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# Vertex-splitting and chromatic index critical graphs

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**Abstract**

We study graphs which are critical with respect to the chromatic index. We relate these to the Overfull Conjecture and we study in particular their construction from regular graphs by subdividing an edge or by splitting a vertex.

In this paper, we consider *simple graphs* (that is graphs which have no loops or multiple edges). An *edge-colouring* of a graph $G$ is a map $\phi : E(G) \to \varphi$, where $\varphi$ is a set of colours and $E(G)$ is the set of edges of $G$, such that no two incident edges receive the same colour. The *chromatic index*, $\chi'(G)$ of $G$ is the least value of $|\varphi|$ for which an edge-colouring of $G$ exists. A well-known theorem of Vizing [13] states that, for a simple graph $G$,

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1,$$

where $\Delta(G)$ denotes the maximum degree of $G$. Graphs for which $\chi'(G) = \Delta(G)$ are said to be *Class* 1, and otherwise they are *Class* 2. A graph $G$ is *critical* if it is Class 2, connected and for each edge $e$ of $G$, $\chi'(G \backslash e) < \chi'(G)$.

A fairly long-standing problem has been the attempt to classify which graphs are Class 1, and which graphs are Class 2. Holyer [10] showed that the problem of determining whether a graph is Class 1 is NP-hard. Notwithstanding this, the Overfull Conjecture of Chetwynd and Hilton [3], if true, would classify all graphs satisfying $\Delta(G) > \frac{1}{3}|V(G)|$ into Class 1 and Class 2 graphs. A graph is called *overfull* if

$$|E(G)| > \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

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It is easy to see that if $G$ is overfull, then $G$ must be Class 2. The Overfull Conjecture is:

**Overfull Conjecture.** *If a simple graph $G$ satisfies $\Delta(G) > \frac{1}{3}|V(G)|$, then $G$ is Class 2 if and only if it contains an overfull subgraph $H$ with $\Delta(G) = \Delta(H)$.*

There is a moderate amount of evidence to support this conjecture; for a review of the evidence, see [7] or [9]. It is worth noting that if $g(n)$ is the number of graphs of order $n$, and $h(n)$ is the number of graphs of order $n$ and maximum degree $\geqslant n/2$, then $g(n)/h(n) \to 1$; this is true whether the graphs counted are labelled or unlabelled (see [6]). Also worth noting is the fact that Niessen [11] recently gave a polynomial algorithm to determine if a graph $G$ satisfying $\Delta(G) \geqslant \frac{1}{2}|V(G)|$ has an overfull subgraph $H$ with $\Delta(G) = \Delta(H)$, and thus demonstrated that, if the Overfull Conjecture is true, then there is a polynomial algorithm to determine if a graph $G$ satisfying $\Delta(G) \geqslant \frac{1}{2}|V(G)|$ is Class 1 or Class 2.

Let $K$ be a connected graph and let $x$ be a vertex of $K$ such that $d_K(x) = m \geqslant 2$. Suppose the neighbourhood $N(x)$ of $x$ is $N(x) = \{x_1, \ldots, x_m\}$. We say that the graph $G$ is obtained from $K$ by *splitting $x$ into two vertices $u$ and $v$* if $V(G) = V(K \backslash x) \cup \{u, v\}$ and $E(G) = E(K \backslash x) \cup uv \cup_{1 \leqslant i \leqslant r} ux_i \cup_{r+1 \leqslant i \leqslant m} vx_i$ for some $1 \leqslant r \leqslant m$.

From this definition, we see that if $G$ is obtained from a connected graph $K$ by inserting a vertex $v$ into any edge whose end vertices are both of degree $\geqslant 2$, then $G$ can be considered as a graph obtained from $K$ by splitting a vertex $x$ into two vertices $x$ and $v$.

In this paper we consider further a question that Yap [15] studied earlier, namely under what circumstances is it true that if a vertex of a regular Class 1 graph $G$ is split into two vertices, then the graph $G^*$ obtained is critical?

Yap [15, 16] proved the following theorem.

**Theorem 1.** *Let $t \geqslant 1$, let $r \geqslant 2$ and let $rt$ be even. If any vertex of the complete $r$-partite graph $K_t^r$ with $t$ vertices in each part is split into two vertices, then the resulting graph is critical.*

In Yap's paper the example in Fig. 1 is given of a graph $G^*$, obtained from a three regular graph by splitting one vertex, where $G^*$ is not critical. In fact, $G^* \backslash pq$ is 3-critical, and can be obtained from Petersen's graph by deleting one vertex.

**Example.**

Although we do not always obtain a critical graph by splitting a vertex, we would like to propose the following conjecture:

**Conjecture 1.** *Let $G$ be a connected regular Class 1 graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Let $G^*$ be the graph obtained from $G$ by splitting one vertex $x$ of $G$ into two vertices $u$ and $v$. Then $G$ is critical.*

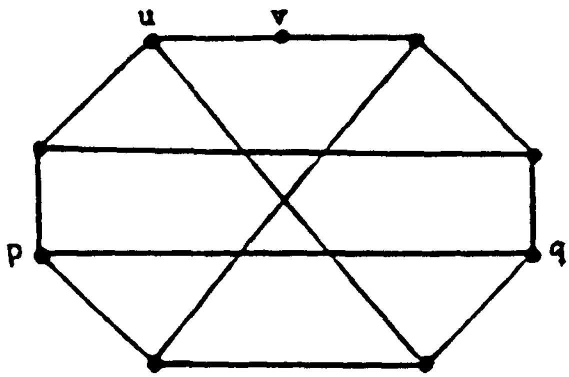


Fig. 1. $G^*$.

The example above shows that the figure $\frac{1}{3}$ cannot be reduced significantly. Note that Yap's theorem, Theorem 1, supports Conjecture 1. The results in this paper all lend further support to this conjecture.

It is easy to see that $G^*$ is Class 2.

**Lemma 1.** *If any vertex $x$ of any connected regular Class 1 graph $G$ with $d(G) \geqslant 2$ is split into two vertices $u$ and $v$ forming $G^*$, then $G^*$ is Class 2.*

**Proof.** It is easy to see that $|V(G)|$ is even and so

$$|E(G^*)| = d(G)\left\lfloor \frac{|V(G)|}{2} \right\rfloor + 1 = \Delta(G^*)\left\lfloor \frac{|V(G^*)|}{2} \right\rfloor + 1,$$

so that $G^*$ is overfull. It is well-known, and easy to see, that any overfull graph is Class 2. Therefore $G^*$ is Class 2. □

To show that $G^*$ is critical we have to show that $G^*\backslash e$ is Class 1 for each edge $e \in E(G^*)$. This is always the case for edges $e$ incident with the new vertices $u$ and $v$.

**Lemma 2.** *If any vertex $x$ of any connected regular Class 1 graph $G$ with $d(G) \geqslant 2$ is split into two vertices $u$ and $v$ forming $G^*$, and if $e$ is an edge of $G^*$ incident with $u$ and $v$, then $G^*\backslash e$ is Class 1.*

**Proof.** If $e = uv$ then any Class 1 edge colouring of $G$ yields a Class 1 edge colouring of $G^*\backslash e$ by allowing each edge to retain its original colouring. If $e$ is incident with $u$, but $e \neq uv$, then we can take the edge colouring of $G^*\backslash v$ just above, and then give $uv$ the colour $e$ had. If $e$ is incident with $v$, the argument is similar. □

Our first result is that the Overfull Conjecture implies Conjecture 1.

**Theorem 2.** *The Overfull Conjecture implies Conjecture 1.*

**Proof.** Let $G$ be a regular connected Class 1 graph satisfying $d(G) > \frac{1}{3}|V(G)|$, and let a vertex $x$ of $G$ be split into two vertices $u$ and $v$. By Lemmas 1 and 2, $G^*$ is Class 2, and if $e \in E(G^*)$ and $e$ is incident with $u$ or $v$ then $G^* \backslash e$ is Class 1. So now let $e$ be non-incident with $u$ or $v$. Let $G_1 = G^* \backslash e$. We need to show that $G_1$ is Class 1.

Suppose on the contrary that $G_1$ is Class 2. Then, according to the Overfull Conjecture, $G_1$ has an overfull subgraph $H$ with $\Delta(H) = \Delta(G_1) = d(G)$. Note that $G_1$ is not overfull, so that $|V(H)| < |V(G_1)|$. If $u \notin V(H)$ then $H$ would be Class 1, since $G$ was Class 1. So we may suppose that $u \in V(H)$, and similarly that $v \in V(H)$. Since $G$ is connected, $G^*$ is connected. Then $G_1$ must be connected unless $e$ was a cut edge; if $e$ were a cut edge in $G_1$ then $e$ must also be a cut edge in $G$. But this would mean that $H$ could be obtained from an even order graph, say $H^*$, by splitting the vertex $x$, and that, within $H^*$, each vertex except one had degree $d(G)$ and that the exceptional vertex had degree $d(G) - 1$. But this would imply that $2|E(H^*)| = (|V(H^*)| - 1)d(G) + (d(G) - 1) = |V(H^*)|d(G) - 1$, which is impossible, since $|V(H^*)|$ is even. Therefore $G_1$ is connected. Therefore $|E(H, G_1 \backslash H)| \geqslant 1$. Consequently,

$$\begin{aligned} 2|E(H)| &= \sum_{v \in V(H)} d_H(v) \\ &\leqslant (2 + \Delta(G)) + (|V(H)| - 2)\Delta(G) - 1 \\ &= 1 + (|V(H)| - 1)\Delta(G). \end{aligned}$$

But since $H$ is overfull, $|V(H)| - 1$ is even, and so

$$2|E(H)| \leqslant (|V(H)| - 1)\Delta(G).$$

But this implies that $H$ is not overfull, a contradiction. □

**Corollary 3.** *The Overfull Conjecture implies that if $G$ is a regular graph of even order satisfying $d(G) \geqslant \frac{1}{2}|V(G)|$, and if $G^*$ is obtained from $G$ by splitting a vertex, then $G^*$ is critical.*

**Proof.** It was shown by Hilton in [8], and independently by Niessen and Volkmann [12], that if the Overfull Conjecture is true, then every regular graph $G$ satisfying $d(G) \geqslant \frac{1}{2}|V(G)|$ is Class 1. It is obviously also connected. The corollary then follows from Theorem 2. □

However, a statement that is very much stronger than Corollary 3 can readily be made. Call a graph $G$ *just overfull* if

$$|E(G)| = \Delta(G)\lfloor \tfrac{1}{2}|V(G)| \rfloor + 1.$$

Note that if $G$ is a regular graph of even order, and $G^*$ is formed from $G$ by splitting a vertex, then $G^*$ is just overfull (see the proof of Lemma 1). Niessen [11] recently showed that if $\Delta(G) \geqslant \frac{1}{2}|V(G)|$ and $G$ is overfull, then $G$ contains no proper induced overfull subgraph $H$ with $\Delta(G) = \Delta(H)$. We can therefore deduce the following stronger result.

**Theorem 4.** *The Overfull Conjecture implies that if $\Delta(G) \geqslant \frac{1}{2}|V(G)|$ then $G$ is critical if and only if $G$ is just overfull.*

**Proof.** First suppose that $G$ is just overfull. Then for any edge $e \in E(G)$, $G \backslash e$ is not overfull. Also, by Niessen's result, $G$ contains no proper induced overfull subgraph $H$ with $\Delta(G) = \Delta(H)$. Consequently, by the Overfull Conjecture, $G \backslash e$ is Class 1. Thus $G$ is critical.

Next suppose that $G$ is critical. Then $G$ is Class 2, so by the Overfull Conjecture $G$ contains an overfull subgraph $H$ with the same maximum degree. If $H \neq G$ then since $G$ is connected, $G$ contains an edge $e$ that is not in $H$. Therefore $G \backslash e$ contains $H$, and so is Class 2, and therefore is not critical, a contradiction. Therefore $H = G$, and every proper subgraph of $G$ is not overfull. Therefore $G$ is just overfull. □

In the rest of the paper the graphs are simple, but we show now that if $G$ is a bipartite multigraph and $G^*$ is obtained from $G$ by splitting a vertex then $G^*$ is critical. First we prove a useful lemma.

**Lemma 3.** *Let $G$ be a connected regular bipartite multigraph of degree $d \geqslant 2$, with independent vertex sets $A$ and $B$. Let $a \in A$ and $b \in B$. Then $G \backslash \{a, b\}$ has a perfect matching.*

**Proof.** Let $A' = A \backslash \{a\}$ and $B' = B \backslash \{b\}$. Let $G_1 = G \backslash \{a, b\}$. Suppose $G_1$ has no perfect matching. Then by Hall's theorem, there is a set $X \subset A$ such that $|N(X)| \geqslant |X| - 1$. Then

$$\begin{aligned} d(|X| - 1) &\geqslant d|N(X)| \\ &\geqslant \sum_{w \in N(X)} d_{G_1}(w) \\ &\geqslant (d-1)|X|, \end{aligned}$$

so $|X| \geqslant d$. There are at most $d$ vertices of degree $d-1$ in $X$, so we obtain in a similar fashion,

$$\begin{aligned} d|N(X)| &\geqslant \sum_{w \in N(X)} d_{G_1}(w) \\ &\geqslant \sum_{w \in X} d_{G_1}(w) \\ &\geqslant d|X| - d. \end{aligned}$$

Therefore, $|N(X)| = |X| - 1$ and equality holds everywhere above. But this implies that $\{b\} \cup N(X) \cup X$ induces in $G$ a connected component with $2d$ vertices containing $a$ and not containing $b$, and thus that $G$ is not connected, a contradiction. □

**Theorem 5.** *Let $G$ be a connected regular bipartite multigraph of degree $d \geqslant 2$. Let $G^*$ be obtained from $G$ by splitting any vertex $x$ into two vertices $u$ and $v$. Then $G^*$ is critical.*

**Proof.** Let $V(G) = A \cup B$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, and each edge of $G$ joins a vertex of $A$ to a vertex of $B$. We may suppose that $x \in A$. By Lemmas 1 and 2, $G^*$ is Class 2 and, if $e$ is an edge that is incident with $u$ or $v$, then $G^* \backslash e$ is Class 1. Now let $e = yz$ where $y \in A$ and $z \in B$, and consider the case when $e$ is not incident with $u, v$. We have to show that $G^* \backslash e$ is Class 1.

By Lemma 3 the graph $G \backslash \{x, z\}$ has a perfect matching $E$. Clearly, $G^* \backslash \{uv \cup E \cup e\}$ has maximum degree $d - 1$, and, being bipartite, is Class 1. Therefore $G^* \backslash e$ is Class 1 also. □

Finally, we show that if $G$ is a regular graph of even order and of sufficently high degree, and $G^*$ is obtained from $G$ by splitting a vertex, then $G$ is critical. Corollary 3 showed that this would follow from the Overfull Conjecture if $d(G) \geqslant \frac{1}{2}|V(G)|$, but here all we can do is to prove it for, approximately, $d(G) \geqslant 0.823|V(G)|$.

Before giving this final result we need to state the following generalization due to Berge [1] of a well-known result of Chvàtal [5].

**Lemma 4.** *Let $G$ be a simple graph of order $n$ with degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. Let $q$ be an integer with $0 \leqslant q \leqslant n - 3$. If, for every $k$ with $q < k < \frac{1}{2}(n + q)$, the following condition holds:*

$$d_{k-q} \leqslant k \Rightarrow d_{n-k} \geqslant n - k + q,$$

*then, for each set $F$ of independent edges with $|F| = q$, there exists a Hamiltonian circuit containing $F$.*

We also need the following result due to Chetwynd and Hilton [4].

**Lemma 5.** *Let $G$ be a regular simple graph of even order satisfying $d(G) \geqslant \frac{1}{2}(\sqrt{7} - 1)|V(G)|$. Then $G$ is Class 1.*

We now give our final result.

**Theorem 6.** *Let $G$ be a regular simple graph of even order satisfying $d(G) \geqslant \frac{1}{2}(\sqrt{7} - 1)|V(G)| + 2$. If $G^*$ is obtained from $G$ by splitting any vertex $x$ into two vertices $u$ and $v$, then $G^*$ is critical.*

**Proof.** By Lemma 5, $G$ is Class 1. It follows by Lemmas 1 and 2 that $G^*$ is Class 2 and that if $e \in E(G^*)$ and $e$ is incident with $u$ and $v$ then $G^* \backslash e$ is Class 1.

So now let $e \in E(G^*)$ and let $e$ be non-incident with $u$ or $v$. Let $e = yz$ and let $G_1 = G^* \backslash \{e\}$. Let $v_1 \in N(v)$. First, choose a perfect matching $M_1$ of $G_1 \backslash \{u, v, y\}$. We show below that $M_1$ can be chosen. Second, choose another perfect matching $M_2$ of $G_1 \backslash (M_1 \cup \{v, v_1, z\})$. We also show that $M_2$ can be chosen. Now consider the graph $G_1 \backslash \{M_1, uv, M_2, vv_1\}$ and form a graph $G'$ by identifying $u$ and $v$; the graph $G'$ is regular of even order $|V(G)|$ and degree $d-2$. Then by Lemma 5, $G'$ is Class 1. It follows that $G_1 \backslash (M_1, uv, M_2, vv_1)$ is Class 1, and then that $G_1$ itself is Class 1.

Now, let us show that $M_1$ and $M_2$ can be chosen. In $G'$, let $u_1 \in N(u)$. Now consider the graph $G$. Recall that $G$ is regular of degree $d$. Let $H$ be a Hamiltonian circuit containing the edges $xu_1, xv_1$ and $e$. This corresponds in $G_1$ to a Hamilton path starting and finishing on $y$ and $z$ containing the edges $u_1u, uv, vv_1$. Clearly, if we choose every other edge of this Hamilton path so that the edge $uv$ is included (possibly interchanging the labels $y$ and $z$), then we obtain $M_1 \cup \{uv\}$. Also, the other set of alternate edges yields $M_2 \cup \{vv_1\}$. To see that $G$ does have a Hamiltonian circuit containing $xu_1$, $xv_1$ and $e$ we apply Lemma 4 with $q = 3$. □

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