^{b}|f(C)-f(x)|\,dx\right] \\
&\le \frac{L}{b-a}\left[\int_a^{V_1}|A-x|\,dx + \int_{V_1}^{V_2}|B-x|\,dx + \int_{V_2}^{b}|C-x|\,dx\right].
\end{aligned} \tag{3.2}$$

Now, using simple calculations, we obtain the following identities $\int_a^{V_1}|A-x|\,dx$, $\int_{V_1}^{V_2}|B-x|\,dx$ and $\int_{V_2}^{b}|C-x|\,dx$.

(1) If $V_1 \le V_2 \le A \le B \le C$, then we have
$$\int_a^{V_1}|A-x|\,dx = \frac{(A-a)^2-(A-V_1)^2}{2}, \qquad \int_{V_1}^{V_2}|B-x|\,dx = \frac{(B-V_1)^2-(B-V_2)^2}{2}$$
and
$$\int_{V_2}^{b}|C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(2) If $V_1 \le A \le V_2 \le B \le C$, then we have
$$\int_a^{V_1}|A-x|\,dx = \frac{(A-a)^2-(A-V_1)^2}{2}, \qquad \int_{V_1}^{V_2}|B-x|\,dx = \frac{(B-V_1)^2-(B-V_2)^2}{2}$$
and
$$\int_{V_2}^{b}|C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(3) If $V_1 \le A \le B \le V_2 \le C$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2-(A-V_1)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(B-V_1)^2+(V_2-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(4) If $V_1 \le A \le B \le C \le V_2$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2-(A-V_1)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(B-V_1)^2+(V_2-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(b-C)^2-(V_2-C)^2}{2}.$$

(5) If $A \le V_1 \le V_2 \le B \le C$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(B-V_1)^2-(B-V_2)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(6) If $A \le V_1 \le B \le V_2 \le C$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(B-V_1)^2+(V_2-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(7) If $A \le V_1 \le B \le C \le V_2$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(B-V_1)^2+(V_2-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(b-C)^2-(V_2-C)^2}{2}.$$

(8) If $A \le B \le V_1 \le V_2 \le C$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(V_2-B)^2-(V_1-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(C-V_2)^2+(b-C)^2}{2}.$$

(9) If $A \le B \le V_1 \le C \le V_2$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(V_2-B)^2-(V_1-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(b-C)^2-(V_2-C)^2}{2}.$$

(10) If $A \le B \le C \le V_1 \le V_2$, then we have

$$\int_a^{V_1} |A-x|\,dx = \frac{(A-a)^2+(V_1-A)^2}{2}, \qquad \int_{V_1}^{V_2} |B-x|\,dx = \frac{(V_2-B)^2-(V_1-B)^2}{2}$$

and

$$\int_{V_2}^{b} |C-x|\,dx = \frac{(b-C)^2-(V_2-C)^2}{2}.$$

Using the inequality (3.2) and the above identities $\int_a^{V_1} |A-x|\,dx$, $\int_{V_1}^{V_2} |B-x|\,dx$ and $\int_{V_2}^{b} |C-x|\,dx$, we derive the inequality (3.1). This completes the proof. □

Under the assumptions of Theorem 2, we have the following Bullen-type inequalities for Lipschitzian functions:

Corollary 3. *In Theorem 2, let $\rho \in \left[\frac{1}{2}, 1\right]$, $A = \rho a + (1-\rho)\, b$, $B = \frac{a+b}{2}$ and $C = (1-\rho)\, a + \rho b$. Then, we have the inequality*

$$\left|\alpha f\left(\rho a + (1-\rho)\, b\right) + \beta f\left(\frac{a+b}{2}\right) + \gamma f\left((1-\rho)\, a + \rho b\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{LN\left(\alpha,\beta\right)(b-a)}{2} \tag{3.3}$$

where $N(\alpha,\beta)$ is defined as follows:

(1) *If $\alpha+\beta \le 1-\rho$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 - (1-\rho-\alpha)^2 + \left(\frac{1}{2}-\alpha\right)^2 - \left(\frac{1}{2}-\alpha-\beta\right)^2 + (\rho-\alpha-\beta)^2.$$

(2) *If $\alpha \le 1-\rho \le \alpha+\beta \le \frac{1}{2}$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 - (1-\rho-\alpha)^2 + \left(\frac{1}{2}-\alpha\right)^2 - \left(\frac{1}{2}-\alpha-\beta\right)^2 + (\rho-\alpha-\beta)^2.$$

(3) *If $\alpha \le 1-\rho \le \frac{1}{2} \le \alpha+\beta \le \rho$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 - (1-\rho-\alpha)^2 + \left(\frac{1}{2}-\alpha\right)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 + (\rho-\alpha-\beta)^2.$$

(4) *If $\alpha \le 1-\rho \le \frac{1}{2} \le \rho \le \alpha+\beta$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 - (1-\rho-\alpha)^2 + \left(\frac{1}{2}-\alpha\right)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 - (\alpha+\beta-\rho)^2.$$

(5) *If $1-\rho \le \alpha \le \alpha+\beta \le \frac{1}{2}$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 + (\alpha-1+\rho)^2 + \left(\frac{1}{2}-\alpha\right)^2 - \left(\frac{1}{2}-\alpha-\beta\right)^2 + (\rho-\alpha-\beta)^2.$$

(6) *If $1-\rho \le \alpha \le \frac{1}{2} \le \alpha+\beta \le \rho$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 + (\alpha-1+\rho)^2 + \left(\frac{1}{2}-\alpha\right)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 + (\rho-\alpha-\beta)^2.$$

(7) *If $1-\rho \le \alpha \le \frac{1}{2} \le \rho \le \alpha+\beta$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 + (\alpha-1+\rho)^2 + \left(\frac{1}{2}-\alpha\right)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 - (\alpha+\beta-\rho)^2.$$

(8) *If $\frac{1}{2} \le \alpha \le \alpha+\beta \le \rho$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 + (\alpha-1+\rho)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 - \left(\alpha-\frac{1}{2}\right)^2 + (\rho-\alpha-\beta)^2.$$

(9) *If $\frac{1}{2} \le \alpha \le \rho \le \alpha+\beta$, then*

$$N(\alpha,\beta) = 2(1-\rho)^2 + (\alpha-1+\rho)^2 + \left(\alpha+\beta-\frac{1}{2}\right)^2 - \left(\alpha-\frac{1}{2}\right)^2 - (\alpha+\beta-\rho)^2.$$

(10) *If* $\rho \le \alpha$, *then*

$$N(\alpha, \beta) = 2(1-\rho)^2 + (\alpha - 1 + \rho)^2 + \left(\alpha + \beta - \frac{1}{2}\right)^2 - \left(\alpha - \frac{1}{2}\right)^2 - (\alpha + \beta - \rho)^2.$$

Corollary 4. *In Corollary 3, let* $\rho = 1, \alpha = \gamma = \frac{\delta}{2}$ *and* $\beta = 1 - \delta$ *with* $\delta \in [0, 1]$. *Then, we have the weighted Bullen-type inequality*

$$\left|\delta \frac{f(a) + f(b)}{2} + (1-\delta) f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(x)dx\right| \le \frac{\delta^2 + (1-\delta)^2}{4} L(b-a) \tag{3.4}$$

for L-Lipschitzian functions.

Remark 3. In the inequality (3.4), let $\delta = 1/2$. Then the inequality (3.4) reduces to Bullen-type inequality

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2}\right] - \frac{1}{(b-a)} \int_a^b f(x)dx\right| \le \frac{L(b-a)}{8} \tag{3.5}$$

for L-Lipschitzian functions.

Remark 4. In the inequality (3.4), let $\delta = 1/3$. Then the inequality (3.4) reduces to Simpson-type inequality

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{(b-a)} \int_a^b f(x)dx\right| \le \frac{5L(b-a)}{36} \tag{3.6}$$

for L-Lipschitzian functions. The inequality (3.6) was proved by Dragomir [15].

4. Some applications for special means

Let us recall the following special means of the two nonnegative number u and v with $\alpha \in [0, 1]$:

(1) The weighted arithmetic mean

$$A_\alpha(u, v) := \alpha u + (1-\alpha) v, \quad u, v \ge 0.$$

(2) The unweighted arithmetic mean

$$A(u, v) := \frac{u+v}{2}, \quad u, v \ge 0.$$

(3) The weighted geometric mean

$$G_\alpha(u, v) := u^\alpha v^{1-\alpha}, \quad u, v > 0.$$

(4) The unweighted geometric mean

$$G(u, v) := \sqrt{uv}, \quad u, v > 0$$

(5) The weighted harmonic mean

$$H_\alpha(u, v) := \left(\frac{\alpha}{u} + \frac{1-\alpha}{v}\right)^{-1}, \quad u, v > 0.$$

(6) The unweighted harmonic mean

$$H(u, v) := \frac{2uv}{u+v}, \quad u, v > 0.$$

(7) The logarithmic mean

$$L(u, v) := \begin{cases} \dfrac{v-u}{\ln v - \ln u} & \text{if } u \ne v \\ u & \text{if } u = v, \end{cases} \quad u, v > 0.$$

(8) The identric mean

$$I = I(u, v) := \begin{cases} \dfrac{1}{e}\left(\dfrac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u \ne v \\ u & \text{if } u = v, \end{cases} \quad u, v > 0.$$

(9) The p-logarithmic mean

$$L_p(u,v) := \begin{cases} \left[\dfrac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right]^{\frac{1}{p}} & \text{if } u \neq v \\ u & \text{if } u = v, \end{cases} \quad u, v > 0,\ p \in (-1,\infty) \setminus \{0\}.$$

To prove the results of this section, we need the following lemma:

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable with $\|f'\|_\infty < \infty$. Then f is an L-Lipschitzian function on $[a,b]$ where $L = \|f'\|_\infty$.*

Proof. The result is obvious by the Mean-Value theorem. We shall omit the details. □

Using Corollaries 1, 3 and 4, Theorem A and Lemma 1, we have the following propositions and remarks about the above special means:

Proposition 1. *In Corollary 1 and Lemma 1, let $r \geq 1$, $a, b \geq 0$ and $f(x) = x^r$ on $[a,b]$. Then we have the inequality*

$$\left|A_\alpha\left(A_\lambda^r(a,b), A_{1-\lambda}^r(a,b)\right) - L_r^r(a,b)\right| \leq \frac{rb^{r-1}M(\alpha,\lambda)(b-a)}{2} \tag{4.1}$$

where $M(\alpha,\lambda)$ is defined as in (2.4).

Proposition 2. *In Corollary 2 and Lemma 1, let $r \geq 1$, $a, b \geq 0$ and $f(x) = x^r$ on $[a,b]$. Then we have the inequalities*

$$\left|A_\alpha\left(a^r, b^r\right) - L_r^r(a,b)\right| \leq \frac{\alpha^2+(1-\alpha)^2}{2} rb^{r-1}(b-a) \tag{4.2}$$

and

$$\left|A_\alpha^r(a,b) - L_r^r(a,b)\right| \leq \frac{\alpha^2+(1-\alpha)^2}{2} rb^{r-1}(b-a). \tag{4.3}$$

Remark 5. Let $\alpha = 1/2$ in the inequalities (4.2) and (4.3). Then, using Hadamard's inequality (1.1), we have Hadamard-type inequalities

$$0 \leq A\left(a^r, b^r\right) - L_r^r(a,b) \leq \frac{rb^{r-1}(b-a)}{4} \tag{4.4}$$

and

$$0 \leq L_r^r(a,b) - A^r(a,b) \leq \frac{rb^{r-1}(b-a)}{4}. \tag{4.5}$$

Proposition 3. *In Corollary 4 and Lemma 1, let $r \geq 1$, $a, b \geq 0$ and $f(x) = x^r$ on $[a,b]$. Then we have the inequality*

$$\left|A_\delta\left(A\left(a^r, b^r\right), A^r(a,b)\right) - L_r^r(a,b)\right| \leq \frac{\delta^2+(1-\delta)^2}{4} rb^{r-1}(b-a). \tag{4.6}$$

Remark 6. Let $\delta = 1/2$ in the inequality (4.6). Then, using Bullen's inequality in the inequality (1.2), we have Bullen-type inequality

$$\begin{aligned} 0 &\leq A\left(A\left(a^r, b^r\right), A^r(a,b)\right) - L_r^r(a,b) \\ &\leq \frac{rb^{r-1}(b-a)}{8}. \end{aligned} \tag{4.7}$$

Proposition 4. *In Corollary 1 and Lemma 1, let $a, b > 0$ and $f(x) = -\ln x$ on $[a,b]$. Then we have the inequality*

$$\left|A_\alpha\left(\ln A_\lambda(a,b), \ln A_{1-\lambda}(a,b)\right) - \ln I(a,b)\right| \leq \frac{M(\alpha,\lambda)(b-a)}{2a} \tag{4.8}$$

where $M(\alpha,\lambda)$ is defined as in (2.4).

Proposition 5. *In Corollary 2 and Lemma 1, let $a, b > 0$ and $f(x) = -\ln x$ on $[a, b]$. Then we have the inequalities*

$$|A_\alpha(\ln a, \ln b) - \ln I(a, b)| \leq \frac{\alpha^2 + (1-\alpha)^2}{2a}(b-a) \tag{4.9}$$

and

$$|\ln A_\alpha(a, b) - \ln I(a, b)| \leq \frac{\alpha^2 + (1-\alpha)^2}{2a}(b-a). \tag{4.10}$$

Remark 7. Let $\alpha = 1/2$ in the inequalities (4.9) and (4.10). Then, using Hadamard's inequality (1.1), we have Hadamard-type inequalities

$$0 \leq \ln I(a, b) - A(\ln a, \ln b) \leq \frac{b-a}{4a} \tag{4.11}$$

and

$$0 \leq \ln A(a, b) - \ln I(a, b) \leq \frac{b-a}{4a}. \tag{4.12}$$

Proposition 6. *In Corollary 4 and Lemma 1, let $a, b > 0$ and $f(x) = -\ln x$ on $[a, b]$. Then we have the inequality*

$$|A_\delta(A(\ln a, \ln b), \ln A(a, b)) - \ln I(a, b)| \leq \frac{\delta^2 + (1-\delta)^2}{4a}(b-a). \tag{4.13}$$

Remark 8. Let $\delta = 1/2$ in the inequality (4.13). Then, using Bullen's inequality in the inequality (1.2), we have Bullen-type inequality

$$0 \leq \ln I(a, b) - A(A(\ln a, \ln b), \ln A(a, b)) \leq \frac{b-a}{8a}. \tag{4.14}$$

Proposition 7. *In Corollary 1 and Lemma 1, let $a, b \in \mathbb{R}$ and $f(x) = e^x$ on $[a, b]$. Then we have the inequality*

$$\left|A_\alpha\left(e^{A_\lambda(a,b)}, e^{A_{1-\lambda}(a,b)}\right) - \frac{e^b - e^a}{b-a}\right| \leq \frac{M(\alpha, \lambda)\, e^b (b-a)}{2} \tag{4.15}$$

where $M(\alpha, \lambda)$ is defined as in (2.4).

Proposition 8. *In Corollary 2 and Lemma 1, let $a, b \in \mathbb{R}$ and $f(x) = e^x$ on $[a, b]$. Then we have the inequalities*

$$\left|A_\alpha\left(e^a, e^b\right) - \frac{e^b - e^a}{b-a}\right| \leq \frac{\alpha^2 + (1-\alpha)^2}{2} e^b (b-a) \tag{4.16}$$

and

$$\left|e^{A_\alpha(a,b)} - \frac{e^b - e^a}{b-a}\right| \leq \frac{\alpha^2 + (1-\alpha)^2}{2} e^b (b-a). \tag{4.17}$$

Remark 9. Let $\alpha = 1/2$ in the inequalities (4.16) and (4.17). Then, using Hadamard's inequality (1.1), we have Hadamard-type inequalities

$$0 \leq A\left(e^a, e^b\right) - \frac{e^b - e^a}{b-a} \leq \frac{e^b (b-a)}{4} \tag{4.18}$$

and

$$0 \leq \frac{e^b - e^a}{b-a} - e^{A(a,b)} \leq \frac{e^b (b-a)}{4}. \tag{4.19}$$

Proposition 9. *In Corollary 4 and Lemma 1, let $a, b \in \mathbb{R}$ and $f(x) = e^x$ on $[a, b]$. Then we have the inequality*

$$\left|A_\delta\left(A\left(e^a, e^b\right), e^{A(a,b)}\right) - \frac{e^b - e^a}{b - a}\right| \le \frac{\delta^2 + (1-\delta)^2}{4} e^b (b - a). \tag{4.20}$$

Remark 10. Let $\delta = 1/2$ in the inequality (4.20). Then, using Bullen's inequality in the inequality (1.2), we have Bullen-type inequality

$$\begin{aligned} 0 &\le A\left(A\left(e^a, e^b\right), e^{A(a,b)}\right) - \frac{e^b - e^a}{b - a} \\ &\le \frac{e^b (b - a)}{8}. \end{aligned} \tag{4.21}$$

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